Dimensionally Reduced Gravity, Hermitian Symmetric Spaces and the Ashtekar Variables

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Abstract

Dimensional reductions of various higher dimensional (super)gravity theories lead to effectively two-dimensional field theories described by gravity coupled G/H nonlinear σ-models. We show that a new set of complexified variables can be introduced when G/H is a Hermitian symmetric space. This generalizes an earlier construction that grew out of the Ashtekar formulation of two Killing vector reduced pure $4d$ general relativity. Apart from giving some new insights into dimensional reductions of higher dimensional (super)gravity theories, these Ashtekar-type variables offer several technical advantages in the context of the exact quantization of these models. As an application, an infinite set of conserved charges is constructed. Our results might serve as a starting point for probing the *quantum* equivalence of the Ashtekar and the metric formalism within a non-trivial midi-superspace model of quantum gravity.

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1 Introduction

Dimensional reductions of higher dimensional (super)gravity theories to two dimensions have been studied from various points of view. In the physically most interesting cases, these theories can be described by an effectively $2d$ field theory consisting of a G/H nonlinear σ-model coupled to $2d$ gravity and a dilaton.

From a particle physicist's perspective, these models provide somewhat extreme examples of so-called 'hidden' symmetries, as they typically arise in dimensionally reduced (super)gravity theories. Whereas such 'hidden' symmetry groups are finite dimensional for reductions to $d \geq 3$ spacetime dimensions, they inflate to infinite dimensional symmetry groups in two dimensions. This phenomenon was first encountered by Geroch [1] in the two Killing vector reduction of source-free $4d$ general relativity, where $G/H = SL(2, \mathbb{R})/SO(2)$, and was later found to be a generic feature of the analogous models with more complicated coset spaces G/H that descend from other higher dimensional (super)gravity theories [2].

From a general relativist's point of view, these models, although being essential truncations of higher dimensional gravity theories, still exhibit a surviving $2d$ diffeomorphism invariance and involve self-interacting *local* degrees of freedom. This raises the hope that their exact quantization might give insights into at least some of the problems of quantum gravity.

What makes such a quantization seem feasible, on the other hand, is precisely the aforementioned rich symmetry structure of these midi-superspace models. On quite general grounds (see eg. [3]), one expects the underlying generators of the Geroch group (resp. its generalizations) to provide an infinite set of nontrivial observables whose Poisson algebra should translate into a spectrum generating algebra in the corresponding quantum theory.

In order to make this idea more explicit, two main routes have been pursued. One approach is to use the formulation of these theories in terms of the conventional metric variables and to exploit the existence of the linear system ('Lax pair') [4, 5] that encodes the metric-based field equations. This was the strategy of refs. [6, 7, 8], where a Poisson (and eventually quantum) realization of the Geroch group could be achieved for various sets of boundary conditions.

An alternative approach is offered by the Hamiltonian formulation of $4d$ general relativity in terms of the connection-type variables put forward by Ashtekar [9]. This direction has been followed in [10, 11, 12, 13], where the two Killing vector reduction of pure $4d$ general relativity has been performed within the Ashtekar formulation.

It is the purpose of this paper to demonstrate that this alternative approach in terms of the Ashtekar variables provides some fruitful insights into both the

dimensional reductions of higher dimensional (super)gravity theories and the non-perturbative quantization of these midi-superspace models in the context of canonical quantum gravity.

Our starting point is the two Killing vector reduction of pure $4d$ Einstein gravity in terms of the Ashtekar variables. As was shown in [11, 14], this reduction naturally leads to an interesting set of $2d$ variables which circumvents several technical difficulties associated with the traditional metric variables. Most importantly, their Poisson brackets are completely ultralocal (ie. they don't contain derivatives of delta-functions), which is in contrast to the σ-model currents in the conventional metric formulation, whose non-ultralocal Poisson brackets require careful treatment in the canonical formalism [6, 7, 8].

From a purely $2d$ point of view, the existence of these alternative variables, and the simplifications they provide, can be traced back to some very special properties of the underlying coset space $SL(2, \mathbb{R})/SO(2)$ [14].

In the first part of this paper, we identify these 'very special' properties as those of Hermitian symmetric spaces (ie., spaces of the form $G/H = G/(H' \times U(1))$). This implies that all higher dimensional (super)gravity theories which lead to Hermitian symmetric space non-linear σ-models in two dimensions admit the construction of these $2d$ Ashtekar-type variables. This suggests an interesting interplay between the Ashtekar formulation, group theory and the dimensional reduction of higher dimensional (super)gravity theories. In fact, our results are consistent with a recent observation made in [15], where it was pointed out that all $2d$ Hermitian symmetric space σ-models have their "genuine" origin in four spacetime dimensions, which is also the critical dimension for the existence of the Ashtekar variables.

In the second part of this paper, we will then have a closer look at the technical advantages of these Ashtekar-type variables in the context the canonical quantization of these models. In particular, we will show that the corresponding field equations also admit the formulation in terms of a linear system. In contrast to the analogous linear system for the metric variables [4, 5], however, this linear system can be written in terms of a *constant* (ie. spacetime *in*dependent) spectral parameter without that unwieldy square roots have to be introduced into the linear system.

As an application of this technical simplification, which seems to be closely related to the ultralocality of the corresponding Poisson structure, we construct an infinite set of conserved charges and determine their Poisson brackets by using techniques similar to those in [6, 7, 8], which in our case, however, simplify considerably.

Apart from offering a complementary approach towards a systematic quantization of an important midi-superspace model, the results of this second part can also be used as a starting point for probing the *quantum* equivalence of the Ashtekar and the

metric formulation within a non-trivial toy model of quantum gravity, since the use of similar techniques in both approaches might facilitate a comparison of the resulting quantum theories.

