

ON SYMMETRIZATION OF $6j$ -SYMBOLS AND LEVIN-WEN HAMILTONIAN

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ABSTRACT. It is known that every ribbon category with unimodality allows symmetrized $6j$ -symbols with full tetrahedral symmetries while a spherical category does not in general. We give an explicit counterexample for this, namely the category \mathcal{E} . We define the mirror conjugate symmetry of $6j$ -symbols instead and show that $6j$ -symbols of any unitary spherical category can be normalized to have this property. As an application, we discuss an exactly soluble model on a honeycomb lattice. We prove that the Levin-Wen Hamiltonian is exactly soluble and hermitian on a unitary spherical category.

1. INTRODUCTION

At first notice that in this paper $6j$ -symbols with 24 full tetrahedral symmetries are called symmetrized $6j$ -symbols rather than normalized $6j$ -symbols as in [T]. However they have the same meaning.

This paper is about spherical fusion categories and their $6j$ -symbols. More specifically the $6j$ -symbols in the well-known examples are invariant under the symmetry group of the tetrahedron. For example, this is the case if the category is unimodal. In [T], V. Turaev showed that a ribbon category with unimodality allows symmetrized $6j$ -symbols (see chapter 6 in [T]), which implies that the state sum model on closed 3-manifold is invariant under the bistellar moves on triangulations. However this need not be the case in general and in section 4.2 we show that a spherical fusion category constructed in [HH] and [HRW] does not admit $6j$ -symbols with tetrahedral symmetry.

We are also interested in applications to topological field theory. There are two standard theories associated with a spherical category. One is the Turaev-Viro theory in [TV]; Barrett and Westbury also constructed an invariant on a spherical category in [BW] and they asserted it is equivalent to the Turaev-Viro theory. The other is the Reshetikhin-Turaev theory based on the quantum double of a spherical category. It is an open problem to determine if these two theories are equal. An alternative approach is the Hamiltonian formulation of the Turaev-Viro model in [LW]. It is expected that the ground states of this model form a modular functor and that this is isomorphic to the Reshetikhin-Turaev theory (For this connection, see section 6 in [RSW]). In [LW] it is assumed that the $6j$ -symbols have tetrahedral symmetry and it is shown that the Hamiltonian is exactly soluble. Levin and Wen asserted in the paper that the Hamiltonian is hermitian.

We also study unitary spherical categories and show in section 3 that any such category admits $6j$ -symbols with so-called mirror conjugate symmetry. Then in section 5 we extend the Levin-Wen Hamiltonian formulation to unitary spherical categories with $6j$ -symbols having mirror conjugate symmetry. This Hamiltonian

is also exactly soluble and hermitian. This is a genuine extension, as the example in [HH] and [HRW] is unitary.

Here are the contents of this paper. In section 2, we recall the definition of a spherical category and make notations for associativity and $6j$ -symbols. Notice that we define two different types of $6j$ -symbols, which is unavoidable because of the lack of symmetries as shown in section 4; Barrett and Westbury did not assume symmetrized $6j$ -symbols in [BW] and also defined two types of $6j$ -symbols. In section 3, we define the mirror conjugate symmetry of $6j$ -symbols and study some other properties of a unitary spherical category. We give normalization conditions on the trivalent basis to obtain the mirror conjugate symmetry of $6j$ -symbols on any unitary spherical category. Section 4 is devoted to symmetrized $6j$ -symbols. We recall the definition of the symmetrized $6j$ -symbols and show that a spherical category does not always allow them by giving a counterexample, category \mathcal{E} (see [HH] and [HRW] for more detail on the structure of category \mathcal{E}). We prove the impossibility of the symmetrization for the category \mathcal{E} , and instead give a normalization having properties studied in section 3. In section 5, we reformulate the Levin-Wen Hamiltonian and show that it is exactly soluble and hermitian on a unitary spherical category. In the proof, the mirror conjugate symmetry plays an important role along with other properties shown in section 3. In this sense the unitary spherical category is good enough to define the Hamiltonian.

This paper gives only a partial result on the Hamiltonian formulation because the study of the ground states still remains to be done. We expect that it would be an interesting further direction.

2. DEFINITIONS AND NOTATIONS

Definition 2.0.1. A *tensor category* is a category \mathcal{C} with a covariant functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a natural isomorphism $\alpha : (\mathcal{C} \otimes \mathcal{C}) \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes (\mathcal{C} \otimes \mathcal{C})$, called associativity, satisfying the pentagon axiom, a tensor unit $\mathbf{1} \in \mathcal{C}$ and natural isomorphisms $\rho : \mathcal{C} \otimes \mathbf{1} \rightarrow \mathcal{C}$ and $\lambda : \mathbf{1} \otimes \mathcal{C} \rightarrow \mathcal{C}$ satisfying the triangle axiom.

A tensor category is called *strict* if α , ρ and λ are identity.

For an algebraically closed field k , a tensor category is *k -linear* if all Hom-spaces are k -linear vector spaces, composition of morphisms are k -bilinear, and \otimes is k -bilinear on morphisms.

In a k -linear tensor category, an object x in \mathcal{C} is said to be *simple* if the map $k \rightarrow \text{End}(x)$, $c \mapsto c \cdot \text{id}_x$, is an isomorphism.

A k -linear tensor category is said to be *semi-simple* if every object is isomorphic to a direct sum of finitely many simple objects.

Right rigidity of a strict tensor category \mathcal{C} means that for each $x \in \mathcal{C}$ there is a right dual object $x^* \in \mathcal{C}$ with a morphism $b_x : \mathbf{1} \rightarrow x \otimes x^*$ and $d_x : x^* \otimes x \rightarrow \mathbf{1}$ such that $(\text{id}_x \otimes d_x) \circ (b_x \otimes \text{id}_x) = \text{id}_x$ and $(d_x \otimes \text{id}_{x^*}) \circ (\text{id}_{x^*} \otimes b_x) = \text{id}_{x^*}$. If every object in \mathcal{C} has its right dual $*$, then we may view $*$ as a contravariant functor and $f^* \in \text{Hom}_{\mathcal{C}}(y^*, x^*)$ is defined by $f^* = (d_y \otimes \text{id}_{x^*}) \circ (\text{id}_{y^*} \otimes f \otimes \text{id}_{x^*}) \circ (\text{id}_{y^*} \otimes b_x)$ for each $f \in \text{Hom}_{\mathcal{C}}(x, y)$. The contravariant functor from *left rigidity* is defined similarly.

A *strict pivotal structure* of a strict tensor category \mathcal{C} means that right dual is equal to left dual as a functor. Note that if a strict category \mathcal{C} has a strict pivotal structure, then $x^{**} = x$ and $f^{**} = f$ for any object x and for any morphism f .

Definition 2.0.2. A tensor category \mathcal{C} is *rigid* if it has right and left rigidity.

A *fusion category* is a k -linear semi-simple rigid tensor category with finitely many isomorphism classes of simple objects, finite dimensional spaces of morphisms, and $\text{End}(\mathbf{1}) \cong k$.

In a strict tensor category \mathcal{C} with a strict pivotal structure, for any endomorphism $f \in \text{End}_{\mathcal{C}}(x)$ the *right trace* is defined by $\text{tr}_r(f) = d_{x^*} \circ (f \otimes \text{id}_{x^*}) \circ b_x$ and the *left trace* is defined by $\text{tr}_l(f) = d_x \circ (\text{id}_{x^*} \otimes f) \circ b_{x^*}$. Category \mathcal{C} is *spherical* if $\text{tr}_r(f) = \text{tr}_l(f)$ for all f , and in this case we denote it by $\text{tr}(f)$. In a spherical category the *quantum dimension* $\dim(x)$ of an object x is defined by $\text{tr}(\text{id}_x)$.

In a spherical category with a strict pivotal structure, we have the following property on trace:

$$(2.0.1) \quad \text{tr}(f \circ g) = \text{tr}(g \circ f)$$

This equality comes from the strict pivotality as follows:

where the second equality comes from the strict pivotality $g = g^{**}$ and the rest are by the rigidity.

2.1. Associativity. In a strict spherical category \mathcal{C} , semi-simplicity of the category allows every morphism to be built up from trivalent morphisms in $\text{Hom}_{\mathcal{C}}$ -spaces of the type $\text{Hom}_{\mathcal{C}}(u \otimes v, w)$ or $\text{Hom}_{\mathcal{C}}(u, v \otimes w)$ for simple objects u, v and w .

Once we fix a trivalent basis for each $\text{Hom}_{\mathcal{C}}$ -space of the type of $\text{Hom}_{\mathcal{C}}(u \otimes v, w)$, then associativity is represented by a matrix F of the following form:

$$F_{uvw}^x : \begin{array}{c} x \\ \beta \\ \begin{array}{ccc} u & v & w \\ \alpha \end{array} \end{array} = \sum_{z, \gamma, \delta} (F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta} \begin{array}{c} x \\ \delta \\ \begin{array}{ccc} u & v & w \\ \gamma \end{array} \end{array}$$

where α, β, γ and δ are basis elements in $\text{Hom}_{\mathcal{C}}(v \otimes w, y)$, $\text{Hom}_{\mathcal{C}}(u \otimes y, x)$, $\text{Hom}_{\mathcal{C}}(u \otimes v, z)$ and $\text{Hom}_{\mathcal{C}}(z \otimes w, x)$, respectively. In this convention F_{uvw}^x is the matrix with entries $(F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta}$ in the $(y\alpha\beta)$ -th column and $(z\gamma\delta)$ -th row.

Using a trivalent basis for each $\text{Hom}_{\mathcal{C}}$ -space of the type of $\text{Hom}_{\mathcal{C}}(u, v \otimes w)$, associativity is similarly represented by a matrix G as follows:

$$G_x^{uvw} : \begin{array}{c} \begin{array}{ccc} u & v & w \\ \alpha \end{array} \\ \beta \\ x \end{array} = \sum_{z, \gamma, \delta} (G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} \begin{array}{c} \begin{array}{ccc} u & v & w \\ \gamma \end{array} \\ \delta \\ x \end{array}$$

2.2. 6j-symbols. We define two different 6j-symbols as follows :

$$\begin{array}{c} u \\ \alpha \\ w \end{array} \begin{array}{c} v \\ \beta \\ x \end{array} = \sum_{z, \gamma, \delta} \left\{ \begin{array}{ccc} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{array} \right\}_+ \begin{array}{c} u \\ \delta \\ w \end{array} \begin{array}{c} v \\ \gamma \\ x \end{array}$$

$$\begin{array}{c} u \\ \beta \\ | \\ y \\ | \\ \alpha \\ | \\ w \end{array} \begin{array}{c} v \\ | \\ x \end{array} = \sum_{z,\gamma,\delta} \left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_- \begin{array}{c} u \\ \diagdown \\ z \\ \diagup \\ w \end{array} \begin{array}{c} v \\ \diagup \\ \delta \\ \diagdown \\ \gamma \\ x \end{array}$$

In this convention, for either the $+$ or $-$ case, $\left\{ \begin{array}{c} u \ v \\ w \ x \end{array} \right\}_\pm$ is the matrix with entries $\left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_\pm$ in the $(y\alpha\beta)$ -th column and $(z\gamma\delta)$ -th row.

3. NORMALIZATION OF TRIVALENT BASIS

In this section, we always assume that every simple object is self-dual and the given unitary spherical category has a strict pivotal structure. The category is said to be unitary if all F -matrices are unitary on a properly normalized trivalent basis. In this section we show how to obtain unitary G -matrices and unitary $6j$ -symbols from unitary F -matrices by choosing algebraic dual basis.

3.1. Algebraic Dual Basis. For each basis vector $\alpha \in \text{Hom}_{\mathcal{C}}(u \otimes v, x)$, we choose the orthogonal algebraic dual basis vector $\bar{\alpha} \in \text{Hom}_{\mathcal{C}}(x, u \otimes v)$ satisfying the following condition

$$(3.1.1) \quad \beta \circ \bar{\alpha} = \delta_{\alpha,\beta} \frac{\sqrt{uv}}{\sqrt{x}} \text{id}_x, \quad \begin{array}{c} x \\ \beta \\ | \\ \alpha \\ | \\ x \end{array} \begin{array}{c} u \\ \diagdown \\ \alpha \\ \diagup \\ v \end{array} = \delta_{\alpha,\beta} \frac{\sqrt{uv}}{\sqrt{x}} \Big|_x$$

Here note that u, v and x on the right hand side of the equality are simple notations for the quantum dimensions of objects u, v and x , and that the bar on the dual basis is omitted in the diagram whenever it is clear from the context. With this dual basis, the following holds :

$$(3.1.2) \quad \begin{cases} \text{tr}(\alpha \circ \bar{\alpha}) = \sqrt{uvx} \ \forall \text{ basis vector } \alpha \in \text{Hom}_{\mathcal{C}}(u \otimes v, x) \\ \text{id}_{u \otimes v} = \sum_{x,\alpha} \frac{\sqrt{x}}{\sqrt{uv}} \bar{\alpha} \circ \alpha \end{cases}$$

$$\begin{array}{c} x \\ \alpha \\ | \\ \alpha \\ | \\ x \end{array} \begin{array}{c} u \\ \diagdown \\ \alpha \\ \diagup \\ v \end{array} = \sqrt{uvx}, \quad \begin{array}{c} | \\ | \end{array} \begin{array}{c} | \\ | \end{array} = \sum_{x,\alpha} \frac{\sqrt{x}}{\sqrt{uv}} \begin{array}{c} u \\ \diagdown \\ \alpha \\ \diagup \\ v \end{array} \begin{array}{c} u \\ \diagdown \\ x \\ \diagup \\ v \end{array} \begin{array}{c} \alpha \\ | \\ \alpha \end{array}$$

where the summation in the second equality runs over all simple objects x and all trivalent basis morphisms $\alpha \in \text{Hom}_{\mathcal{C}}(u \otimes v, x)$.

