JUMPING NUMBERS OF A SIMPLE COMPLETE IDEAL IN A TWO-DIMENSIONAL REGULAR LOCAL RING

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1 Introduction

In recent years multiplier ideals have emerged as an important tool in algebraic geometry and commutative algebra. For the history of multiplier ideals and for more details concerning their general theory we refer to [10]. Given an ideal I in a regular local ring, its multiplier ideals $\mathcal{J}(cI)$ form a family parametrized by non-negative rational numbers c. This family is totally ordered by inclusion and the parametrization is order reversing, i.e., $c < c'$ implies $\mathcal{J}(c'I) \subset \mathcal{J}(cI)$. The jumping numbers associated to the ideal I are those rational numbers ξ satisfying $\mathcal{J}(\xi I) \subsetneq \mathcal{J}((\xi - \varepsilon)I)$ for all $\varepsilon > 0$. Jumping numbers of an ideal encode information about the singularities of the corresponding subscheme. The first one of these numbers, the log-canonical threshold, has been much studied in birational geometry. A good source for information about jumping numbers is the fundamental article [6] by Ein, Lazarsfeld, Smith and Varolin.

The purpose of the present manuscript is to determine the jumping numbers of a simple complete ideal $\mathfrak{a}$ in a two dimensional regular local ring α, and investigate their connection to other singularity invariants associated to the ideal. Recall that an ideal is called simple if it is not a product of two proper ideals. Simple ideals play a fundamental role in Zariski's theory of complete ideals, his famous theorem about the unique factorization of complete ideals saying that every complete ideal can be expressed uniquely as a product of simple ideals. Zariski thought of complete ideals as linear systems of curves satisfying "infinitely near base conditions". His theorem about the unique factorization of a complete ideal then corresponds to the factorization of a curve into irreducible branches. This leads us to consider the jumping numbers of an analytically irreducible plane curve. In particular, we are interested to compare the jumping numbers of a simple complete ideal to those of the analytically irreducible plane curve defined by a general element of the ideal.

Based on the proximity relations between infinitely near points, our methods are very much arithmetical in nature. The advantage of this approach is that tecniques needed are elementary. Moreover, our results hold in every characteristic.

Let us explain our results in more detail. We first define the notion of a log-canonical threshold of an ideal with respect to another ideal. It turns out in Proposition 6.7 that jumping numbers are log-canonical thresholds with respect to suitable ideals. We then utilize the proximity relations to calculate these numbers (see Propositions 7.2 and 7.5). In our main Theorem 8.3 we give the promised formula for the jumping numbers. Consider the composition of point blow-ups

$$\mathcal{X} = \mathcal{X}_{n+1} \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_2} \mathcal{X}_2 \xrightarrow{\pi_1} \mathcal{X}_1 = \operatorname{Spec} \alpha, \tag{1}$$

obtained by blowing up the base points of $\mathfrak{a}$. It turns out that the set of jumping numbers is $\mathcal{H}_{\mathfrak{a}} = H_0 \cup \cdots \cup H_{g^*}$, where the sets $H_0, \ldots, H_{g^*}$ correspond to the stars of the dual graph arising from the configuration of the irreducible exceptional divisors on $\mathcal{X}$. A star is a vertex associated to an irreducible exceptional divisor which intersects more than two other irreducible exceptional divisors (see page 15). More precisely, assume that $(a_1, \ldots, a_n)$ is the point basis of $\mathfrak{a}$ (see page 5) and let $\{\gamma_1, \ldots, \gamma_{g^*}\}$ be the set of indices corresponding to the star vertices. Set $\gamma_0 = 1$ and $\gamma_{g^*+1} = n$, and for every $\nu = 0, \ldots, g^*$ write

$$b_\nu := \frac{a_1^2 + \cdots + a_{\gamma_{\nu+1}}^2}{a_{\gamma_\nu}}.$$

Then

$$H_\nu := \left\{ \frac{s+1}{a_{\gamma_\nu}} + \frac{t+1}{b_\nu} + \frac{m}{a_{\gamma_{\nu+1}}} \,\middle|\, s,t,m \in \mathbb{N}, \frac{s+1}{a_{\gamma_\nu}} + \frac{t+1}{b_\nu} \le \frac{1}{a_{\gamma_{\nu+1}}} \right\}$$

for $\nu = 0, \ldots, g^* - 1$, while

$$H_{g^*} := \left\{ \frac{s+1}{a_{\gamma_{g^*}}} + \frac{t+1}{b_{g^*}} \,\middle|\, s,t \in \mathbb{N} \right\}.$$

We point out in Remark 8.2 that the numbers $a_{\gamma_0}, b_0, b_1, \ldots$ above are in fact the "Zariski exponents" of the ideal. They are conventionally denoted by $\bar{\beta}_\nu$ while the integers a_{γ_ν} are often denoted by e_ν for every ν.