The organization of the paper is as follows: For convenience of the reader, we briefly recall, in section 2, the results of the two Killing vector reduction of pure $4d$ general relativity in terms of metric and Ashtekar variables and show how the resulting $2d$ formulations are related to each other. In section 3 we will embed the (metric-based) two Killing vector reduction of general relativity into the more general class of $2d$ gravity coupled G/H nonlinear σ-models as they arise in dimensional reductions of various (super)gravity models. This section will mainly serve to establish our notation. Section 4 then introduces the alternative variables as they grew out of the Ashtekar formulation and generalizes their construction to arbitrary Hermitian symmetric coset spaces. The linear system encoding the field equations of these alternative variables will be displayed in Section 5, which also contains a comparison with the Lax pair for the metric variables and a construction of an infinite set of conserved non-local charges.Section 6 concludes with a short discussion of our results.

2 Motivation: Two Killing vector reduced pure $4d$ gravity

Before considering more general dimensionally reduced gravity models, and in order to motivate the more abstract constructions in the rest of this paper, let us briefly recall the two Killing vector reduction of source-free $4d$ general relativity in terms of metric and Ashtekar variables (for details, see [10, 11, 13, 14]).

Assuming the existence of two commuting, spacelike and 2-surface orthogonal[2] Killing vector fields, one may choose local coordinates $(t, x, y, z) \equiv (x^0, x^1, x^2, x^3)$ such that the Killing vectors are given by $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ and the metric G_{MN} attains the following form

$$G_{MN} = G_{MN}(t, x) = \begin{pmatrix} e^{2\sigma}\eta_{\mu\nu} & 0 \\ 0 & \rho g_{\bar{m}\bar{n}} \end{pmatrix}, \tag{2.1}$$

where $M, N, \ldots \in \{t, x, y, z\}$; $\mu, \nu, \ldots \in \{t, x\}$; $\bar{m}, \bar{n}, \ldots \in \{y, z\}$; $\det g_{\bar{m}\bar{n}} = 1$; $\eta_{\mu\nu} = \text{diag}(-1, +1)$ and σ and ρ are some functions of (t, x). Rewriting the (2×2)-

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[2]By 2-surface orthogonality we mean that the two-dimensional planes orthogonal to the planes spanned by the two Killing vectors are integrable. In contrast to *hyper*surface orthogonality, this is a very weak (in fact, often even redundant) assumption, which removes some topological degrees of freedom right from the beginning.

matrix g in terms of *unimodular* 'zweibeins' $e_{\bar{m}}^{\bar{a}} \in SL(2, \mathbb{R})$ $(\bar{a}, \bar{b}, \ldots \in \{2, 3\})$

$$g_{\bar{m}\bar{n}} = e_{\bar{m}}^{\bar{a}} \delta_{\bar{a}\bar{b}} e_{\bar{n}}^{\bar{b}}, \tag{2.2}$$

it becomes clear that g parametrizes the coset space $SL(2, \mathbb{R})/SO(2)$.

The Einstein equations imply $\eta^{\mu\nu} \partial_\mu \partial_\nu \rho = 0$ and a first order equation for the conformal factor σ. The non-trivial dynamics is captured by the field g. Its field equation reads

$$\partial_0 J_0 - \partial_1 J_1 = 0. \tag{2.3}$$

where the currents J_μ are defined by

$$J_\mu := \rho g^{-1} \partial_\mu g, \tag{2.4}$$

and therefore obey the following compatibility condition

$$\partial_0 J_1 - \partial_1 J_0 + \frac{1}{\rho}[J_0, J_1] - \frac{\partial_0 \rho}{\rho} J_1 + \frac{\partial_1 \rho}{\rho} J_0 = 0. \tag{2.5}$$

The underlying symmetry-reduced $2d$ Lagrangian induces Poisson brackets between the currents J_μ, which contain non-ultralocal terms (see section 3.2).

In terms of the Ashtekar formulation, on the other hand, the symmetry reduction looks as follows.

Starting point is the parametrization of the phase space of $4d$ general relativity in terms of the inverse dreibein density $\tilde{E}_a^m$ and the Ashtekar connection A_{nb}, where $a, b, \ldots = 1, 2, 3$ are the internal $SO(3)$-indices, whereas $m, n, \ldots = x, y, z$.

$\tilde{E}_a^m$ and A_{nb} are canonically conjugate

$$\{\tilde{E}_a^m(x), A_{nb}(x')\} = -i\, \delta_{ab} \delta_n^m \delta^{(3)}(x - x')$$

and subject to the first class constraints

$$
\begin{aligned}
\mathcal{H} &:= & \varepsilon^{abc} F_{mn}^a \tilde{E}^{mb} \tilde{E}^{nc} \approx 0 \tag{2.6} \\
\mathcal{C}_m &:= & F_{mn}^a \tilde{E}^{na} \approx 0 \tag{2.7} \\
\mathcal{G}^a &:= & D_m \tilde{E}^{ma} \approx 0, \tag{2.8}
\end{aligned}
$$

where D_m and F_{mn}^a denote the covariant derivative with respect to the connection A_{nb} and the corresponding field strength, respectively.

Again, one chooses adapted coordinates such that the Killing vectors are given by $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. Imposing various further gauge fixings and solving the resulting second

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class constraints, one eventually arrives at a reduced phase space consisting of the canonical pairs $(\tilde{E}_1^x, A_{x1})$ and $(\tilde{E}_{\bar{a}}^{\bar{m}}, A_{\bar{n}\bar{b}})$, $(\bar{m}, \bar{n}, \ldots = y, z$ and $\bar{a}, \bar{b}, \ldots = 2, 3)$, which are subject to three surviving constraints, one of them being an $SO(2)$-remnant of the $SO(3)$-Gauss law constraint (2.8). The non-trivial dynamics is carried by $\tilde{E}_{\bar{a}}^{\bar{m}}$ and $A_{\bar{m}\bar{a}}$.