Using this convention on the algebraic dual basis, $6j$ -symbols can be obtained from F matrices, or vice versa, by the formulas

$$(3.1.3) \quad \left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_+ = \frac{\sqrt{yz}}{\sqrt{vw}} (F_{uyx})^{v\beta\delta} \left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_- = \frac{\sqrt{yz}}{\sqrt{ux}} ((F_{wyz})^{-1})^{u\beta\delta}$$

3.2. Mirror Conjugate Symmetry. We may ask what happens to the transformation rules when we take the mirror image of a given diagram about the horizontal axis. Mirror conjugate symmetry means that in the mirror image we have the conjugate coefficients for each transformation. It will be shown in the next subsection that every unitary spherical category with strict pivotality has this property. Because every transformation of a diagram can be obtained by a sequence of associativities or $6j$ -symbols we need only to study those. In terms of associativities, the mirror conjugate symmetry is expressed as follows :

$$(G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} = \overline{(F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta}}$$

For the $6j$ -symbols, note that the $(+)$ $6j$ -symbol and $(-)$ $6j$ -symbol are the mirror images of each other. Thus equality

$$\left\{ \begin{matrix} w & x & y(\beta\alpha) \\ u & v & z(\delta\gamma) \end{matrix} \right\}_- = \overline{\left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+}$$

implies the mirror conjugate symmetry.

3.3. Some Properties. We are considering a spherical category \mathcal{C} in which all associativity matrices F are unitary and the algebraic dual bases satisfy the condition (3.1.1).

Theorem 3.3.1. *We have*

- (1) *the category \mathcal{C} has mirror conjugate symmetry, and*
- (2) *$6j$ -symbols form unitary matrices.*

Proof. (1) From the condition (3.1.1) of the algebraic dual basis,

$$\begin{array}{c} x \\ | \\ \beta' \\ / \quad \backslash \\ y \quad \alpha' \\ / \quad \backslash \\ u \quad v \quad w \\ / \quad \backslash \\ y \quad \alpha \\ | \\ \beta \\ | \\ x \end{array} = \delta_{y,y'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \frac{\sqrt{vw}}{\sqrt{y}} \frac{\sqrt{uy}}{\sqrt{x}} \Big|_x$$

On the other hand, using the associativities F and G ,

$$\begin{array}{c} x \\ | \\ \beta' \\ / \quad \backslash \\ y \quad \alpha' \\ / \quad \backslash \\ u \quad v \quad w \\ / \quad \backslash \\ y \quad \alpha \\ | \\ \beta \\ | \\ x \end{array} = \sum_{z,\gamma,\delta} (F_{uvw}^x)^{y'\alpha'\beta'} (G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} \begin{array}{c} x \\ | \\ \delta \\ / \quad \backslash \\ \gamma \quad z \\ / \quad \backslash \\ u \quad v \quad w \\ / \quad \backslash \\ \gamma \quad z \\ | \\ \delta \\ | \\ x \end{array}$$

$$= \sum_{z,\gamma,\delta} (F_{uvw}^x)_{z\gamma\delta}^{y'\alpha'\beta'} (G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} \frac{\sqrt{uv}}{\sqrt{z}} \frac{\sqrt{zw}}{\sqrt{x}} \Big|_x$$

So we have $\delta_{y,y'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'} = \sum_{z,\gamma,\delta} (F_{uvw}^x)_{z\gamma\delta}^{y'\alpha'\beta'} (G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha}$ while unitarity of the F matrix implies $\sum_{z,\gamma,\delta} (F_{uvw}^x)_{z\gamma\delta}^{y'\alpha'\beta'} (F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta} = \delta_{y,y'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$. From the uniqueness of $F^{-1} = F^\dagger$, we conclude $(G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} = \overline{(F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta}}$.

The mirror conjugate symmetry of 6j-symbols can be shown easily using the formulas (3.1.3),

$$\left\{ \begin{matrix} w & x & y(\alpha\beta) \\ u & v & z(\delta\gamma) \end{matrix} \right\}_- = \frac{\sqrt{yz}}{\sqrt{vw}} \left(\widetilde{F_{uwx}^z} \right)_{v\beta\delta}^{w\alpha\gamma} = \frac{\sqrt{yz}}{\sqrt{vw}} \overline{(F_{uwx}^z)_{w\alpha\gamma}^{v\beta\delta}} = \overline{\left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+}.$$

(2) For the unitarity of 6j-symbols, we need to show

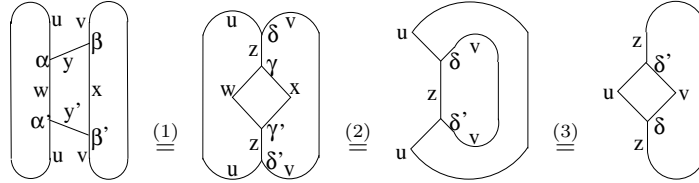
$$\sum_{z,\gamma,\delta} \left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+ \overline{\left\{ \begin{matrix} u & v & y'(\alpha'\beta') \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+} = \delta_{y,y'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$$

and by the mirror conjugate symmetry we need to show equivalently

$$\sum_{z,\gamma,\delta} \left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+ \left\{ \begin{matrix} w & x & y'(\beta'\alpha') \\ u & v & z(\delta\gamma) \end{matrix} \right\}_- = \delta_{y,y'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}.$$

We evaluate the following diagram in two different ways and then compare them.

On one hand,



where, for each step, the contribution is

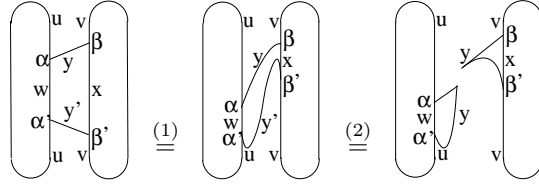
$$(1) = \sum_{z,\gamma,\gamma',\delta,\delta'} \left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+ \left\{ \begin{matrix} w & x & y'(\beta'\alpha') \\ u & v & z(\delta'\gamma') \end{matrix} \right\}_- \text{ applying two 6j-symbols,}$$

$$(2) = \delta_{\gamma,\gamma'} \frac{\sqrt{wx}}{\sqrt{z}} \text{ using (3.1.1) and sphericity. The equality (3) comes from}$$

$$(2.0.1), \text{ and the last diagram is equal to } \delta_{\delta,\delta'} \sqrt{uvwz} \text{ by (3.1.2), thus overall}$$

$$\text{we have } \sqrt{uvwz} \sum_{z,\gamma,\delta} \left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+ \left\{ \begin{matrix} w & x & y'(\beta'\alpha') \\ u & v & z(\delta\gamma) \end{matrix} \right\}_-.$$

On the other hand,



where (1) is the rigidity of y' , (2) = $\delta_{y,y'} \frac{1}{y}$ using (3.1.2), and the last diagram is equal to $\delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \sqrt{uvw} \sqrt{yvx}$ using the sphericity, (2.0.1), and (3.1.2), thus overall we have $\delta_{y,y'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \sqrt{uvw} \sqrt{yvx}$. \square

Corollary 3.3.2. *Let a diagram has two parts as shown below with sum over all basis elements α of $\text{Hom}_{\mathcal{C}}(u \otimes x, v)$ (or β of $\text{Hom}_{\mathcal{C}}(x \otimes v, u)$). Then we can transform the two parts in the diagram simultaneously with sum over all basis γ of $\text{Hom}_{\mathcal{C}}(u \otimes v, x)$ as follows:*

$$\sum_{\alpha} \left(\begin{array}{c} x \\ \diagup \quad \diagdown \\ u \quad v \\ \alpha \end{array} \quad \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \alpha \\ x \end{array} \right) = \sum_{\gamma} \left(\begin{array}{c} x \\ \diagup \quad \diagdown \\ u \quad v \\ \gamma \end{array} \quad \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \gamma \\ x \end{array} \right) = \sum_{\beta} \left(\begin{array}{c} x \\ \diagup \quad \diagdown \\ u \quad v \\ \beta \end{array} \quad \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \beta \\ x \end{array} \right)$$

Proof. This is a direct consequence of the mirror conjugate symmetry and unitarity of 6j-symbols. For the first equality, the coefficient of the transformation is

$$\sum_{\alpha,\gamma,\delta} \left\{ \begin{array}{c} 1 \quad x \quad u(\alpha) \\ u \quad v \quad x(\gamma) \end{array} \right\}_- \left\{ \begin{array}{c} u \quad v \quad u(\alpha) \\ 1 \quad x \quad x(\delta) \end{array} \right\}_+ = \sum_{\gamma,\delta} \left(\sum_{\alpha} \overline{\left\{ \begin{array}{c} u \quad v \quad u(\alpha) \\ 1 \quad x \quad x(\gamma) \end{array} \right\}_+} \left\{ \begin{array}{c} u \quad v \quad u(\alpha) \\ 1 \quad x \quad x(\delta) \end{array} \right\}_+ \right) = \delta_{\gamma,\delta} \sum_{\gamma} 1.$$

The proof for the second one is similar. \square

In particular, this corollary implies that $\text{id}_{u \otimes v}$ can be expressed in a different way from the second equality of (3.1.2), as follows:

$$(3.3.1) \quad \begin{array}{c} | \\ | \\ u \quad v \end{array} = \sum_{x,\gamma} \frac{\sqrt{x}}{\sqrt{uv}} \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \gamma \\ x \\ \diagup \quad \diagdown \\ \gamma \\ u \quad v \end{array} \quad \text{or} \quad \sum_{x,\gamma} \frac{\sqrt{x}}{\sqrt{uv}} \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \gamma \\ x \\ \diagup \quad \diagdown \\ \gamma \\ u \quad v \end{array}$$

Another consequence of the unitarity and mirror conjugate symmetry is the following corollary which plays an important role when we discuss the Levin-Wen Hamiltonian in the next section.

Corollary 3.3.3. *The following equality holds:*

$$\frac{\sqrt{g'}}{\sqrt{g}} \sum_{s,\eta} \sqrt{s} \begin{array}{c} g' \quad g' \\ \diagdown \quad \diagup \\ \eta \quad \eta \\ s \\ \diagup \quad \diagdown \\ g \quad g \end{array} = g' \begin{array}{c} g' \\ \diagdown \quad \diagup \\ g \quad g \end{array}$$

where the summation runs over all simple objects s and all trivalent basis morphisms $\eta \in \text{Hom}_{\mathcal{C}}(g \otimes s, g')$.

Proof. Since $** = \text{id}$ on morphisms, we may replace the right half of the given diagram by its double dual,

then we set

and take the trace after composing $\gamma \in \text{Hom}_{\mathcal{C}}(t, g \otimes g)$ and $\delta \in \text{Hom}_{\mathcal{C}}(g' \otimes g', t)$ for each t, γ and δ .

Now we claim that $C_{t, (\gamma, \delta)} = g'$ only for the case $t = \mathbf{1}$ and is 0 otherwise. This can be shown by deforming the diagram on the left hand side as follows (dotted box indicates the place deformed for each step) :

where for each step, the contribution is (1) $= \sum_{\epsilon} \frac{\sqrt{g}}{\sqrt{sg'}}$ using (3.1.2), (2) $= g' \left\{ \begin{matrix} g & g & s(\eta\epsilon) \\ g' & g' & \mathbf{1} \end{matrix} \right\}_+$, and (3) $= \left\{ \begin{matrix} g' & g' & s(\epsilon\eta) \\ g & g & t(\gamma\delta) \end{matrix} \right\}_-$. Thus with the initial coefficient $\frac{\sqrt{g}}{\sqrt{s}} \sum_{s, \eta} \sqrt{s}$ we have

$$C_{t(\gamma,\delta)} = g' \sum_{s,\eta,\varepsilon} \left\{ \begin{matrix} g & g & s(\eta\varepsilon) \\ g' & g' & \mathbf{1} \end{matrix} \right\}_+ \left\{ \begin{matrix} g' & g' & s(\varepsilon\eta) \\ g & g & t(\gamma\delta) \end{matrix} \right\}_- = g' \sum_{s,\eta,\varepsilon} \left\{ \begin{matrix} g & g & s(\eta\varepsilon) \\ g' & g' & \mathbf{1} \end{matrix} \right\}_+ \overline{\left\{ \begin{matrix} g & g & s(\eta\varepsilon) \\ g' & g' & t(\delta\gamma) \end{matrix} \right\}_+}$$

$$= \delta_{t,\mathbf{1}} \cdot g'$$

where the second equality comes from the mirror conjugate symmetry and the last one from the unitarity of 6j-symbols. \square

4. NON-SYMMETRIZED 6j-SYMBOLS

In [T], V. Turaev showed that a ribbon category with so-called unimodality allows symmetrized 6j-symbols (see chapter 6 in [T]). Braiding in a ribbon category plays an important role to prove this. A question now is whether or not this is possible for a spherical category which does not allow braiding in general. In this section, we give a counterexample to this question, which is the category \mathcal{E} . The category \mathcal{E} in [HH] is an example of a unitary spherical category with strict pivotal structure (all F -matrices in [HH] are given again in the appendix). In the following we show that the category \mathcal{E} does not allow the symmetrized 6j-symbols. Furthermore, we normalize the trivalent basis morphisms in each $\text{Hom}_{\mathcal{E}}$ space to have the properties in section 3 including unitary F -matrices, while F -matrices in [HH] are not yet unitary.