As a consequence of the main result, it turns out in Theorem 8.17 that the set of jumping numbers gives equivalent data to the information obtained from the point basis. The proof of this result is based on our Theorem 8.18, which shows that the three smallest jumping numbers determine the order of the ideal.

In fact, to obtain the point basis from the jumping numbers we only need to know the elements $\xi'_\nu = 1/a_{\gamma_\nu} + 1/b_\nu = \min H_\nu$ in every set $H_0, \ldots, H_{g^*}$. This is proved in Corollary 8.15. We shall observe in Proposition 8.14 that $\xi'_\nu = \min\{\xi \in \mathcal{H}_{\mathfrak{a}} \mid \xi \ge 1/a_{\gamma_\nu}\}$. Note that ξ'_0 is the log-canonical threshold of the ideal, which is already enough to determine the point basis and thereby all the jumping numbers, in the case the ideal is monomial. In a way, we may regard the sequence $(\xi'_0, \ldots, \xi'_{g^*})$ as a generalization of the log-canonical threshold. In Corollary 8.12 we give also a formula $e(\mathfrak{a}) = (\xi' - 1)^{-1}$, where $\xi' := \min\{\xi \in \mathcal{H}_{\mathfrak{a}} \mid \xi > 1\}$, for the Hilbert-Samuel multiplicity of the ideal $\mathfrak{a}$.

Utilizing the equivalence between simple complete ideals and analytically irreducible plane curves, we can now determine the jumping numbers of an analytically irreducible plane curve as well. It follows from [10, Proposition 9.2.28] (see also Proposition 9.3) that the jumping numbers of our simple complete ideal $\mathfrak{a}$ coincide in the interval $[0, 1[$ to those of the analytically irreducible plane curve corresponding to a "general" element of $\mathfrak{a}$. In fact, these numbers determine the jumping numbers of $\mathfrak{a}$ as soon as the integer n appearing in the resolution (1) is known (see Theorem 9.9 and Remark 9.10). It is also worth to note that the word "general" can be interpreted here in the sense of Spivakovsky: Consider the resolution (1) above. Following [20, Definition 7.1] and [3, Definition 1], an element $f \in \mathfrak{a}$ is defined to be general, if the corresponding curve $\mathcal{C}_f$ is analytically irreducible and the strict transform of $\mathcal{C}_f$ intersects the strict transform of any exceptional divisor passing through the center of π_n transversely at this point.

Our formula for the jumping numbers of an arbitrary analytically irreducible plane curve in Theorem 9.4 shows that the jumping numbers depend only on the equisingularity class of the curve. Remarkably, it turns out in Theorem 9.8 that the jumping numbers actually determine the equisingularity class. Thus one can say that information about the topological type of the curve singularity is encoded in the set of the jumping numbers.

One should note that in the case of an analytically irreducible plane curve singularity over the complex numbers our formula can be obtained more directly using the general theory of jumping numbers. Indeed, in [21, p. 1191] and [22, p. 390] Vaquié observed that the jumping numbers can be read off from the Hodge-theoretic spectrum defined by Steenbrink and Varchenko. This spectrum has in turn been calculated by several authors, starting from an unpublished preprint of M. Saito [17, Theorem 1.5]. One should also note that Igusa found a formula for the log-canonical threshold of an analytically irreducible plane curve with an isolated singularity (see [8]), and that this result was generalized to reducible curves by Kuwata in [9, Theorem 1.2].

After finishing this paper I received a manuscript [18] by Smith and Thompson, which treats a similar problem from a different perspective.

2 Preliminaries on complete ideals

To begin with, we will briefly review some basic facts from the Zariski-Lipman theory of complete ideals. For more details, we refer to [2], [13], [14], [15] and [23]. Let $\mathcal{K}$ be a field. A two-dimensional regular local ring with the fraction field $\mathcal{K}$ is called a *point*. The maximal ideal of a point α is denoted by $\mathfrak{m}_\alpha$.