Defining the $SO(2)$-invariant contractions[3]

$$K_{\bar{m}}^{\bar{n}} := -i\, A_{\bar{m}\bar{a}}\tilde{E}_{\bar{a}}^{\bar{n}} \tag{2.9}$$

$$J_{\bar{m}}^{\bar{n}} := -\varepsilon^{\bar{a}\bar{b}}A_{\bar{m}\bar{a}}\tilde{E}_{\bar{b}}^{\bar{n}}, \tag{2.10}$$

with $\varepsilon^{\bar{a}\bar{b}} = -\varepsilon^{\bar{b}\bar{a}}$, $\varepsilon^{23} = +1$, the traceless (2×2)-matrices

$$A_0 = (A_0)_{\bar{n}\bar{m}} := K_{\bar{m}}^{\bar{n}} - \frac{1}{2}K_{\bar{p}}^{\bar{p}}\delta_{\bar{m}}^{\bar{n}} \tag{2.11}$$

$$A_1 = (A_1)_{\bar{n}\bar{m}} := J_{\bar{m}}^{\bar{n}} - \frac{1}{2}J_{\bar{p}}^{\bar{p}}\delta_{\bar{m}}^{\bar{n}} \tag{2.12}$$

are found to obey

$$\partial_0 A_0 - \partial_1 A_1 = 0 \tag{2.13}$$

$$\partial_0 A_1 - \partial_1 A_0 + \frac{1}{\rho}[A_0, A_1] = 0, \tag{2.14}$$

which has a striking similarity with (2.3) and (2.5). As opposed to the currents J_μ, however, the Poisson brackets between the matrices A_μ are completely ultralocal (see section 4).

The link between the variables J_μ and A_μ was found in ref [14]:

$$A_0 = \frac{1}{2}J_0 + \frac{i}{2}\varepsilon\partial_x(\rho g)$$

$$A_1 = \frac{1}{2}J_1 + \frac{i}{2}\varepsilon\partial_t(\rho g), \tag{2.15}$$

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.16}$$

Using this relation, it can be verified that the field equations (2.13), (2.14) are equivalent to (2.3), (2.5) and that the non-ultralocal terms in the Poisson brackets cancel

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[3]For later convenience, our prefactors are slightly different from those of ref [14]

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for the linear combinations (2.15) due to some special properties of the involved (2×2)-matrices.

Having a closer look at the linear combinations (2.15), however, also raises some questions. Whereas the first term J_μ is clearly Lie algebra (ie.$\mathfrak{sl}(2,\mathbb{R})$)-valued (cf. eq. (2.4)), the second term looks a bit odd. It consists of the derivative of a group element g multiplied by a matrix ε, which in general does not have any abstract Lie algebraic meaning. It just happens that g is symmetric and ε is antisymmetric, so that the tracelessness of their products allow their interpretation as $\mathfrak{sl}(2,\mathbb{R})$-valued quantities. At first sight, this seems to make it unlikely that similar variables can be constructed for coset spaces other than $SL(2,\mathbb{R})/SO(2)$ and that the A_μ should perhaps more be considered as a mathematical curiosity of $SL(2,\mathbb{R})/SO(2)$ without a fundamental meaning.

The following two sections, however, will show that this is not true and that the variables A_μ of (2.15) can indeed be generalized to a large number of coset spaces. We will not attempt to construct these analogs via dimensional reduction of matter coupled gravity models in terms of their corresponding $4d$ Ashtekar formulation. Instead, we will work entirely in two dimensions using the relation (2.15) as a model. This approach has the advantage that we can make use of the powerful group theoretical structure underlying the $2d$ coset space nonlinear σ-models whose explicit parametrization in terms of their $4d$ ancestors is sometimes quite intricate [16, 17].

3 The conventional formulation for arbitrary symmetric spaces

In this section, we briefly recapitulate the standard formulation of more general $2d$ nonlinear coset space σ-models as they typically arise in dimensional reductions of higher dimensional (super)gravity theories. This generalizes (the metric version of) the construction for $SL(2,\mathbb{R})/SO(2)$ given in the previous section.

Let G be a simple Lie group with involutive automorphism τ ($\tau^2 = 1$, $\tau \neq 1$) such that $H = \{g \in G : \tau(g) = g\}$ is the maximal compact subgroup of G and the coset space G/H is a noncompact Riemannian symmetric space [18].

The involution τ induces a decomposition of the underlying Lie algebra $\mathfrak{g}$ of G

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k},$$

where $\mathfrak{h} := \{\xi \in \mathfrak{g} : \tau(\xi) = \xi\}$ is the Lie algebra of H and $\mathfrak{k} := \{\xi \in \mathfrak{g} : \tau(\xi) = -\xi\}$ coincides with the orthogonal complement of $\mathfrak{h}$ with respect to the Cartan-Killing form of $\mathfrak{g}$. The generators of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ will be denoted by T_M, T_m and T_a, respectively.

Let Σ be a 2-dimensional Lorentzian manifold with local coordinates $(x^\mu) = (x^0, x^1) = (t, x)$. G/H-valued fields on Σ can be parametrized by G-valued functions

$$\mathcal{V} : \Sigma \to G$$

that are subject to the local gauge freedom of left H-multiplication (in addition to the global action of G on G/H):

$$\mathcal{V}(x^\mu) \to h(x^\mu)\mathcal{V}(x^\mu)g^{-1}, \qquad h \in H, \quad g \in G. \tag{3.1}$$

The 'vielbein' $\mathcal{V}$ is the proper generalization of the 'unimodular zweibein' $e_{\bar{m}}^{\bar{a}}$ introduced in eq. (2.2) for the case $G/H = SL(2, \mathbb{R})/SO(2)$. The Lie algebra valued currents $\partial_\mu \mathcal{V} \cdot \mathcal{V}^{-1}$ decompose as

$$\partial_\mu \mathcal{V} \cdot \mathcal{V}^{-1} = Q_\mu + P_\mu, \qquad Q_\mu \in \mathfrak{h}, \quad P_\mu \in \mathfrak{k} \tag{3.2}$$

with the transformation laws (induced by (3.1))

$$\begin{aligned} Q_\mu &\to hQ_\mu h^{-1} + \partial_\mu h \cdot h^{-1} \\ P_\mu &\to hP_\mu h^{-1}. \end{aligned}$$