Notice that here we use the left multiplication convention instead of the right multiplication convention used in [HH]. So we need to consider the transpose of each F -matrix in [HH].

4.1. Symmetrized 6j-symbols. If the (+)6j-symbol is equal to the (-)6j-symbol, we define a new 6j-symbol by

$$\left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\} = \left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+ = \left\{ \begin{matrix} u & v & y(\beta\alpha) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_-$$

For the symmetrized 6j-symbol we require 24 tetrahedral symmetries generated by the following:

$$\left\{ \begin{matrix} u & v & y \\ w & x & z \end{matrix} \right\} = \left\{ \begin{matrix} v & u & y \\ x & w & z \end{matrix} \right\} = \left\{ \begin{matrix} x & w & y \\ v & u & z \end{matrix} \right\} = \left\{ \begin{matrix} y & x & w \\ u & z & v \end{matrix} \right\} \frac{\sqrt{yz}}{\sqrt{vw}}$$

where the above is a simple expression for the case of 1-dimensional $\text{Hom}_{\mathcal{C}}$ -spaces. For multi dimensional $\text{Hom}_{\mathcal{C}}$ -space case, we need to specify trivalent basis as before.

For a unimodal ribbon category, it is always possible for us to have symmetrized 6j-symbols on an appropriate basis by [T].

4.2. Impossibility of Symmetrization for the Category \mathcal{E} . Symmetrized 6j symbols have property $\left\{ \begin{matrix} a & d & e \\ b & c & - \end{matrix} \right\}_+ = \left\{ \begin{matrix} a & d & e \\ b & c & - \end{matrix} \right\}_-$. Otherwise the horizontal line in the middle does not have any meaning.

$$\begin{array}{c} a \\ | \\ b \end{array} \begin{array}{c} e \\ / \\ | \\ c \end{array} \begin{array}{c} d \\ | \\ c \end{array} = \begin{array}{c} a \\ | \\ b \end{array} \begin{array}{c} e \\ \backslash \\ | \\ c \end{array} \begin{array}{c} d \\ | \\ c \end{array}$$

The category \mathcal{E} , however, does not allow it. In other words, no matter how one normalizes the trivalent basis, the equality can not be obtained. Supposing it, we easily get a contradiction as follows:

Suppose we have new basis for each $\text{Hom}_{\mathcal{E}}$ space with such a property. Then we express each side of the above equality as a linear combination of the old basis elements and then compare them.

Let $w_{xx}^1, w_1^{xx}, \{w_1, w_2\}, \{w^1, w^2\}$ be new bases of $\text{Hom}_{\mathcal{E}}(x \otimes x, \mathbf{1}), \text{Hom}_{\mathcal{E}}(\mathbf{1}, x \otimes x), \text{Hom}_{\mathcal{E}}(x \otimes x, x), \text{Hom}_{\mathcal{E}}(x, x \otimes x)$, respectively, and let $w_{xx}^1 = f \cdot v_{xx}^1, w_1^{xx} = f' \cdot v_1^{xx}, w_1 = k \cdot v_1 + l \cdot v_2, w_2 = m \cdot v_1 + n \cdot v_2, w^1 = k' \cdot v^1 + l' \cdot v^2, w^2 = m' \cdot v^1 + n' \cdot v^2$ for some nonzero f, f' and invertible $\begin{bmatrix} k & m \\ l & n \end{bmatrix}, \begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix}$ where the basis elements denoted by v are the old basis used in [HH]. In the following diagram, the thickened graphs denote these new basis elements. Note that we are using the same convention for diagrams as in [HH], that is, the solid line indicates the object x , and dotted line indicates the object y .

$$\begin{aligned} \text{thick } \wedge &= f \wedge, & \text{thick } \vee &= f' \vee \\ \text{thick } \begin{array}{c} \diagup \\ \diagdown \end{array} &= k \begin{array}{c} \diagup \\ \diagdown \end{array} + l \begin{array}{c} \diagup \\ \diagdown \end{array}, & \text{thick } \begin{array}{c} \diagdown \\ \diagup \end{array} &= m \begin{array}{c} \diagdown \\ \diagup \end{array} + n \begin{array}{c} \diagdown \\ \diagup \end{array} \\ \text{thick } \begin{array}{c} \diagdown \\ \diagup \end{array} &= k' \begin{array}{c} \diagdown \\ \diagup \end{array} + l' \begin{array}{c} \diagdown \\ \diagup \end{array}, & \text{thick } \begin{array}{c} \diagup \\ \diagdown \end{array} &= m' \begin{array}{c} \diagup \\ \diagdown \end{array} + n' \begin{array}{c} \diagup \\ \diagdown \end{array} \end{aligned}$$

Now consider the case that $a = \mathbf{1}$ and everything else is x in the above symmetry.

$$\left(\text{thick } \begin{array}{c} \diagup \\ \diagdown \end{array}, \text{thick } \begin{array}{c} \diagdown \\ \diagup \end{array} \right) = \left(\text{thick } \begin{array}{c} \diagdown \\ \diagup \end{array}, \text{thick } \begin{array}{c} \diagup \\ \diagdown \end{array} \right)$$

If we express the left hand side in terms of the old basis v_1, v_2 , then we have a matrix expression $\begin{bmatrix} k & m \\ l & n \end{bmatrix}$. For the right hand side, we need to use associativity $(F_{xxx}^x)^{-1}$ to get $\frac{f}{2 \dim(x)} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix}$ on the same old basis. Thus the symmetry means $\begin{bmatrix} k & m \\ l & n \end{bmatrix} = \frac{f}{2 \dim(x)} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix}$, or equivalently $\begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix} = \frac{\dim(x)}{f} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k & m \\ l & n \end{bmatrix}$.

Next, consider the case that $c = \mathbf{1}$ and everything else is x .

$$\left(\text{thick } \begin{array}{c} \diagup \\ \diagdown \end{array}, \text{thick } \begin{array}{c} \diagdown \\ \diagup \end{array} \right) = \left(\text{thick } \begin{array}{c} \diagdown \\ \diagup \end{array}, \text{thick } \begin{array}{c} \diagup \\ \diagdown \end{array} \right)$$

From this symmetry, we have $\begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix} = f' e^{5\pi i/6} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} k & m \\ l & n \end{bmatrix}$. Comparing the two matrix equalities, we have $\frac{\dim(x)}{f} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = f' e^{5\pi i/6} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$, and it is easy to see the impossibility.

4.3. Normalization for the Category \mathcal{E} . Fix $d = \dim(x) = 1 + \sqrt{3}$ and $v = \sqrt{d}$, and then we define new basis elements(thick ones) as follows:

$$\begin{aligned} \text{thick } \begin{array}{c} \diagup \\ \diagdown \end{array}^j &= \sqrt{v} \begin{array}{c} \diagup \\ \diagdown \end{array}^j, & \text{thick } \begin{array}{c} \diagdown \\ \diagup \end{array}^j &= \sqrt{v} \begin{array}{c} \diagdown \\ \diagup \end{array}^j \quad \text{for each } j \in \{1, 2\} \\ \text{thick } \wedge &= \sqrt{2d} \wedge, & \text{thick } \vee &= \frac{1}{\sqrt{2}} \vee, & \text{thick } \begin{array}{c} \vdots \\ \wedge \end{array} &= \sqrt{2d} \begin{array}{c} \vdots \\ \wedge \end{array}, & \text{thick } \begin{array}{c} \vee \\ \vdots \end{array} &= \frac{1}{\sqrt{2}} \begin{array}{c} \vee \\ \vdots \end{array} \\ \text{and keep } \begin{array}{c} \diagdown \\ \diagup \end{array}, \begin{array}{c} \diagup \\ \diagdown \end{array}, \begin{array}{c} \vdots \\ \vee \end{array}, \begin{array}{c} \vee \\ \vdots \end{array}, \begin{array}{c} \vdots \\ \wedge \end{array}, \begin{array}{c} \wedge \\ \vdots \end{array} & \text{as before.} \end{aligned}$$

It is a straightforward computation to see that these basis elements allow all properties in section 3, especially the property 3.1.1. We list the associativity

matrices F obtained by this normalization below, all of which are unitary. The $6j$ -symbols can be computed by the formulas 3.1.3

$$\begin{aligned}
 F_{yyy}^y &= F_{xyy}^x = F_{yxy}^x = F_{xyx}^1 = F_{xxy}^1 = F_{xyx}^y = F_{yxx}^1 = F_{yxx}^y = 1, \\
 F_{xyx}^y &= F_{xyx}^x = -1, \\
 F_{xyx}^x &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, F_{xxy}^x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, F_{yxx}^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\
 F_{xxx}^1 &= \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, F_{xxx}^y = \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, \\
 F_{xxx}^x &= \begin{bmatrix} \frac{1}{d} & \frac{1}{d} & \frac{1}{\sqrt{2v}} & \frac{1}{\sqrt{2v}} & \frac{1}{\sqrt{2v}} & \frac{-1}{\sqrt{2v}} \\ \frac{1}{d} & \frac{-1}{d} & \frac{1}{\sqrt{2v}} & \frac{1}{\sqrt{2v}} & \frac{-1}{\sqrt{2v}} & \frac{1}{\sqrt{2v}} \\ \frac{1}{\sqrt{2v}} e^{-\frac{5\pi i}{6}} & \frac{1}{\sqrt{2v}} e^{-\frac{5\pi i}{6}} & \frac{1}{\sqrt{2d}} e^{-\frac{5\pi i}{12}} & \frac{1}{2} e^{\frac{\pi i}{3}} & \frac{1}{\sqrt{2d}} e^{-\frac{5\pi i}{12}} & \frac{1}{2} e^{-\frac{2\pi i}{3}} \\ \frac{1}{\sqrt{2v}} e^{-\frac{\pi i}{3}} & \frac{1}{\sqrt{2v}} e^{-\frac{\pi i}{3}} & \frac{1}{2} e^{\frac{5\pi i}{6}} & \frac{1}{\sqrt{2d}} e^{\frac{\pi i}{12}} & \frac{1}{2} e^{\frac{5\pi i}{6}} & \frac{1}{\sqrt{2d}} e^{-\frac{11\pi i}{12}} \\ \frac{1}{\sqrt{2v}} e^{-\frac{\pi i}{3}} & \frac{1}{\sqrt{2v}} e^{\frac{2\pi i}{3}} & \frac{1}{\sqrt{2d}} e^{\frac{\pi i}{12}} & \frac{1}{2} e^{\frac{5\pi i}{6}} & \frac{1}{\sqrt{2d}} e^{-\frac{11\pi i}{12}} & \frac{1}{2} e^{\frac{5\pi i}{6}} \\ \frac{1}{\sqrt{2v}} e^{-\frac{5\pi i}{6}} & \frac{1}{\sqrt{2v}} e^{\frac{\pi i}{6}} & \frac{1}{2} e^{\frac{\pi i}{3}} & \frac{1}{\sqrt{2d}} e^{-\frac{5\pi i}{12}} & \frac{1}{2} e^{-\frac{2\pi i}{3}} & \frac{1}{\sqrt{2d}} e^{-\frac{5\pi i}{12}} \end{bmatrix}
 \end{aligned}$$

5. EXACTLY SOLUBLE LATTICE MODELS

The exactly soluble Hamiltonian on a honeycomb lattice model is studied in [LW] and in chapter 11 of [We], in which they assume the symmetrized $6j$ -symbols. They claim that the symmetrized $6j$ -symbols with a so-called unitary condition imply that the Hamiltonian is exactly soluble and hermitian. In this section we study the same model but on a unitary spherical category.

We prove the following theorem in this section:

Theorem 5.0.1. *On a unitary spherical category with strict pivotal structure, the Levin-Wen Hamiltonian H on the honeycomb lattice model has the following properties:*

- (1) B_P 's and E_I 's commute with each other and hence H is exactly soluble.
- (2) B_P 's are projectors.
- (3) H is hermitian.