Instead of the redundant parametrization $\mathcal{V}$, one can also use the alternative variable

$$M := \tau(\mathcal{V}^{-1})\mathcal{V}, \tag{3.3}$$

which is manifestly H-invariant (and G- covariant). For $G/H = SL(2, \mathbb{R})/SO(2)$, τ is given by $\tau(\mathcal{V}) = (\mathcal{V}^T)^{-1}$, ie. M generalizes the (unimodular) metric block $g_{\bar{m}\bar{n}}$ (cf. (2.2)). The corresponding currents

$$M^{-1}\partial_\mu M = 2\mathcal{V}^{-1}P_\mu \mathcal{V} \tag{3.4}$$

transform under (3.1) according to

$$M^{-1}\partial_\mu M \to g^{-1}(M^{-1}\partial_\mu M)g. \tag{3.5}$$

3.1 The field equations

Like the matrix $e_{\bar{m}}^{\bar{a}}$ (resp. $g_{\bar{m}\bar{n}}$) in Section 2, $\mathcal{V}$ (resp. M) will from now on be considered as a matrix-valued function on Σ, based on some faithful representation of G.

Dimensional reduction of higher dimensional (super) gravity theories to two dimensions leads to nonlinear G/H coset space σ-models coupled to $2d$ gravity and a dilatonic scalar field (plus possible fermionic fields). The dynamically nontrivial part of (the bosonic sector of) these $2d$ field theories is encoded in the kinetic energy term for the coset fields, which is of the following generic form[4]

$$\mathcal{L} \;=\; \frac{1}{8}\rho\,\mathrm{tr}[M^{-1}\partial_\mu M M^{-1}\partial^\mu M] \tag{3.6}$$

$$\;=\; \frac{1}{2}\rho\,\mathrm{tr}[P_\mu P^\mu], \tag{3.7}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1,+1)$ is used to raise and lower the $2d$ worldsheet indices, and ρ is the dilaton field solving the free wave equation $\Box\rho = 0$.

The corresponding field equations for $M(x^\mu)$ read

$$\partial^\mu J_\mu = 0, \tag{3.8}$$

where the currents J_μ are defined by

$$J_\mu := \rho M^{-1}\partial_\mu M \tag{3.9}$$

and therefore obey the following integrability condition

$$\partial_0 J_1 - \partial_1 J_0 + \frac{1}{\rho}[J_0, J_1] - \frac{\partial_0\rho}{\rho}J_1 + \frac{\partial_1\rho}{\rho}J_0 = 0. \tag{3.10}$$

In terms of the variables P_μ and Q_μ, the equation of motion becomes

$$D^\mu(\rho P_\mu) = 0 \tag{3.11}$$

with the H-covariant derivative

$$D_\mu P_\nu := \partial_\mu P_\nu - [Q_\mu, P_\nu],$$

whereas the definition (3.2) of Q_μ and P_ν entails the integrability conditions

$$D_0 P_1 - D_1 P_0 = 0 \tag{3.12}$$

$$\partial_0 Q_1 - \partial_1 Q_0 + [P_1, P_0] + [Q_1, Q_0] = 0. \tag{3.13}$$

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[4] As in (2.1), we assume (at least locally) world-sheet coordinates in which the world-sheet metric differs from the flat $2d$ Minkowski metric only by a conformal factor (which drops out in the part of the Lagrangian we are considering here)

3.2 The Poisson structure

Extracting the Poisson brackets from the Lagrangian (3.7) requires some care due to the coset properties of the field variables. We will simply quote the results and refer to ref. [8] for a detailed description of this procedure.

Let us first introduce some notation. For any $(n \times n)$-matrices $A = A_{\alpha\beta}$ and $B = B_{\gamma\delta}$ we define

$$\overset{1}{A} := A \otimes \mathbf{1}, \qquad \overset{2}{A} := \mathbf{1} \otimes A$$

and similarly for B[5].

For the Poisson brackets between the components of A and B, we introduce [19]

$$\{\overset{1}{A}, \overset{2}{B}\}_{\alpha\beta,\gamma\delta} := \{A_{\alpha\beta}, B_{\gamma\delta}\}.$$

Finally, β^{MN} denotes the inverse of

$$\beta_{MN} := \operatorname{tr}(T_M T_N).$$

The non-vanishing Poisson brackets for the set of variables (P_μ, Q_μ) can then be concisely written as

$$\{\overset{1}{P_0}(x), \overset{2}{P_1}(y)\} = -\frac{1}{\rho(x)} \left([\Omega_{\mathfrak{k}}, \overset{1}{Q_1}] \delta(x-y) + \Omega_{\mathfrak{k}} \partial_x \delta(x-y) \right) \tag{3.14}$$

$$\{\overset{1}{P_0}(x), \overset{2}{Q_1}(y)\} = -\frac{1}{\rho}[\Omega_{\mathfrak{h}}, \overset{1}{P_1}]\delta(x-y), \tag{3.15}$$

where $\Omega_{\mathfrak{k}} := \beta^{mn} T_m \otimes T_n$ and $\Omega_{\mathfrak{h}} := \beta^{ab} T_a \otimes T_b$. (Note that the arguments of functions multiplying derivatives of $\delta(x-y)$ have to be treated with some care.)

Remembering $\partial_1 \mathcal{V} \mathcal{V}^{-1} = Q_1 + P_1$, eqs. (3.14) and (3.15) imply

$$\{\overset{1}{P_0}(x), \overset{2}{\mathcal{V}}(y)\} = \frac{1}{\rho}\Omega_{\mathfrak{k}} \overset{2}{\mathcal{V}} \delta(x-y). \tag{3.16}$$

Using these Poisson brackets, one can then also determine the Poisson brackets between the J_μ:

$$\{\overset{1}{J_0}(x), \overset{2}{J_0}(y)\} = 2[\Omega_{\mathfrak{g}}, \overset{1}{J_0}]\delta(x-y) \tag{3.17}$$

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[5]In components, $(A \otimes \mathbf{1})_{\alpha\beta,\gamma\delta} \equiv A_{\alpha\beta}\delta_{\gamma\delta}$ and $(\mathbf{1} \otimes A)_{\alpha\beta,\gamma\delta} \equiv A_{\gamma\delta}\delta_{\alpha\beta}$ etc.

$$\{\overset{1}{J_1}(x), \overset{2}{J_0}(y)\} = 2[\Omega_{\mathfrak{g}}, \overset{1}{J_1}]\delta(x-y)$$
$$-4\left(\rho \overset{1}{\mathcal{V}^{-1}}\overset{2}{\mathcal{V}^{-1}} \Omega_{\mathfrak{k}} \overset{1}{\mathcal{V}}\overset{2}{\mathcal{V}}\right)(x) \cdot \partial_x \delta(x-y) \qquad (3.18)$$

$$\{\overset{1}{J_1}(x), \overset{2}{J_1}(y)\} = 0, \qquad (3.19)$$

where $\Omega_{\mathfrak{g}} := \beta^{MN} T_M \otimes T_N = \Omega_{\mathfrak{k}} + \Omega_{\mathfrak{h}}$ denotes the Casimir element of $\mathfrak{g}$. Note that for both sets of variables, (P_μ, Q_μ) as well as (J_μ), the Poisson brackets involve non-ultralocal terms (i.e. terms containing derivatives of $\delta(x-y)$).

4 Ashtekar-type variables for Hermitian symmetric coset spaces

As has been explained in section 2, the nonlinear σ-model based on the coset space $SL(2,\mathbb{R})/SO(2)$ admits the construction of an alternative set of variables, which have a natural embedding into the Ashtekar formulation of $4d$ general relativity. In view of the attractive features of these variables (slightly simplified equations of motion plus ultralocal Poisson brackets), it is natural to ask whether they can be generalized to other symmetric spaces G/H.

A first hint comes from the coset space $SL(2,\mathbb{R})/SO(2)$ itself. It is a so-called Hermitian symmetric space, ie. a symmetric space that admits a complex structure (see eg. [18] for a precise definition). Since the variables (2.15) are obviously based on a complexification, one might suspect that Hermitian symmetric spaces could provide natural candidates for a generalization of the relation (2.15). We will now show that this is indeed the case.

The Hermiticity of a Hermitian symmetric space G/H is reflected in its peculiar group theoretical structure: All Hermitian symmetric spaces G/H are of the form

$$G/H = G/(H' \times U(1))$$

with some compact group H'.

It is the (properly normalized) $U(1)$-generator u which induces the complex structure on the coset space G/H:

$$[u, [u, k]] = -k, \text{ for all } k \in \mathfrak{k}. \qquad (4.1)$$

Having a closer look at the mechanism that led to a cancellation of the non-ultralocal terms in the Poisson brackets for the combinations (2.15) in the

$SL(2, \mathbb{R})/SO(2)$-model, one finds that it is precisely the property (4.1) (plus the trivial identity $[u, \mathfrak{h}] = 0$) which is needed for ultralocality. Thus, Hermitian symmetric spaces provide exactly the right amount of additional structure that allows the extension of the construction (2.15) beyond $SL(2, \mathbb{R})/SO(2)$.

Let us now become more explicit. Consider

$$A_0 \; := \; \rho \mathcal{V}^{-1} \left[P_0 + i[u, P_1] + i\frac{\partial_x \rho}{\rho} u \right] \mathcal{V} \tag{4.2}$$

$$\equiv \; \frac{1}{2} J_0 + i\partial_x (\rho \mathcal{V}^{-1} u \mathcal{V}) \tag{4.3}$$

$$A_1 \; := \; \rho \mathcal{V}^{-1} \left[P_1 + i[u, P_0] + i\frac{\partial_t \rho}{\rho} u \right] \mathcal{V} \tag{4.4}$$

$$\equiv \; \frac{1}{2} J_1 + i\partial_t (\rho \mathcal{V}^{-1} u \mathcal{V}). \tag{4.5}$$

These variables are manifestly $\mathfrak{g}^{\mathbb{C}}$-valued, where $\mathfrak{g}^{\mathbb{C}}$ denotes the complexification of the Lie algebra $\mathfrak{g}$. Remembering $[u, \mathfrak{h}] = 0$, it is also easy to see that the A_μ are H-gauge invariant (cf. (3.1), (3.5)), which generalizes the $SO(2)$-invariance of the A_μ of section 2.

As in the case $SL(2, \mathbb{R})/SO(2)$, the advantage of these complexified potentials is twofold. First, consider the equations of motion. A straightforward calculation reveals (cf. (3.11) and (3.12))

$$\partial^\mu A_\mu \; = \; \mathcal{V}^{-1} \left(D^\mu(\rho P_\mu) + i\rho[u, \varepsilon^{\mu\nu} D_\mu P_\nu] \right) \mathcal{V}$$
$$= \; 0$$
$$\partial_0 A_1 - \partial_1 A_0 + \frac{1}{\rho}[A_0, A_1] \; = \; -\mathcal{V}^{-1}(\rho \varepsilon^{\mu\nu} D_\mu P_\nu + i[u, D^\mu(\rho P_\mu)]$$
$$+ i\Box \rho u) \mathcal{V}$$
$$= \; 0 \tag{4.6}$$

with $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$, $\varepsilon^{10} = +1$. Obviously, the equations of motion for A_μ are again of a similar but somewhat simpler form compared to the equations (3.8) and (3.10) for the currents J_μ, although both sets of equations are completely equivalent. In particular, terms like the ones in (3.10) involving the logarithmic derivatives of ρ are absent. It is *precisely* due to these latter terms that the spectral parameter of the linear system encoding (3.8) and (3.10) has to have an irrational x^μ-dependence [4, 5], as will become clear in the next section.