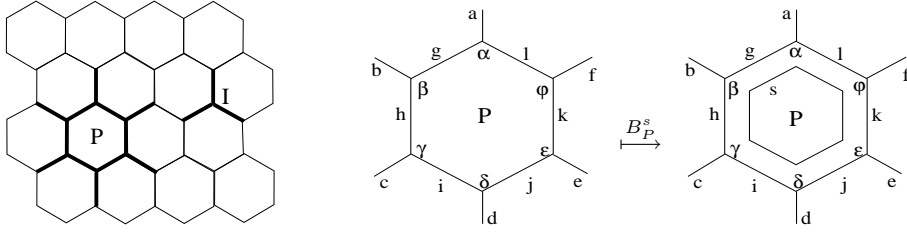
Commutativity is clear. In particular the commutativity of B_{P_1} and B_{P_2} is an easy conclusion from the topological consideration. We prove the rest of the theorem in section 5.2 and 5.3.

5.1. Definitions. The Levin-Wen model is the usual spin model on the honeycomb lattice with edges decorated by simple objects of a category \mathcal{C} . In the case that any fusion coefficient $N_{a,b}^c = \dim \text{Hom}_{\mathcal{C}}(a \otimes b, c)$ is bigger than 1, we need to distinguish the corresponding vertex by decorating it with basis morphisms of the $\text{Hom}_{\mathcal{C}}$ -space, so the Hilbert space is $\otimes_{\text{edges}} \mathbb{C}^n \otimes_{\text{vertices}} \mathbb{C}^{N_{a,b}^c}$ where n is the rank of the given category and three edges colored by simple objects a, b , and c meet at a vertex.

The exactly soluble Hamiltonian on the honeycomb lattice model is defined by

$$H = \sum_I (1 - E_I) + \sum_P (1 - B_P), \quad B_P = \sum_s \frac{s}{D^2} B_P^s$$

B_P^s is an operator acting on 12 links and 6 trivalent vertices around the hexagon P by introducing an extra loop labeled by simple object s in P . E_I acts on the vertex I such that it is the identity if the vertex I is admissible and zero otherwise (see [LW] and [We] for detail).

FIGURE 1. Honeycomb lattice model and operator B_P^s

5.2. B_P is a **projector**. The following computation shows that $(B_P)^2 = B_P$:

$$(B_P)^2 \left(\text{hexagon} \right) = \frac{1}{D^4} \sum_{s,t} st \text{ (nested hexagons)} = \frac{1}{D^2} \sum_f f \text{ (hexagon with f)} = B_P \left(\text{hexagon} \right)$$

where the second equality comes from

$$\text{(nested hexagons)} = \sum_{f,\eta} \frac{\sqrt{f}}{\sqrt{st}} \text{ (hexagon with f and } \eta) = \sum_{f,\eta} \frac{\sqrt{f}}{\sqrt{st}} \text{ (hexagon with } \eta \text{ and } f) = \sum_f N_{st}^f \text{ (hexagon with } f)$$

$$\text{and } \sum_{s,t} st \sum_f N_{st}^f = \sum_f \sum_s s \left(\sum_t t N_{sf}^t \right) = \sum_f \sum_s s^2 f = \sum_f D^2 f.$$

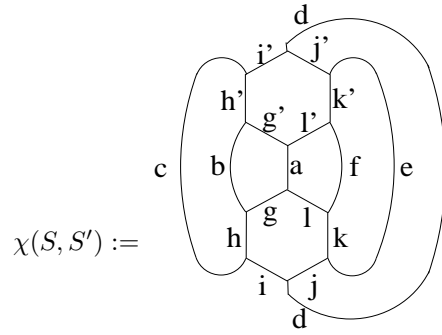
5.3. **Hamiltonian H is hermitian.** Let

$$\text{(hexagon with labels a-l)} \xrightarrow{B_P} \sum C \text{ (hexagon with labels a'-l')}$$

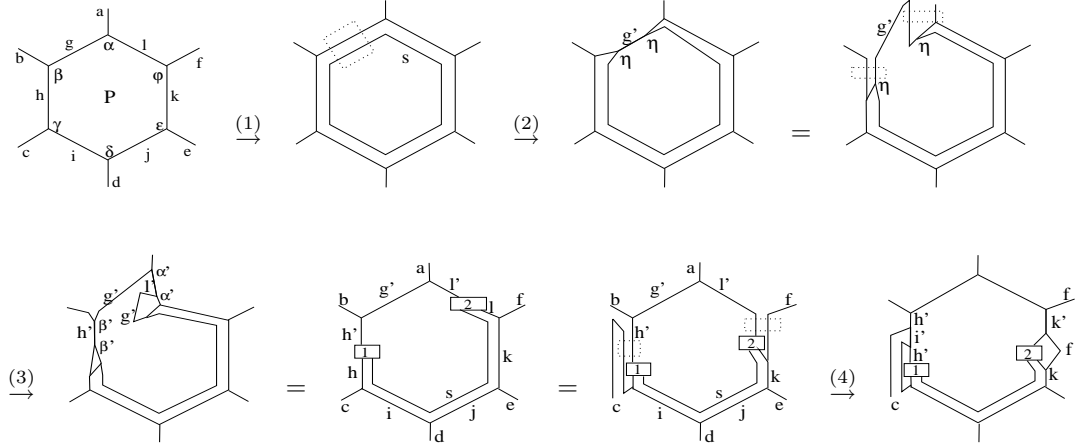
where the sum is over all possible $g', h', i', j', k', l', \alpha', \beta', \gamma', \delta', \epsilon', \varphi'$ and the coefficient C is a function on 30 labels in both diagrams. Note that if any label other than the 18 ones around the hexagonal face is different in both diagrams, then the coefficient is equal to zero. Once we fix two states shown as above, let us call them S and S' which are identical outside the diagram. Then $C = C(S, S')$.

It is sufficient to show that the operator B_P is hermitian, that is, $C(S, S')$ and $C(S', S)$ are complex conjugate to each other.

We claim that $C(S, S') = \frac{1}{D^2} \frac{1}{\sqrt{abcdef}} \chi(S, S')$ where $\chi(S, S') \in \mathbb{C}$ is defined by the trace of a morphism in $\text{Hom}_{\mathcal{C}}(d, d)$ as follows:



We have omitted the labels for trivalent vertices which are the same as in the states S or S' above determined by the three edges on each vertex. Note that the hermitian property for the Hamiltonian is obtained easily from this claim by the mirror conjugate symmetry since $\chi(S', S)$ is the mirror image of $\chi(S, S')$. The proof of the claim is done by picture calculus as below. In this picture calculus, each step indicated by an arrow has to be a sum of possibly many states, but for the computation of $C(S, S')$ for fixed states S and S' , the diagram following the arrow is the only state contributing to the computation, and every other state in the summation is irrelevant.



where for each step the contributions are as follows: (1) = $\sum_s \frac{s}{D^2}$ applying the operator B_P , (2) = $\sum_\eta \frac{\sqrt{g'}}{\sqrt{gs}}$ using (3.1.2), (3) = $\frac{\sqrt{h'}}{\sqrt{bg'}} \frac{\sqrt{l'}}{\sqrt{g'a}}$ using (3.1.2) and (3.3.1), (4) = $\frac{\sqrt{i'}}{\sqrt{ch'}} \frac{\sqrt{k'}}{\sqrt{l'f}}$ using (3.3.1), (5) = $\frac{\sqrt{j'}}{\sqrt{k'e}}$ using (3.3.1), and finally (6) = $\frac{\sqrt{d}}{\sqrt{i'j'}}$ using (3.1.2). So the overall coefficient is $\left(\frac{\sqrt{d}}{g'\sqrt{abcef}} \frac{1}{D^2}\right) \frac{\sqrt{g'}}{\sqrt{g}} \sum_{s,\eta} \sqrt{s}$ along with the diagram in box 6. The following is a deformation of the diagram in box 6 with scalar $\frac{\sqrt{g'}}{\sqrt{g}} \sum_{s,\eta} \sqrt{s}$ which completes the proof.

$$= g' \frac{1}{d} \chi(S, S') \text{id}_d$$

where the second equality comes from Lemma 3.3.3.

6. APPENDIX

The category \mathcal{E} is a spherical category with three simple objects, $\{\mathbf{1}, x, y\}$, and fusion rules:

$$x \otimes y = y \otimes x = x, x \otimes x = \mathbf{1} \oplus 2x \oplus y, y \otimes y = \mathbf{1}$$

The following is the list of F -matrices obtained by transposing the ones in [HH] since we are using left multiplication convention in this paper.

$$F_{yyy}^y = F_{xyy}^x = F_{yyx}^x = F_{xyx}^{\mathbf{1}} = F_{xxy}^{\mathbf{1}} = F_{xyx}^y = F_{yxx}^{\mathbf{1}} = F_{yxx}^y = 1,$$

$$\begin{aligned}
 F_{xyx}^y &= F_{yxy}^x = -1, \\
 F_{xyx}^x &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, F_{xxy}^x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, F_{yxx}^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, F_{xxx}^1 = \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \\
 F_{xxx}^y &= \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, \\
 F_{xxx}^x &= \begin{bmatrix} \frac{-1+\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2} & 1 & 1 & 1 & -1 \\ \frac{-1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} & 1 & 1 & -1 & 1 \\ \frac{1-\sqrt{3}}{4} e^{\pi i/6} & \frac{1-\sqrt{3}}{4} e^{\pi i/6} & -\frac{1}{2}(e^{\pi i/6}-1) & \frac{1}{2} e^{\pi i/3} & -\frac{1}{2}(e^{\pi i/6}-1) & -\frac{1}{2} e^{\pi i/3} \\ \frac{1-\sqrt{3}}{4} e^{2\pi i/3} & \frac{1-\sqrt{3}}{4} e^{2\pi i/3} & \frac{1}{2} e^{5\pi i/6} & \frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2} e^{5\pi i/6} & -\frac{1}{2}(e^{-\pi i/3}+i) \\ \frac{1-\sqrt{3}}{4} e^{2\pi i/3} & -\frac{1-\sqrt{3}}{4} e^{2\pi i/3} & \frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2} e^{5\pi i/6} & -\frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2} e^{5\pi i/6} \\ \frac{1-\sqrt{3}}{4} e^{\pi i/6} & -\frac{1-\sqrt{3}}{4} e^{\pi i/6} & \frac{1}{2} e^{\pi i/3} & -\frac{1}{2}(e^{\pi i/6}-1) & -\frac{1}{2} e^{\pi i/3} & -\frac{1}{2}(e^{\pi i/6}-1) \end{bmatrix} \\
 (F_{xxx}^x)^{-1} &= \begin{bmatrix} \frac{1}{1+\sqrt{3}} & \frac{1}{1+\sqrt{3}} & e^{5\pi i/6} & e^{\pi i/3} & e^{\pi i/3} & e^{5\pi i/6} \\ \frac{1}{1+\sqrt{3}} & -\frac{1}{1+\sqrt{3}} & e^{5\pi i/6} & e^{\pi i/3} & e^{-2\pi i/3} & e^{-\pi i/6} \\ \frac{1}{2(1+\sqrt{3})} & \frac{1}{2(1+\sqrt{3})} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{5\pi i/12} & \frac{1}{2} e^{-5\pi i/6} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{-\pi i/12} & \frac{1}{2} e^{-\pi i/3} \\ \frac{1}{2(1+\sqrt{3})} & \frac{1}{2(1+\sqrt{3})} & \frac{1}{2} e^{-\pi i/3} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{-\pi i/12} & \frac{1}{2} e^{-5\pi i/6} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{5\pi i/12} \\ \frac{1}{2(1+\sqrt{3})} & -\frac{1}{2(1+\sqrt{3})} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{5\pi i/12} & \frac{1}{2} e^{-5\pi i/6} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{11\pi i/12} & \frac{1}{2} e^{2\pi i/3} \\ -\frac{1}{2(1+\sqrt{3})} & \frac{1}{2(1+\sqrt{3})} & \frac{1}{2} e^{2\pi i/3} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{11\pi i/12} & \frac{1}{2} e^{-5\pi i/6} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{5\pi i/12} \end{bmatrix}
 \end{aligned}$$

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ON SYMMETRIZATION OF $6j$ -SYMBOLS AND LEVIN-WEN HAMILTONIAN

SEUNG-MOON HONG

ABSTRACT. It is known that every ribbon category with unimodality allows symmetrized $6j$ -symbols with full tetrahedral symmetries while a spherical category does not in general. We give an explicit counterexample for this, namely the category \mathcal{E} . We define the mirror conjugate symmetry of $6j$ -symbols instead and show that $6j$ -symbols of any unitary spherical category can be normalized to have this property. As an application, we discuss an exactly soluble model on a honeycomb lattice. We prove that the Levin-Wen Hamiltonian is exactly soluble and hermitian on a unitary spherical category.

1. INTRODUCTION

At first notice that in this paper $6j$ -symbols with 24 full tetrahedral symmetries are called symmetrized $6j$ -symbols rather than normalized $6j$ -symbols as in [T]. However they have the same meaning.