Another simplification occurs with respect to the Poisson brackets. Using the techniques of the previous section, one finds, after some algebra,

$$
\begin{aligned}
\{\overset{1}{A_0}(x), \overset{2}{A_0}(y)\} &= [\Omega_{\mathfrak{g}}, \overset{1}{A_0}\,]\delta(x-y) \\
\{\overset{1}{A_1}(x), \overset{2}{A_0}(y)\} &= [\Omega_{\mathfrak{g}}, \overset{1}{A_1}\,]\delta(x-y) \\
\{\overset{1}{A_1}(x), \overset{2}{A_1}(y)\} &= [\Omega_{\mathfrak{g}}, \overset{1}{A_0}\,]\delta(x-y),
\end{aligned} \tag{4.7}
$$

which is completely ultralocal as opposed to the Poisson brackets between the P_μ or the J_μ. The (quite non-trivial) cancellation of the non-ultralocal terms is due to a subtle and well-balanced interplay between the different terms appearing in the definition of the A_μ (4.2)-(4.5) and the specific properties of the generator u.

This interplay does not work anymore for the Poisson brackets between the A_μ and their complex conjugates, where the non-ultralocal terms can no longer be by-passed. However, the Poisson bracket between A_μ and $\bar{A}_\mu$ is not needed in a canonical formulation along the lines of [9], which can be seen from the original definition of the A_μ for $SL(2, \mathbb{R})/SO(2)$ in terms of the $4d$ Ashtekar variables (eqs. (2.9) - (2.12)): Obviously, the complex conjugate of A_μ involves the complex conjugate ($4d$) Ashtekar connection, whose Poisson bracket with the original ($4d$) Ashtekar connection is
a) already non-ultralocal in four dimensions and
b) completely irrelevant for this formulation of general relativity [9].

We conclude this section with some more "phenomenological" remarks. In order to get an impression of what we are talking about, let us first give a complete list of the possible non-compact Hermitian symmetric spaces [18] (The corresponding compact versions would also allow the construction of A_μ-like quantities, but compact coset spaces cannot occur in dimensional reductions, as they cannot contain the non-compact space $SL(2, \mathbb{R})/SO(2)$ from pure $4d$ gravity as a subspace.)

- $Sp(2n, \mathbb{R})/U(n)$

- $SO^*(2n)/U(n)$

- $SU(n, m)/S(U(n) \times U(m))$

- $SO(n, 2)/SO(n) \times SO(2)$

- $E_{6(-14)}/SO(10) \times SO(2)$

- $E_{7(-25)}/E_6 \times U(1)$

As for the higher dimensional origin of these theories, an interesting observation was made in a recent paper by Cremmer, Julia, Lü and Pope [15]. There it was pointed out that these $2d$ theories with Hermitian symmetric coset spaces G/H all have their 'oxidation endpoint' in four dimensions, ie. the highest possible dimension for a theory that, upon dimensional reduction, leads to the above Hermitian symmetric space nonlinear σ-models is four (generic coset space models can usually be 'oxidized' to much higher dimensions $d \leq 11$).

The corresponding four-dimensional theories that lead to Hermitian symmetric spaces in two dimensions are well-known and can be found in [16, 17]. Some interesting special cases are:

- $Sp(2, \mathbb{R})/U(1) \cong SL(2, \mathbb{R})/SO(2)$
  As seen in section 2, this coset space arises in the two Killing vector reduction of pure $4d$ general relativity.

- $SU(2, 1)/S(U(2) \times U(1))$ and the higher dimensional analogs $SU(n, 1)/S(U(n) \times U(1))$
  They result from $4d$ Maxwell-Einstein gravity, respectively its generalizations with $(n-1)$ vector fields.

- $SO(3, 2)/SO(3) \times SO(2) \cong Sp(4, \mathbb{R})/U(2)$
  This coset space occurs in the dimensional reduction of $4d$ Maxwell-Einstein-dilaton-axion theory, where the two *four-dimensional* scalar fields parametrize $SO(2, 1)/SO(2) \cong SL(2, \mathbb{R})/SO(2)$

- $SO(8, 2)/SO(8) \times SO(2)$
  The corresponding $4d$ ancestor of this model is (the bosonic sector of) $d = 4$, $\mathcal{N} = 4$ supergravity.

We finally note that if, like in the fundamental representations of $Sp(2n, \mathbb{R})$, $SO^*(2n)$ and $SU(m, m)$, u is represented by an invertible matrix with $u^{-1}ku = -k$ for all $k \in \mathfrak{k}$, the involution τ is given by

$$\tau(\xi) = u^{-1}\xi u, \quad \xi \in \mathfrak{g}.$$

In such a case, the A_μ can be written as

$$
\begin{aligned}
A_0 &= \frac{1}{2}J_0 + iu \cdot \partial_x(\rho M) \\
A_1 &= \frac{1}{2}J_1 + iu \cdot \partial_t(\rho M).
\end{aligned}
\tag{4.8}
$$

In the fundamental representation of $SU(m, m)$, u is given by

$$u = \frac{i}{2} \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_m \end{pmatrix},$$

whereas for $Sp(2n, \mathbb{R})$ and $SO^*(2n)$ one has

$$u = \frac{1}{2} \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

so that eqs. (4.8) reduce to the form (2.15) for the two Killing vector reduction of pure $4d$ general relativity, where $G/H = SL(2, \mathbb{R})/SO(2) \cong Sp(2, \mathbb{R})/U(1)$.