This paper is about spherical fusion categories and their $6j$ -symbols. More specifically the $6j$ -symbols in the well-known examples are invariant under the symmetry group of the tetrahedron. For example, this is the case if the category is unimodal. In [T], V. Turaev showed that a ribbon category with unimodality allows symmetrized $6j$ -symbols (see chapter 6 in [T]), which implies that the state sum model on closed 3-manifold is invariant under the bistellar moves on triangulations. However this need not be the case in general and in section 4.2 we show that a spherical fusion category constructed in [HH] and [HRW] does not admit $6j$ -symbols with tetrahedral symmetry.

We are also interested in applications to topological field theory. There are two standard theories associated with a spherical category. One is the Turaev-Viro theory in [TV]; Barrett and Westbury also constructed an invariant on a spherical category in [BW] and they asserted it is equivalent to the Turaev-Viro theory. The other is the Reshetikhin-Turaev theory based on the quantum double of a spherical category. It is an open problem to determine if these two theories are equal. An alternative approach is the Hamiltonian formulation of the Turaev-Viro model in [LW]. It is expected that the ground states of this model form a modular functor and that this is isomorphic to the Reshetikhin-Turaev theory (For this connection, see section 6 in [RSW]). In [LW] it is assumed that the $6j$ -symbols have tetrahedral symmetry and it is shown that the Hamiltonian is exactly soluble. Levin and Wen asserted in the paper that the Hamiltonian is hermitian.

We also study unitary spherical categories and show in section 3 that any such category admits $6j$ -symbols with so-called mirror conjugate symmetry. Then in section 5 we extend the Levin-Wen Hamiltonian formulation to unitary spherical categories with $6j$ -symbols having mirror conjugate symmetry. This Hamiltonian

is also exactly soluble and hermitian. This is a genuine extension, as the example in [HH] and [HRW] is unitary.

Here are the contents of this paper. In section 2, we recall the definition of a spherical category and make notations for associativity and $6j$ -symbols. Notice that we define two different types of $6j$ -symbols, which is unavoidable because of the lack of symmetries as shown in section 4; Barrett and Westbury did not assume symmetrized $6j$ -symbols in [BW] and also defined two types of $6j$ -symbols. In section 3, we define the mirror conjugate symmetry of $6j$ -symbols and study some other properties of a unitary spherical category. We give normalization conditions on the trivalent basis to obtain the mirror conjugate symmetry of $6j$ -symbols on any unitary spherical category. Section 4 is devoted to symmetrized $6j$ -symbols. We recall the definition of the symmetrized $6j$ -symbols and show that a spherical category does not always allow them by giving a counterexample, category \mathcal{E} (see [HH] and [HRW] for more detail on the structure of category \mathcal{E}). We prove the impossibility of the symmetrization for the category \mathcal{E} , and instead give a normalization having properties studied in section 3. In section 5, we reformulate the Levin-Wen Hamiltonian and show that it is exactly soluble and hermitian on a unitary spherical category. In the proof, the mirror conjugate symmetry plays an important role along with other properties shown in section 3. In this sense the unitary spherical category is good enough to define the Hamiltonian.

This paper gives only a partial result on the Hamiltonian formulation because the study of the ground states still remains to be done. We expect that it would be an interesting further direction.

2. DEFINITIONS AND NOTATIONS

Definition 2.0.1. A *tensor category* is a category \mathcal{C} with a covariant functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a natural isomorphism $\alpha : (\mathcal{C} \otimes \mathcal{C}) \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes (\mathcal{C} \otimes \mathcal{C})$, called associativity, satisfying the pentagon axiom, a tensor unit $\mathbf{1} \in \mathcal{C}$ and natural isomorphisms $\rho : \mathcal{C} \otimes \mathbf{1} \rightarrow \mathcal{C}$ and $\lambda : \mathbf{1} \otimes \mathcal{C} \rightarrow \mathcal{C}$ satisfying the triangle axiom.

A tensor category is called *strict* if α , ρ and λ are identity.

For an algebraically closed field k , a tensor category is *k -linear* if all Hom-spaces are k -linear vector spaces, composition of morphisms are k -bilinear, and \otimes is k -bilinear on morphisms.

In a k -linear tensor category, an object x in \mathcal{C} is said to be *simple* if the map $k \rightarrow \text{End}(x)$, $c \mapsto c \cdot \text{id}_x$, is an isomorphism.

A k -linear tensor category is said to be *semi-simple* if every object is isomorphic to a direct sum of finitely many simple objects.

Right rigidity of a strict tensor category \mathcal{C} means that for each $x \in \mathcal{C}$ there is a right dual object $x^* \in \mathcal{C}$ with a morphism $b_x : \mathbf{1} \rightarrow x \otimes x^*$ and $d_x : x^* \otimes x \rightarrow \mathbf{1}$ such that $(\text{id}_x \otimes d_x) \circ (b_x \otimes \text{id}_x) = \text{id}_x$ and $(d_x \otimes \text{id}_{x^*}) \circ (\text{id}_{x^*} \otimes b_x) = \text{id}_{x^*}$. If every object in \mathcal{C} has its right dual $*$, then we may view $*$ as a contravariant functor and $f^* \in \text{Hom}_{\mathcal{C}}(y^*, x^*)$ is defined by $f^* = (d_y \otimes \text{id}_{x^*}) \circ (\text{id}_{y^*} \otimes f \otimes \text{id}_{x^*}) \circ (\text{id}_{y^*} \otimes b_x)$ for each $f \in \text{Hom}_{\mathcal{C}}(x, y)$. The contravariant functor from *left rigidity* is defined similarly.

A *strict pivotal structure* of a strict tensor category \mathcal{C} means that right dual is equal to left dual as a functor. Note that if a strict category \mathcal{C} has a strict pivotal structure, then $x^{**} = x$ and $f^{**} = f$ for any object x and for any morphism f .

Definition 2.0.2. A tensor category \mathcal{C} is *rigid* if it has right and left rigidity.

A *fusion category* is a k -linear semi-simple rigid tensor category with finitely many isomorphism classes of simple objects, finite dimensional spaces of morphisms, and $\text{End}(\mathbf{1}) \cong k$.

In a strict tensor category \mathcal{C} with a strict pivotal structure, for any endomorphism $f \in \text{End}_{\mathcal{C}}(x)$ the *right trace* is defined by $\text{tr}_r(f) = d_{x^*} \circ (f \otimes \text{id}_{x^*}) \circ b_x$ and the *left trace* is defined by $\text{tr}_l(f) = d_x \circ (\text{id}_{x^*} \otimes f) \circ b_{x^*}$. Category \mathcal{C} is *spherical* if $\text{tr}_r(f) = \text{tr}_l(f)$ for all f , and in this case we denote it by $\text{tr}(f)$. In a spherical category the *quantum dimension* $\dim(x)$ of an object x is defined by $\text{tr}(\text{id}_x)$.

In a spherical category with a strict pivotal structure, we have the following property on trace:

$$(2.0.1) \quad \text{tr}(f \circ g) = \text{tr}(g \circ f)$$

This equality comes from the strict pivotality as follows:

where the second equality comes from the strict pivotality $g = g^{**}$ and the rest are by the rigidity.

2.1. Associativity. In a strict spherical category \mathcal{C} , semi-simplicity of the category allows every morphism to be built up from trivalent morphisms in $\text{Hom}_{\mathcal{C}}$ -spaces of the type $\text{Hom}_{\mathcal{C}}(u \otimes v, w)$ or $\text{Hom}_{\mathcal{C}}(u, v \otimes w)$ for simple objects u, v and w .

Once we fix a trivalent basis for each $\text{Hom}_{\mathcal{C}}$ -space of the type of $\text{Hom}_{\mathcal{C}}(u \otimes v, w)$, then associativity is represented by a matrix F of the following form:

$$F_{uvw}^x : \begin{array}{c} x \\ \beta \\ \begin{array}{ccc} u & v & w \\ \alpha \end{array} \end{array} = \sum_{z, \gamma, \delta} (F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta} \begin{array}{c} x \\ \delta \\ \begin{array}{ccc} u & v & w \\ \gamma \end{array} \end{array}$$

where α, β, γ and δ are basis elements in $\text{Hom}_{\mathcal{C}}(v \otimes w, y)$, $\text{Hom}_{\mathcal{C}}(u \otimes y, x)$, $\text{Hom}_{\mathcal{C}}(u \otimes v, z)$ and $\text{Hom}_{\mathcal{C}}(z \otimes w, x)$, respectively. In this convention F_{uvw}^x is the matrix with entries $(F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta}$ in the $(y\alpha\beta)$ -th column and $(z\gamma\delta)$ -th row.

Using a trivalent basis for each $\text{Hom}_{\mathcal{C}}$ -space of the type of $\text{Hom}_{\mathcal{C}}(u, v \otimes w)$, associativity is similarly represented by a matrix G as follows:

$$G_x^{uvw} : \begin{array}{c} \begin{array}{ccc} u & v & w \\ \alpha \end{array} \\ \beta \\ x \end{array} = \sum_{z, \gamma, \delta} (G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} \begin{array}{c} \begin{array}{ccc} u & v & w \\ \gamma \end{array} \\ \delta \\ x \end{array}$$

2.2. 6j-symbols. We define two different 6j-symbols as follows :

$$\begin{array}{c} u \\ \alpha \\ \begin{array}{ccc} y & \beta & v \\ x \end{array} \end{array} = \sum_{z, \gamma, \delta} \left\{ \begin{array}{ccc} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{array} \right\}_+ \begin{array}{c} u \\ \delta \\ \begin{array}{ccc} z & \gamma & v \\ x \end{array} \end{array}$$

$$\begin{array}{c} u \\ \beta \\ | \\ y \\ | \\ \alpha \\ x \end{array} \begin{array}{c} v \\ \\ \\ \\ \\ \end{array} = \sum_{z,\gamma,\delta} \left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_- \quad \begin{array}{c} u \\ \\ z \\ \delta \\ \gamma \\ x \\ \\ v \end{array}$$

In this convention, for either the $+$ or $-$ case, $\left\{ \begin{array}{c} u \ v \\ w \ x \end{array} \right\}_\pm$ is the matrix with entries $\left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_\pm$ in the $(y\alpha\beta)$ -th column and $(z\gamma\delta)$ -th row.

3. NORMALIZATION OF TRIVALENT BASIS

In this section, we always assume that every simple object is self-dual and the given unitary spherical category has a strict pivotal structure. The category is said to be unitary if all F -matrices are unitary on a properly normalized trivalent basis. In this section we show how to obtain unitary G -matrices and unitary $6j$ -symbols from unitary F -matrices by choosing algebraic dual basis.

3.1. Algebraic Dual Basis. For each basis vector $\alpha \in \text{Hom}_{\mathcal{C}}(u \otimes v, x)$, we choose the orthogonal algebraic dual basis vector $\bar{\alpha} \in \text{Hom}_{\mathcal{C}}(x, u \otimes v)$ satisfying the following condition

$$(3.1.1) \quad \beta \circ \bar{\alpha} = \delta_{\alpha,\beta} \frac{\sqrt{uv}}{\sqrt{x}} \text{id}_x, \quad \begin{array}{c} x \\ \beta \\ \diagdown \quad / \\ u \quad v \\ \diagup \quad \diagdown \\ x \quad \alpha \end{array} \Big|_x = \delta_{\alpha,\beta} \frac{\sqrt{uv}}{\sqrt{x}}$$

Here note that u, v and x on the right hand side of the equality are simple notations for the quantum dimensions of objects u, v and x , and that the bar on the dual basis is omitted in the diagram whenever it is clear from the context. With this dual basis, the following holds :

$$(3.1.2) \quad \begin{cases} \text{tr}(\alpha \circ \bar{\alpha}) = \sqrt{uvx} \ \forall \text{ basis vector } \alpha \in \text{Hom}_{\mathcal{C}}(u \otimes v, x) \\ \text{id}_{u \otimes v} = \sum_{x,\alpha} \frac{\sqrt{x}}{\sqrt{uv}} \bar{\alpha} \circ \alpha \end{cases}$$

$$\begin{array}{c} x \\ \alpha \\ \diagdown \quad / \\ u \quad v \\ \diagup \quad \diagdown \\ x \quad \alpha \end{array} = \sqrt{uvx}, \quad \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} | \\ | \\ | \end{array} = \sum_{x,\alpha} \frac{\sqrt{x}}{\sqrt{uv}} \begin{array}{c} u \quad v \\ \alpha \\ \diagup \quad \diagdown \\ x \\ \alpha \\ \diagdown \quad / \\ u \quad v \end{array}$$

where the summation in the second equality runs over all simple objects x and all trivalent basis morphisms $\alpha \in \text{Hom}_{\mathcal{C}}(u \otimes v, x)$.

Using this convention on the algebraic dual basis, $6j$ -symbols can be obtained from F matrices, or vice versa, by the formulas

$$(3.1.3) \quad \left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_+ = \frac{\sqrt{yz}}{\sqrt{vw}} (F_{uyx})^{v\beta\delta}, \quad \left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_- = \frac{\sqrt{yz}}{\sqrt{ux}} ((F_{wyz})^{-1})^{u\beta\delta}$$

3.2. Mirror Conjugate Symmetry. We may ask what happens to the transformation rules when we take the mirror image of a given diagram about the horizontal axis. Mirror conjugate symmetry means that in the mirror image we have the conjugate coefficients for each transformation. It will be shown in the next subsection that every unitary spherical category with strict pivotality has this property. Because every transformation of a diagram can be obtained by a sequence of associativities or $6j$ -symbols we need only to study those. In terms of associativities, the mirror conjugate symmetry is expressed as follows :

$$(G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} = \overline{(F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta}}$$

For the $6j$ -symbols, note that the $(+)$ $6j$ -symbol and $(-)$ $6j$ -symbol are the mirror images of each other. Thus equality

$$\left\{ \begin{matrix} w & x & y(\beta\alpha) \\ u & v & z(\delta\gamma) \end{matrix} \right\}_- = \overline{\left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+}$$

implies the mirror conjugate symmetry.

3.3. Some Properties. We are considering a spherical category \mathcal{C} in which all associativity matrices F are unitary and the algebraic dual bases satisfy the condition (3.1.1).

Theorem 3.3.1. *We have*

- (1) *the category \mathcal{C} has mirror conjugate symmetry, and*
- (2) *$6j$ -symbols form unitary matrices.*

Proof. (1) From the condition (3.1.1) of the algebraic dual basis,

$$\begin{array}{c} x \\ | \\ \beta' \\ / \quad \backslash \\ y \quad \alpha' \\ / \quad \backslash \\ u \quad v \quad w \\ / \quad \backslash \\ y \quad \alpha \\ | \\ \beta \\ | \\ x \end{array} = \delta_{y,y'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \frac{\sqrt{vw}}{\sqrt{y}} \frac{\sqrt{uy}}{\sqrt{x}} \Big|_x$$

On the other hand, using the associativities F and G ,

$$\begin{array}{c} x \\ | \\ \beta' \\ / \quad \backslash \\ y \quad \alpha' \\ / \quad \backslash \\ u \quad v \quad w \\ / \quad \backslash \\ y \quad \alpha \\ | \\ \beta \\ | \\ x \end{array} = \sum_{z,\gamma,\delta} (F_{uvw}^x)_{z\gamma\delta}^{y'\alpha'\beta'} (G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} \begin{array}{c} x \\ | \\ \delta \\ / \quad \backslash \\ \gamma \quad z \\ / \quad \backslash \\ u \quad v \quad w \\ / \quad \backslash \\ \gamma \quad z \\ | \\ \delta \\ | \\ x \end{array}$$

$$= \sum_{z,\gamma,\delta} (F_{uvw}^x)^{y'\alpha'\beta'} (G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} \frac{\sqrt{uv}}{\sqrt{z}} \frac{\sqrt{zw}}{\sqrt{x}} \Big|_x$$

So we have $\delta_{y,y'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'} = \sum_{z,\gamma,\delta} (F_{uvw}^x)^{y'\alpha'\beta'} (G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha}$ while unitarity of the F matrix implies $\sum_{z,\gamma,\delta} (F_{uvw}^x)^{y'\alpha'\beta'} (F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta} = \delta_{y,y'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$. From the uniqueness of $F^{-1} = F^\dagger$, we conclude $(G_x^{uvw})_{z\delta\gamma}^{y\beta\alpha} = \overline{(F_{uvw}^x)_{z\gamma\delta}^{y\alpha\beta}}$.

The mirror conjugate symmetry of $6j$ -symbols can be shown easily using the formulas (3.1.3),

$$\left\{ \begin{matrix} w & x & y(\alpha\beta) \\ u & v & z(\delta\gamma) \end{matrix} \right\}_- = \frac{\sqrt{yz}}{\sqrt{vw}} \left(\widetilde{F_{uwx}^z} \right)_{v\beta\delta}^{w\alpha\gamma} = \frac{\sqrt{yz}}{\sqrt{vw}} \overline{(F_{uwx}^z)_{w\alpha\gamma}^{v\beta\delta}} = \overline{\left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+}.$$

(2) For the unitarity of $6j$ -symbols, we need to show

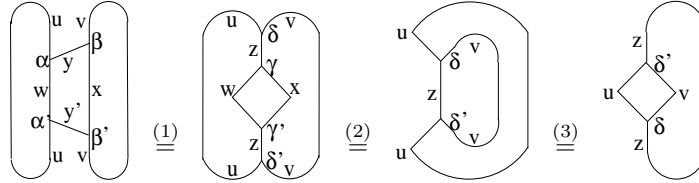
$$\sum_{z,\gamma,\delta} \left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+ \overline{\left\{ \begin{matrix} u & v & y'(\alpha'\beta') \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+} = \delta_{y,y'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$$

and by the mirror conjugate symmetry we need to show equivalently

$$\sum_{z,\gamma,\delta} \left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+ \left\{ \begin{matrix} w & x & y'(\beta'\alpha') \\ u & v & z(\delta\gamma) \end{matrix} \right\}_- = \delta_{y,y'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}.$$

We evaluate the following diagram in two different ways and then compare them.

On one hand,



where, for each step, the contribution is

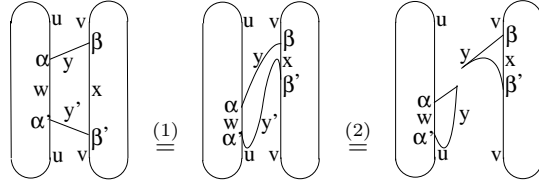
$$(1) = \sum_{z,\gamma,\gamma',\delta,\delta'} \left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+ \left\{ \begin{matrix} w & x & y'(\beta'\alpha') \\ u & v & z(\delta'\gamma') \end{matrix} \right\}_- \text{ applying two } 6j\text{-symbols,}$$

$$(2) = \delta_{\gamma,\gamma'} \frac{\sqrt{wx}}{\sqrt{z}} \text{ using (3.1.1) and sphericity. The equality (3) comes from}$$

$$(2.0.1), \text{ and the last diagram is equal to } \delta_{\delta,\delta'} \sqrt{uvwz} \text{ by (3.1.2), thus overall}$$

$$\text{we have } \sqrt{uvwz} \sum_{z,\gamma,\delta} \left\{ \begin{matrix} u & v & y(\alpha\beta) \\ w & x & z(\gamma\delta) \end{matrix} \right\}_+ \left\{ \begin{matrix} w & x & y'(\beta'\alpha') \\ u & v & z(\delta\gamma) \end{matrix} \right\}_-.$$

On the other hand,



where (1) is the rigidity of y' , (2) $= \delta_{y,y'} \frac{1}{y}$ using (3.1.2), and the last diagram is equal to $\delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \sqrt{uvw} \sqrt{yvx}$ using the sphericity, (2.0.1), and (3.1.2), thus overall we have $\delta_{y,y'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \sqrt{uvw} \sqrt{yvx}$. \square

Corollary 3.3.2. *Let a diagram has two parts as shown below with sum over all basis elements α of $\text{Hom}_{\mathcal{C}}(u \otimes x, v)$ (or β of $\text{Hom}_{\mathcal{C}}(x \otimes v, u)$). Then we can transform the two parts in the diagram simultaneously with sum over all basis γ of $\text{Hom}_{\mathcal{C}}(u \otimes v, x)$ as follows:*

$$\sum_{\alpha} \left(\begin{array}{c} x \\ \diagup \quad \diagdown \\ u \quad v \\ \alpha \end{array} \quad \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \alpha \\ x \end{array} \right) = \sum_{\gamma} \left(\begin{array}{c} x \\ \diagup \quad \diagdown \\ u \quad v \\ \gamma \end{array} \quad \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \gamma \\ x \end{array} \right) = \sum_{\beta} \left(\begin{array}{c} x \\ \diagup \quad \diagdown \\ u \quad v \\ \beta \end{array} \quad \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \beta \\ x \end{array} \right)$$

Proof. This is a direct consequence of the mirror conjugate symmetry and unitarity of 6j-symbols. For the first equality, the coefficient of the transformation is

$$\sum_{\alpha,\gamma,\delta} \left\{ \begin{array}{c} 1 \quad x \quad u(\alpha) \\ u \quad v \quad x(\gamma) \end{array} \right\}_- \left\{ \begin{array}{c} u \quad v \quad u(\alpha) \\ 1 \quad x \quad x(\delta) \end{array} \right\}_+ = \sum_{\gamma,\delta} \left(\sum_{\alpha} \overline{\left\{ \begin{array}{c} u \quad v \quad u(\alpha) \\ 1 \quad x \quad x(\gamma) \end{array} \right\}_+} \left\{ \begin{array}{c} u \quad v \quad u(\alpha) \\ 1 \quad x \quad x(\delta) \end{array} \right\}_+ \right) = \delta_{\gamma,\delta} \sum_{\gamma} 1.$$

The proof for the second one is similar. \square

In particular, this corollary implies that $\text{id}_{u \otimes v}$ can be expressed in a different way from the second equality of (3.1.2), as follows:

$$(3.3.1) \quad \begin{array}{c} | \\ | \\ u \quad v \end{array} = \sum_{x,\gamma} \frac{\sqrt{x}}{\sqrt{uv}} \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \gamma \\ x \\ \diagup \quad \diagdown \\ \gamma \\ u \quad v \end{array} \quad \text{or} \quad \sum_{x,\gamma} \frac{\sqrt{x}}{\sqrt{uv}} \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ \gamma \\ x \\ \diagup \quad \diagdown \\ \gamma \\ u \quad v \end{array}$$

Another consequence of the unitarity and mirror conjugate symmetry is the following corollary which plays an important role when we discuss the Levin-Wen Hamiltonian in the next section.

Corollary 3.3.3. *The following equality holds:*

$$\frac{\sqrt{g'}}{\sqrt{g}} \sum_{s,\eta} \sqrt{s} \begin{array}{c} g' \quad g' \\ \diagdown \quad \diagup \\ \eta \quad \eta \\ s \\ \diagup \quad \diagdown \\ g \quad g \end{array} = g' \begin{array}{c} g' \\ \diagdown \quad \diagup \\ g \quad g \end{array}$$

where the summation runs over all simple objects s and all trivalent basis morphisms $\eta \in \text{Hom}_{\mathcal{C}}(g \otimes s, g')$.

Proof. Since $** = \text{id}$ on morphisms, we may replace the right half of the given diagram by its double dual,

then we set

and take the trace after composing $\gamma \in \text{Hom}_{\mathcal{C}}(t, g \otimes g)$ and $\delta \in \text{Hom}_{\mathcal{C}}(g' \otimes g', t)$ for each t, γ and δ .

Now we claim that $C_{t, \gamma, \delta} = g'$ only for the case $t = \mathbf{1}$ and is 0 otherwise. This can be shown by deforming the diagram on the left hand side as follows (dotted box indicates the place deformed for each step) :

where for each step, the contribution is (1) = $\sum_{\varepsilon} \frac{\sqrt{g'}}{\sqrt{sg'}}$ using (3.1.2), (2) = $g' \left\{ \begin{matrix} g & g & s(\eta\varepsilon) \\ g' & g' & \mathbf{1} \end{matrix} \right\}_+$, and (3) = $\left\{ \begin{matrix} g' & g' & s(\varepsilon\eta) \\ g & g & t(\gamma\delta) \end{matrix} \right\}_-$. Thus with the initial coefficient $\frac{\sqrt{g'}}{\sqrt{g}} \sum_{s, \eta} \sqrt{s}$ we have

$$C_{t(\gamma, \delta)} = g' \sum_{s, \eta, \varepsilon} \left\{ \begin{array}{c} g \ g \ s(\eta\varepsilon) \\ g' \ g' \ \mathbf{1} \end{array} \right\}_+ \left\{ \begin{array}{c} g' \ g' \ s(\varepsilon\eta) \\ g \ g \ t(\gamma\delta) \end{array} \right\}_- = g' \sum_{s, \eta, \varepsilon} \left\{ \begin{array}{c} g \ g \ s(\eta\varepsilon) \\ g' \ g' \ \mathbf{1} \end{array} \right\}_+ \overline{\left\{ \begin{array}{c} g \ g \ s(\eta\varepsilon) \\ g' \ g' \ t(\delta\gamma) \end{array} \right\}_+}$$

$$= \delta_{t, \mathbf{1}} \cdot g'$$

where the second equality comes from the mirror conjugate symmetry and the last one from the unitarity of 6j-symbols. \square

4. NON-SYMMETRIZED 6j-SYMBOLS

In [T], V. Turaev showed that a ribbon category with so-called unimodality allows symmetrized 6j-symbols (see chapter 6 in [T]). Braiding in a ribbon category plays an important role to prove this. A question now is whether or not this is possible for a spherical category which does not allow braiding in general. In this section, we give a counter example to this question, which is the category \mathcal{E} . The category \mathcal{E} in [HH] is an example of a unitary spherical category with strict pivotal structure (all F -matrices in [HH] are given again in the appendix). In the following we show that the category \mathcal{E} does not allow the symmetrized 6j symbols. Furthermore, we normalize the trivalent basis morphisms in each $\text{Hom}_{\mathcal{E}}$ space to have the properties in section 3 including unitary F -matrices, while F -matrices in [HH] are not yet unitary.