To sum up, we have obviously found the direct generalization of the Ashtekar-type variables (2.15) for arbitrary Hermitian symmetric spaces G/H. Although all the $4d$ ancestors of these models are known, it is an open question at this point whether the A_μ for $G/H \neq SL(2, \mathbb{R})/SO(2)$ also have their natural origin in a corresponding $4d$ Ashtekar formulation. On the other hand, from a purely practical point of view, this is not very important. One can simply try to work with these variables without really having to know where they might come from. Adopting this attitude for the moment, we will now have a closer look at the integrability of the field equations.

5 The Lax pair for the complexified potentials

It is a well-known fact [4, 5] that the field equations (3.8), (3.10) for the currents J_μ, respectively (3.11)-(3.13) for Q_μ and P_μ, are completely integrable in the sense that they can be written as the compatibility condition of a system of linear differential equations ('Lax pair').

These linear systems can be cast into a very compact form if they are written in terms of light cone coordinates $x^\pm := (x^0 \pm x^1)$ with $\partial_\pm := \frac{1}{2}(\partial_0 \pm \partial_1)$ and analogously $V_\pm := \frac{1}{2}(V_0 \pm V_1)$ for any V_μ.

Let $\tilde{\rho}$ be the harmonic conjugate to the dilaton field ρ

$$\partial_\mu \tilde{\rho} = \varepsilon_{\mu\nu} \partial^\nu \rho,$$

whose (local) existence is guaranteed by $\Box \rho = 0$.

Consider now the function (the 'variable spectral parameter')

$$\gamma(t, x; w) := \frac{1}{\rho}(w + \tilde{\rho} - \sqrt{(w + \tilde{\rho})^2 - \rho^2}), \tag{5.1}$$

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where w is a constant and the implicit (t, x)-dependence is via ρ and $\tilde{\rho}$.

(5.1) can be inverted

$$w = \frac{1}{2}\rho\left(\gamma + \frac{1}{\gamma}\right) - \tilde{\rho}$$

and implies

$$\gamma^{-1}\partial_\pm\gamma = \frac{(1 \mp \gamma)}{(1 \pm \gamma)}\rho^{-1}\partial_\pm\rho.$$

This particular spacetime dependence of γ ensures that the compatibility condition of the linear system

$$\partial_\pm\hat{\mathcal{V}}\hat{\mathcal{V}}^{-1} = Q_\pm + \frac{1 \mp \gamma}{1 \pm \gamma}P_\pm \tag{5.2}$$

for the G-valued function $\hat{\mathcal{V}} = \hat{\mathcal{V}}(t, x; \gamma(t, x; w))$ implies the field equations (3.11)-(3.13) for the currents P_μ and Q_μ.

Similarly, the linear system for the G-valued function $\Psi = \Psi(t, x; \gamma(t, x; w))$

$$\Psi^{-1}\partial_\pm\Psi = \frac{J_\pm}{\rho(1 \pm \gamma)} \tag{5.3}$$

can be easily verified to imply

$$\partial^\mu J_\mu = 0 \tag{5.4}$$

$$\partial_0 J_1 - \partial_1 J_0 + \frac{1}{\rho}[J_0, J_1] - \frac{\partial_0\rho}{\rho}J_1 + \frac{\partial_1\rho}{\rho}J_0 = 0, \tag{5.5}$$

ie. the field equations (3.8) and (3.10) for the currents J_μ.

In both linear systems, γ plays the rôle of a spacetime dependent spectral parameter, whereas w can be interpreted as a 'hidden' constant spectral parameter. In order to avoid the explicit appearance of the square roots in the linear system, the latter is usually stated in terms of the variable spectral parameter γ. In [6] it was shown that it is exactly this spacetime dependence of γ which serves as a natural (classical) regulator that removes certain ambiguities in the Poisson brackets between transition matrices that are caused by the non-ultralocal terms in the Poisson brackets (3.14) resp. (3.18). These transition matrices are closely related to the infinite number of conserved charges that generate the Geroch group on the phase space. Quantization of the Poisson structure of these charges led to certain twisted Yangian algebras [8].

It would now be extremely interesting to see whether one arrives at similar structures if one quantizes the system in terms of the generalized Ashtekar variables A_μ.

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In order to do so, it would obviously be very convenient if one could also make use of a linear system for the equations of motion

$$\partial^\mu A_\mu = 0 \tag{5.6}$$

$$\partial_0 A_1 - \partial_1 A_0 + \frac{1}{\rho}[A_0, A_1] = 0, \tag{5.7}$$

for the complexified potentials A_μ. In fact, one might suspect that the simplicity of these equations should be reflected in a simpler linear system compared to the ones for the variables (J_μ) or (P_μ, Q_μ). In particular, one might hope that the spacetime dependence of the spectral parameter might simplify. And indeed, this is exactly what happens. The linear system encoding (5.6)-(5.7) has the same form as the system (5.3) for the equations (5.4)-(5.5):

$$U^{-1}\partial_\pm U = \frac{A_\pm}{\rho(1 \pm \lambda)}, \tag{5.8}$$

but the variable spectral parameter λ is now given by

$$\lambda(t, x; v) := \frac{1}{\rho}(v + \tilde{\rho}) \tag{5.9}$$

with the 'hidden' constant spectral parameter

$$v = \rho\lambda - \tilde{\rho}$$

and the resulting differential equation

$$\partial_\pm \lambda = \pm(1 \mp \lambda)\rho^{-1}\partial_\pm \rho.$$

This shows that the square roots in the spacetime dependence of the variable spectral parameter γ (eq. (5.1)) for the J_μ is precisely due to the extra terms in (5.5) which involve the logarithmic derivatives of ρ and which are missing in (5.7). In view of the lack of these square roots, there is no reason anymore to 'hide' the constant spectral parameter v, and one can write the linear system (5.8) entirely in terms of v without introducing square roots:

$$U^{-1}\partial_\pm U = \frac{A_\pm}{\rho \pm (v + \tilde{\rho})}. \tag{5.10}$$

This meshes nicely with the ultralocality of the Poisson brackets (4.7) between the A_μ, for now a regularization mechanism induced by a particular spacetime dependence

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in order to remove ambiguities in the Poisson structure of the transition matrices is not necessary anymore.