Notice that here we are using left multiplication convention while [HH] is on the right multiplication convention. So we need to consider the transpose of each F -matrix in [HH].

4.1. Symmetrized 6j-symbols. If we have that (+)6j-symbol is equal to (-)6j-symbol, we define 6j-symbol by

$$\left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\} = \left\{ \begin{array}{c} u \ v \ y(\alpha\beta) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_+ = \left\{ \begin{array}{c} u \ v \ y(\beta\alpha) \\ w \ x \ z(\gamma\delta) \end{array} \right\}_-$$

By the symmetrized 6j-symbol we ask 24 tetrahedral symmetries generated by the following:

$$\left\{ \begin{array}{c} u \ v \ y \\ w \ x \ z \end{array} \right\} = \left\{ \begin{array}{c} v \ u \ y \\ x \ w \ z \end{array} \right\} = \left\{ \begin{array}{c} x \ w \ y \\ v \ u \ z \end{array} \right\} = \left\{ \begin{array}{c} y \ x \ w \\ u \ z \ v \end{array} \right\} \frac{\sqrt{yz}}{\sqrt{vw}}$$

where the above is a simple expression for the case of 1-dimensional $\text{Hom}_{\mathcal{C}}$ -spaces. For multi dimensional $\text{Hom}_{\mathcal{C}}$ -space case, we need to specify trivalent basis as before.

For a unimodal ribbon category, it is always possible for us to have symmetrized 6j-symbols on an appropriate basis by [T].

4.2. Impossibility of Symmetrization for the Category \mathcal{E} . Symmetrized 6j symbols have property $\left\{ \begin{array}{c} a \ d \ e \\ b \ c \ - \end{array} \right\}_+ = \left\{ \begin{array}{c} a \ d \ e \\ b \ c \ - \end{array} \right\}_-$. Otherwise horizontal line in the middle does not have any meaning.

$$\begin{array}{c} a \quad | \quad d \\ \quad | \quad / \quad | \\ b \quad | \quad e \quad | \quad c \end{array} = \begin{array}{c} a \quad | \quad d \\ \quad | \quad \backslash \quad | \\ b \quad | \quad e \quad | \quad c \end{array}$$

The category \mathcal{E} , however, does not allow it. In other words, no matter how one normalize the trivalent basis, the equality can not be obtained. Supposing it, we easily get a contradiction as follow:

Suppose we have new basis for each $\text{Hom}_{\mathcal{E}}$ space with such a property, then we express each side of the above equality as a linear combination of the old basis and then compare them.

Let $w_{xx}^1, w_1^{xx}, \{w_1, w_2\}, \{w^1, w^2\}$ be new bases of $\text{Hom}_{\mathcal{E}}(x \otimes x, \mathbf{1}), \text{Hom}_{\mathcal{E}}(\mathbf{1}, x \otimes x), \text{Hom}_{\mathcal{E}}(x \otimes x, x), \text{Hom}_{\mathcal{E}}(x, x \otimes x)$, respectively, and let $w_{xx}^1 = f \cdot v_{xx}^1, w_1^{xx} = f' \cdot v_1^{xx}, w_1 = k \cdot v_1 + l \cdot v_2, w_2 = m \cdot v_1 + n \cdot v_2, w^1 = k' \cdot v^1 + l' \cdot v^2, w^2 = m' \cdot v^1 + n' \cdot v^2$ for some nonzero f, f' and invertible $\begin{bmatrix} k & m \\ l & n \end{bmatrix}, \begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix}$ where the basis elements denoted by v are the old basis used in [HH]. In the following diagram, the thicken graphs denote these new basis elements. Note that we are using the same convention for diagrams as in [HH], that is, solid line indicates the object x , and dotted line does the object y .

$$\begin{aligned} \text{thick } \wedge &= f \wedge, & \text{thick } \vee &= f' \vee \\ \text{thick } \text{dot} \wedge &= k \text{dot} \wedge + l \text{dot} \wedge, & \text{thick } \text{dot} \vee &= m \text{dot} \vee + n \text{dot} \vee \\ \text{thick } \text{dot} \wedge &= k' \text{dot} \wedge + l' \text{dot} \wedge, & \text{thick } \text{dot} \vee &= m' \text{dot} \vee + n' \text{dot} \vee \end{aligned}$$

Now consider the case that $a = \mathbf{1}$ and everything else is x in the above symmetry.

$$\left(\text{thick } \text{dot} \wedge, \text{thick } \text{dot} \vee \right) = \left(\text{thick } \wedge, \text{thick } \vee \right)$$

If we express the left hand side in terms of the old basis v_1, v_2 , then we have a matrix expression $\begin{bmatrix} k & m \\ l & n \end{bmatrix}$. For the right hand side, we need to use associativity $(F_{xxx}^x)^{-1}$ to get $\frac{f}{2 \dim(x)} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix}$ on the same old basis. Thus the symmetry means $\begin{bmatrix} k & m \\ l & n \end{bmatrix} = \frac{f}{2 \dim(x)} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix}$, equivalently $\begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix} = \frac{\dim(x)}{f} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k & m \\ l & n \end{bmatrix}$.

Next, consider the case that $c = \mathbf{1}$ and everything else is x .

$$\left(\text{thick } \wedge, \text{thick } \vee \right) = \left(\text{thick } \text{dot} \wedge, \text{thick } \text{dot} \vee \right)$$

From this symmetry, we have $\begin{bmatrix} k' & m' \\ l' & n' \end{bmatrix} = f' e^{5\pi i/6} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} k & m \\ l & n \end{bmatrix}$. Comparing the two matrix equalities, we have $\frac{\dim(x)}{f} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = f' e^{5\pi i/6} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$, and it is easy to see the impossibility.

4.3. Normalization for the Category \mathcal{E} . Fix $d = \dim(x) = 1 + \sqrt{3}$ and $v = \sqrt{d}$, and then we define new basis (thick ones) as follow:

$$\begin{aligned} \text{thick } \wedge^j &= \sqrt{v} \wedge^j, & \text{thick } \vee^j &= \sqrt{v} \vee^j \quad \text{for each } j \in \{1, 2\} \\ \text{thick } \wedge &= \sqrt{2d} \wedge, & \text{thick } \vee &= \frac{1}{\sqrt{2}} \vee, & \text{thick } \text{dot} \wedge &= \sqrt{2d} \text{dot} \wedge, & \text{thick } \text{dot} \vee &= \frac{1}{\sqrt{2}} \text{dot} \vee \\ \text{and keep } \text{dot} \wedge, \text{dot} \vee, \text{dot} \wedge, \text{dot} \vee, \text{dot} \wedge, \text{dot} \vee & \text{ as before.} \end{aligned}$$

It is straightforward computation to see that these basis elements allow all properties in section 3, especially the property 3.1.1. In the below we list the associativity matrices F obtained by this normalization, all of which are unitary. The $6j$ -symbols can be computed by the formulas 3.1.3

$$\begin{aligned}
 F_{yyy}^y &= F_{xyy}^x = F_{yyx}^x = F_{xyx}^1 = F_{xxy}^1 = F_{xyx}^y = F_{yxx}^1 = F_{yxx}^y = 1, \\
 F_{xyx}^y &= F_{yxy}^x = -1, \\
 F_{xyx}^x &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, F_{xxy}^x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, F_{yxx}^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\
 F_{xxx}^1 &= \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, F_{xxx}^y = \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, \\
 F_{xxx}^x &= \begin{bmatrix} \frac{1}{d} & \frac{1}{d} & \frac{1}{\sqrt{2v}} & \frac{1}{\sqrt{2v}} & \frac{1}{\sqrt{2v}} & \frac{-1}{\sqrt{2v}} \\ \frac{1}{d} & \frac{-1}{d} & \frac{1}{\sqrt{2v}} & \frac{1}{\sqrt{2v}} & \frac{-1}{\sqrt{2v}} & \frac{1}{\sqrt{2v}} \\ \frac{1}{\sqrt{2v}} e^{-\frac{5\pi i}{6}} & \frac{1}{\sqrt{2v}} e^{-\frac{5\pi i}{6}} & \frac{1}{\sqrt{2d}} e^{-\frac{5\pi i}{12}} & \frac{1}{2} e^{\frac{\pi i}{3}} & \frac{1}{\sqrt{2d}} e^{-\frac{5\pi i}{12}} & \frac{1}{2} e^{-\frac{2\pi i}{3}} \\ \frac{1}{\sqrt{2v}} e^{-\frac{\pi i}{3}} & \frac{1}{\sqrt{2v}} e^{-\frac{\pi i}{3}} & \frac{1}{2} e^{\frac{5\pi i}{6}} & \frac{1}{\sqrt{2d}} e^{\frac{\pi i}{12}} & \frac{1}{2} e^{\frac{5\pi i}{6}} & \frac{1}{\sqrt{2d}} e^{-\frac{11\pi i}{12}} \\ \frac{1}{\sqrt{2v}} e^{-\frac{\pi i}{3}} & \frac{1}{\sqrt{2v}} e^{\frac{2\pi i}{3}} & \frac{1}{\sqrt{2d}} e^{\frac{\pi i}{12}} & \frac{1}{2} e^{\frac{5\pi i}{6}} & \frac{1}{\sqrt{2d}} e^{-\frac{11\pi i}{12}} & \frac{1}{2} e^{\frac{5\pi i}{6}} \\ \frac{1}{\sqrt{2v}} e^{-\frac{5\pi i}{6}} & \frac{1}{\sqrt{2v}} e^{\frac{\pi i}{6}} & \frac{1}{2} e^{\frac{\pi i}{3}} & \frac{1}{\sqrt{2d}} e^{-\frac{5\pi i}{12}} & \frac{1}{2} e^{-\frac{2\pi i}{3}} & \frac{1}{\sqrt{2d}} e^{-\frac{5\pi i}{12}} \end{bmatrix}
 \end{aligned}$$

5. EXACTLY SOLUBLE LATTICE MODELS

The exactly soluble Hamiltonian on the honeycomb lattice model is studied in [LW] and in chapter 11 of [We], in which they assume the symmetrized $6j$ -symbols. They claimed that the symmetrized $6j$ -symbols with a so called unitary condition implies that the Hamiltonian is exactly soluble and hermitian. In this section we study the same model but on a unitary spherical category.

We prove the following theorem in this section:

Theorem 5.0.1. *On a unitary spherical category with strict pivotal structure, the Levin-Wen Hamiltonian H on the honeycomb lattice model has the following properties:*

- (1) B_P 's and E_I 's commute with each other and hence H is exactly soluble.
- (2) B_P 's are projectors.
- (3) H is hermitian.

Commutativity is clear. In particular the commutativity of B_{P_1} and B_{P_2} is an easy conclusion from the topological consideration. We prove the rest of the theorem in section 5.2 and 5.3.

5.1. Definitions. The Levin-Wen's model is the usual spin model on the honeycomb lattice with edges decorated by simple objects of a category \mathcal{C} . In the case that any fusion coefficient $N_{a,b}^c = \dim \text{Hom}_{\mathcal{C}}(a \otimes b, c)$ is bigger than 1, we need to distinguish the corresponding vertex by decorating it with basis morphisms of the $\text{Hom}_{\mathcal{C}}$ -space, so the Hilbert space is $\otimes_{\text{edges}} \mathbb{C}^n \otimes_{\text{vertices}} \mathbb{C}^{N_{a,b}^c}$ where n is the rank of the given category and three edges colored by simple objects a, b , and c meet at a vertex.

The exactly soluble Hamiltonian on the honeycomb lattice model is defined by

$$H = \sum_I (1 - E_I) + \sum_P (1 - B_P), \quad B_P = \sum_s \frac{s}{D^2} B_P^s$$

where B_P^s is an operator acting on 12 links and 6 trivalent vertices around the hexagon P by introducing an extra loop labeled by simple object s in P , and E_I

acts on the vertex I such that it is the identity if the vertex I is admissible and zero otherwise (see [LW] and [We] for detail).

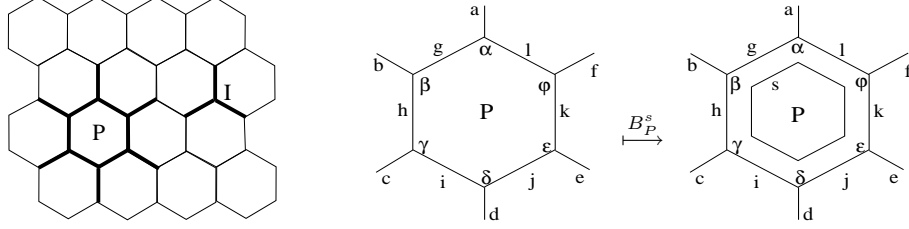


FIGURE 1. Honeycomb lattice model and operator B_P^s

5.2. B_P is a projector. The following computation shows that $(B_P)^2 = B_P$:

$$(B_P)^2 \left(\begin{array}{c} \text{hexagon} \\ \text{with } s, t \end{array} \right) = \frac{1}{D^4} \sum_{s,t} st \begin{array}{c} \text{hexagon} \\ \text{with } s, t, f \end{array} = \frac{1}{D^2} \sum_f f \begin{array}{c} \text{hexagon} \\ \text{with } f \end{array} = B_P \left(\begin{array}{c} \text{hexagon} \\ \text{with } s, t \end{array} \right)$$

where the second equality comes from

$$\begin{array}{c} \text{hexagon} \\ \text{with } s, t \end{array} = \sum_{f,\eta} \frac{\sqrt{f}}{\sqrt{st}} \begin{array}{c} \text{hexagon} \\ \text{with } s, t, f, \eta \end{array} = \sum_{f,\eta} \frac{\sqrt{f}}{\sqrt{st}} \begin{array}{c} \text{hexagon} \\ \text{with } s, t, f, \eta \end{array} = \sum_f N_{st}^f \begin{array}{c} \text{hexagon} \\ \text{with } f \end{array},$$

$$\text{and } \sum_{s,t} st \sum_f N_{st}^f = \sum_f \sum_s s \left(\sum_t t N_{sf}^t \right) = \sum_f \sum_s s^2 f = \sum_f D^2 f.$$

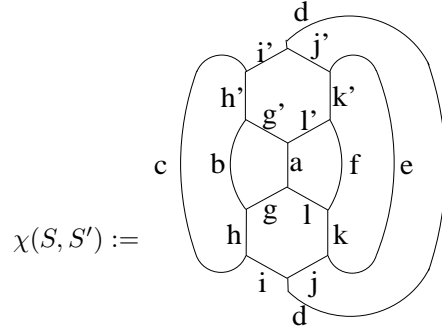
5.3. Hamiltonian H is hermitian. Let

$$\begin{array}{c} \text{hexagon} \\ \text{with } a, b, c, d, e, f \end{array} \xrightarrow{B_P} \sum C \begin{array}{c} \text{hexagon} \\ \text{with } a', b', c', d', e', f \end{array}$$

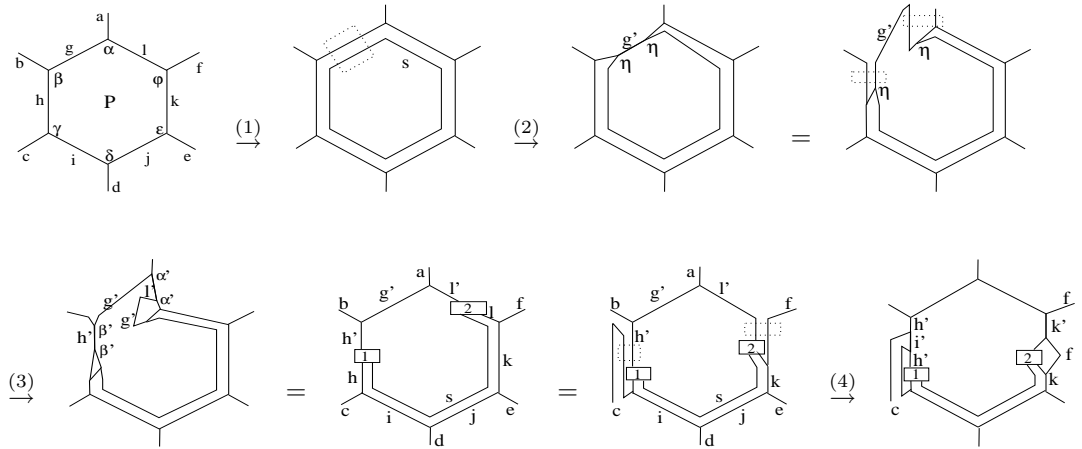
where the sum is over all possible $g', h', i', j', k', l', a', \beta', \gamma', \delta', \epsilon', \varphi'$ and the coefficient C is a function on 30 labels in both diagrams. Note that if any label other than the 18 ones around the hexagonal face is different in both diagrams, then the coefficient is equal to zero. Once we fix two states shown as above, let us call them by S and S' which are identical outside, then $C = C(S, S')$.

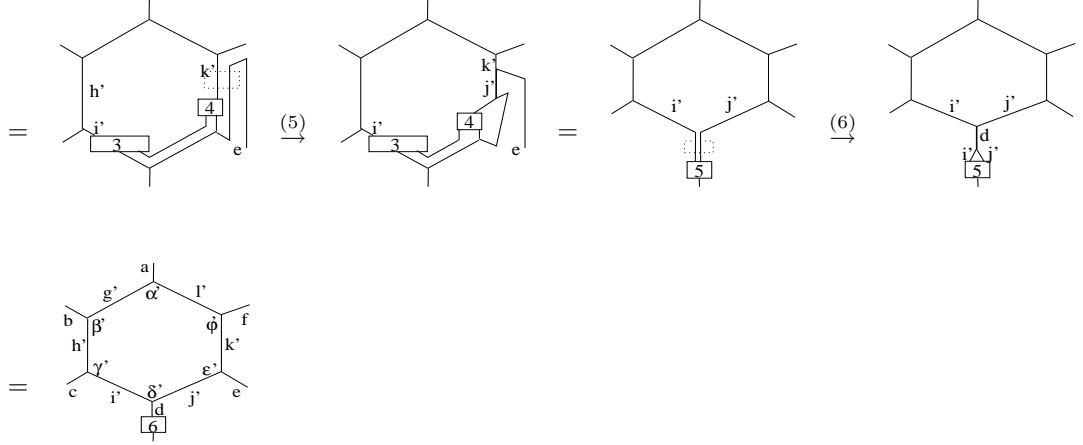
It is sufficient to show that the operator B_P is hermitian, that is, $C(S, S')$ and $C(S', S)$ are complex conjugate to each other.

We claim that $C(S, S') = \frac{1}{D^2} \frac{1}{\sqrt{abcdef}} \chi(S, S')$ where $\chi(S, S') \in \mathbb{C}$ is defined by the trace of a morphism in $\text{Hom}_{\mathcal{C}}(d, d)$ as follows:

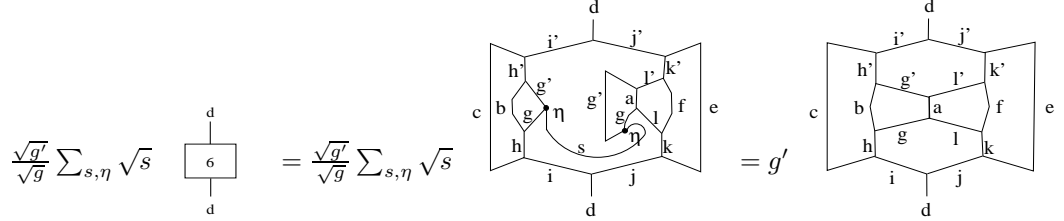


where we omitted the labels for trivalent vertices which are the same as in the states S or S' above determined by the three edges on each vertex. Note that being hermitian for the Hamiltonian is obtained easily from this claim by the mirror conjugate symmetry since $\chi(S', S)$ is the mirror image of $\chi(S, S')$. The proof of the claim is done by picture calculus as below. In this picture calculus, each step indicated by an arrow has to be a sum of possibly many states, but for the computation of $C(S, S')$ for fixed states S and S' , the diagram following the arrow is the only state contributing to the computation, and every other state in the summation is irrelevant.





where for each step the contribution is (1) = $\sum_s \frac{s}{D^2}$ applying the operator B_P , (2) = $\sum_\eta \frac{\sqrt{g'}}{\sqrt{gs}}$ using (3.1.2), (3) = $\frac{\sqrt{h'}}{\sqrt{bg'}} \frac{\sqrt{l'}}{\sqrt{g'a}}$ using (3.1.2) and (3.3.1), (4) = $\frac{\sqrt{i'}}{\sqrt{ch'}} \frac{\sqrt{k'}}{\sqrt{l'f}}$ using (3.3.1), (5) = $\frac{\sqrt{j'}}{\sqrt{k'e}}$ using (3.3.1), and finally (6) = $\frac{\sqrt{d}}{\sqrt{i'j'}}$ using (3.1.2). So the overall coefficient is $\left(\frac{\sqrt{d}}{g' \sqrt{abcef}} \frac{1}{D^2}\right) \frac{\sqrt{g'}}{\sqrt{g}} \sum_{s,\eta} \sqrt{s}$ along with the diagram in box 6. The following is a deformation of the diagram in box 6 with scalar $\frac{\sqrt{g'}}{\sqrt{g}} \sum_{s,\eta} \sqrt{s}$ which completes the proof.



$$= g' \frac{1}{d} \chi(S, S') \text{id}_d$$

where the second equality comes from Lemma 3.3.3.

6. APPENDIX

The category \mathcal{E} is a spherical category with three simple objects, $\{\mathbf{1}, x, y\}$, and fusion rules:

$$x \otimes y = y \otimes x = x, x \otimes x = \mathbf{1} \oplus 2x \oplus y, y \otimes y = \mathbf{1}$$

The following is the list of F -matrices obtained by transposing the ones in [HH] since we are using left multiplication convention in this paper.

$$F_{yy}^y = F_{xy}^x = F_{yx}^x = F_{xyx}^{\mathbf{1}} = F_{xxy}^{\mathbf{1}} = F_{xxy}^y = F_{yxx}^{\mathbf{1}} = F_{yxx}^y = 1,$$

$$\begin{aligned}
F_{xyx}^y &= F_{yxy}^x = -1, \\
F_{xyx}^x &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, F_{xxy}^x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, F_{yxx}^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, F_{xxx}^1 = \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \\
F_{xxx}^y &= \frac{1}{\sqrt{2}} e^{7\pi i/12} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, \\
F_{xxx}^x &= \begin{bmatrix} \frac{-1+\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2} & 1 & 1 & 1 & -1 \\ \frac{-1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} & 1 & 1 & -1 & 1 \\ \frac{1-\sqrt{3}}{4} e^{\pi i/6} & \frac{1-\sqrt{3}}{4} e^{\pi i/6} & -\frac{1}{2}(e^{\pi i/6}-1) & \frac{1}{2} e^{\pi i/3} & -\frac{1}{2}(e^{\pi i/6}-1) & -\frac{1}{2} e^{\pi i/3} \\ \frac{1-\sqrt{3}}{4} e^{2\pi i/3} & \frac{1-\sqrt{3}}{4} e^{2\pi i/3} & \frac{1}{2} e^{5\pi i/6} & \frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2} e^{5\pi i/6} & -\frac{1}{2}(e^{-\pi i/3}+i) \\ \frac{1-\sqrt{3}}{4} e^{2\pi i/3} & -\frac{1-\sqrt{3}}{4} e^{2\pi i/3} & \frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2} e^{5\pi i/6} & -\frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2} e^{5\pi i/6} \\ \frac{1-\sqrt{3}}{4} e^{\pi i/6} & -\frac{1-\sqrt{3}}{4} e^{\pi i/6} & \frac{1}{2} e^{\pi i/3} & -\frac{1}{2}(e^{\pi i/6}-1) & -\frac{1}{2} e^{\pi i/3} & -\frac{1}{2}(e^{\pi i/6}-1) \end{bmatrix} \\
(F_{xxx}^x)^{-1} &= \begin{bmatrix} \frac{1}{1+\sqrt{3}} & \frac{1}{1+\sqrt{3}} & e^{5\pi i/6} & e^{\pi i/3} & e^{\pi i/3} & e^{5\pi i/6} \\ \frac{1}{1+\sqrt{3}} & -\frac{1}{1+\sqrt{3}} & e^{5\pi i/6} & e^{\pi i/3} & e^{-2\pi i/3} & e^{-\pi i/6} \\ \frac{1}{2(1+\sqrt{3})} & \frac{1}{2(1+\sqrt{3})} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{5\pi i/12} & \frac{1}{2} e^{-5\pi i/6} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{-\pi i/12} & \frac{1}{2} e^{-\pi i/3} \\ \frac{1}{2(1+\sqrt{3})} & \frac{1}{2(1+\sqrt{3})} & \frac{1}{2} e^{-\pi i/3} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{-\pi i/12} & \frac{1}{2} e^{-5\pi i/6} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{5\pi i/12} \\ \frac{1}{2(1+\sqrt{3})} & -\frac{1}{2(1+\sqrt{3})} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{5\pi i/12} & \frac{1}{2} e^{-5\pi i/6} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{11\pi i/12} & \frac{1}{2} e^{2\pi i/3} \\ -\frac{1}{2(1+\sqrt{3})} & \frac{1}{2(1+\sqrt{3})} & \frac{1}{2} e^{2\pi i/3} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{11\pi i/12} & \frac{1}{2} e^{-5\pi i/6} & \frac{1}{\sqrt{2}(1+\sqrt{3})} e^{5\pi i/12} \end{bmatrix}
\end{aligned}$$

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