In fact, it is now comparatively easy to verify that the transition matrices

$$
\begin{aligned}
T(v) \;\; &:= \;\; U^{-1}(t, x = -\infty; v) U(t, y = \infty; v) & (5.11) \\
&= \;\; \mathcal{P} \exp \left[\int_{-\infty}^{\infty} dx \frac{\rho A_1 - (v + \tilde{\rho}) A_0}{\rho^2 - (v + \tilde{\rho})^2} \right],
\end{aligned}
$$

where $\mathcal{P}$ denotes the path-ordered exponential, have the Poisson algebra

$$
\{ \overset{1}{T}(v), \overset{2}{T}(w) \} = \frac{1}{v - w} \left[\overset{1}{T}(v) \overset{2}{T}(w), \Omega_{\mathfrak{g}} \right]. \tag{5.12}
$$

Formally, this Poisson algebra looks the same as the one found in the metric formalism [6], yet the respective transition matrices themselves might a priori have nothing to do with each other.

The definition (5.11) of the transition matrices and the linear system (5.10) imply that the $T(v)$ are time-independent for sufficiently rapidly decreasing boundary conditions and therefore comprise an infinite set of conserved non-local charges parametrized by a parameter v.

A formal expansion

$$
T(v) = \mathbf{1} + \frac{T_1}{v} + \frac{T_2}{v^2} + \dots
$$

yields for the first three non-trivial charges

$$
T_1 \;\; = \;\; \int_{-\infty}^{\infty} dx A_0 \tag{5.13}
$$

$$
T_2 \;\; = \;\; \int_{-\infty}^{\infty} dx (-\rho A_1 - \tilde{\rho} A_0) + \int_{-\infty}^{\infty} dx A_0 \int_{x}^{\infty} dy A_0 \tag{5.14}
$$

$$
\begin{aligned}
T_3 \;\; = \;\; & \int_{-\infty}^{\infty} dx \left[2\rho\tilde{\rho} A_1 + (\rho^2 + \tilde{\rho}^2) A_0 \right] + \int_{-\infty}^{\infty} dx A_0 \int_{x}^{\infty} dy (-\rho A_1 - \tilde{\rho} A_0) \\
& + \int_{-\infty}^{\infty} dx (-\rho A_1 - \tilde{\rho} A_0) \int_{x}^{\infty} dy A_0 + \int_{-\infty}^{\infty} dx A_0 \int_{x}^{\infty} dy A_0 \int_{y}^{\infty} dz A_0. \tag{5.15}
\end{aligned}
$$

(Using the equations of motion (5.6) - (5.7), the time independence of these charges can also be verified directly.)

It is instructive to compare these charges with the analogous charges found in the metric approach [6]. The first charge T_1 is the same as the corresponding charge in [6], since the imaginary part of A_0 is a total derivative and the real part is just

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$J_0/2$ (see eq. (4.3)). Interestingly enough, the second charge, T_2, also reproduces the corresponding charge in [6], when the definition (4.3)-(4.5) is inserted into (5.14) and some partial integrations are performed. Since these manipulations, in particular, the above series expansion of $T(v)$, are rather formal and might possibly require some more care, we leave a further comparison of these two sets of charges, or rather the corresponding transition matrices, as an interesting problem for the future.

6 Conclusion and open problems

In this article we have given a reformulation of a certain class of dimensionally reduced (super)gravity theories. These theories are described by $2d$ dilaton gravity coupled G/H nonlinear σ-models for which $G/H = G/(H' \times U(1))$ is a Hermitian symmetric space. This construction was motivated by the formulation of the two Killing vector reduction of pure $4d$ gravity in terms of the Ashtekar variables, where the coset space is $SL(2, \mathbb{R})/SO(2)$. In this sense, our construction can be understood as a direct (ie. two-dimensional) generalization of the dimensionally reduced Ashtekar formulation to arbitrary Hermitian symmetric coset spaces. At this point, however, this is, strictly speaking, still an analogy, and it would be very interesting to see whether the variables constructed in this paper really have a natural embedding into the corresponding $4d$ Ashtekar formulations also for $G/H \neq SL(2, \mathbb{R})/SO(2)$.

The theories with Hermitian symmetric coset spaces in $2d$ have recently been found to be very special also in that they all have their 'oxidation endpoints' in $4d$. Since the Ashtekar formulation of general relativity is essentially tied to four dimensions as well, there seems to be a certain consistency with our results. Having a closer look at the simplest cases like eg. $4d$ Maxwell-Einstein gravity might perhaps help to understand these connections a little bit better.

For purely practical purposes, on the other hand, the exact knowledge of a potential $4d$ origin is not very relevant. One can simply start with the two-dimensional theories, perform the transformation to the new set of variables and then try to exploit some of their attractive properties.

These attractive properties are twofold:
1) The Poisson brackets are completely ultralocal.
2) The equations of motion are of a slightly simpler form. As we have shown, a linear system can also be constructed for these equations of motion, and the simpler form of the field equations is reflected in a simpler (in fact, easily removable) spacetime dependence of the spectral parameter. The mere existence of this linear system enables one to apply the same techniques that have been used in the metric formulation

[6, 7, 8].

Making this idea explicit, we exploited the technical advantages of the new variables and calculated the Poisson brackets between the corresponding transition matrices. In the metric picture, an analogous calculation had to deal with an intricate mechanism that removed the ambiguities due to the non-ultralocal Poisson brackets. In terms of the Ashtekar-type variables, however, this ambiguity was absent right from the beginning.

If one can find a nice way to implement the reality conditions for the A_μ, one might now be able to directly probe the *quantum* equivalence of the metric and the self-dual connection approach to quantum gravity within a nontrivial toy model that still exhibits many of the most important properties of full $4d$ general relativity.

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