Write $\operatorname{ord}_\alpha$ for the unique valuation of $\mathcal{K}$ such that for every $x \in \alpha \setminus \{0\}$

$$\operatorname{ord}_\alpha(x) = \max\{\nu \mid x \in \mathfrak{m}_\alpha^\nu\}.$$

A point β is *infinitely near* to a point α, if

$$\beta \supset \alpha.$$

Then the residue field extension $\alpha/\mathfrak{m}_\alpha \subset \beta/\mathfrak{m}_\beta$ is finite. In the following we will always consider points infinitely near to a fixed point, which has an algebraically closed residue field $\Bbbk$.

Take an element $x \in \mathfrak{m}_\alpha \setminus \mathfrak{m}_\alpha^2$. A *quadratic transform* of the point α is a localization of the ring $\alpha[\mathfrak{m}_\alpha/x]$ at a maximal ideal of $\alpha[\mathfrak{m}_\alpha/x]$. Any two points $\alpha \subset \beta$ can be connected by a unique sequence of quadratic transforms $\alpha = \alpha_1 \subset \cdots \subset \alpha_n = \beta$.

Assume that a point β is a quadratic transform of the point α. Then the ideal $\mathfrak{m}_\alpha\beta$ is generated by a single element, say $b \in \beta$. Let $\mathfrak{a}$ be an ideal in α. The *transform* of an $\mathfrak{a}$ at β is $\mathfrak{a}^\beta := b^{-\operatorname{ord}_\alpha(\mathfrak{a})}\mathfrak{a}\beta$. For any point $\beta \supset \alpha$, the transform $\mathfrak{a}^\beta$ is then defined inductively using the sequence of quadratic transforms connecting α and β. It follows that if $\gamma \supset \beta \supset \alpha$ are any points, then $\mathfrak{a}^\gamma = (\mathfrak{a}^\beta)^\gamma$. If $\mathfrak{a}$ has a finite colength, then so does $\mathfrak{a}^\beta$. Moreover, if $\mathfrak{a}$ is complete, i.e., integrally closed, then so is $\mathfrak{a}^\beta$.

The non-negative integer $\operatorname{ord}_\beta(\mathfrak{a}^\beta)$ is called the *multiplicity* of $\mathfrak{a}$ *at* β. In the case this is strictly positive we say that the point $\beta \supset \alpha$ is a *base point* of the ideal $\mathfrak{a}$. The *support* of $\mathfrak{a}$ is the set of the base points, which is known to be a finite set. The *point basis* of the ideal $\mathfrak{a}$ is the family of multiplicities

$$\mathbf{B}(\mathfrak{a}) := \{\operatorname{ord}_\beta(\mathfrak{a}^\beta) \mid \beta \supset \alpha\}.$$

Since the transform preserves products, i.e., $(\mathfrak{a}\mathfrak{b})^\beta = \mathfrak{a}^\beta\mathfrak{b}^\beta$ for any finite colength ideals $\mathfrak{a}, \mathfrak{b} \subset \alpha$, we have $\mathbf{B}(\mathfrak{a}\mathfrak{b}) = \mathbf{B}(\mathfrak{a}) + \mathbf{B}(\mathfrak{b})$. If in addition $\mathfrak{a}$ and $\mathfrak{b}$ are complete, then the condition $\operatorname{ord}_\beta(\mathfrak{a}^\beta) \geq \operatorname{ord}_\beta(\mathfrak{b}^\beta)$ for every point $\beta \supset \alpha$ implies $\mathfrak{a} \subset \mathfrak{b}$. Moreover, $\mathfrak{a} = \mathfrak{b}$ exactly when $\operatorname{ord}_\beta(\mathfrak{a}^\beta) = \operatorname{ord}_\beta(\mathfrak{b}^\beta)$ for every point $\beta \supset \alpha$.

By the famous result of Zariski [23, p. 385] a complete ideal in α factorizes uniquely into a product of simple complete ideals. Recall that an ideal is simple, if it is not a product of two proper ideals. There is a one to one correspondence between the simple complete ideals of finite colength in α and the points containing α (see e.g. [14, p. 226]). The simple ideal corresponding to a point $\beta \supset \alpha$ is then the unique ideal $\mathfrak{a}$ in α, whose transform at β is the maximal ideal of β. The base points of $\mathfrak{a}$ are totally ordered by inclusion, and if $\alpha = \alpha_1 \subset \cdots \subset \alpha_n$ is the sequence of the base points of $\mathfrak{a}$, then

$\alpha_n = \beta$. The point basis of $\mathfrak{a}$ is the vector $I := (a_1, \dots, a_n) \in \mathbb{N}^n$, where $a_i := \mathrm{ord}_{\alpha_i}(\mathfrak{a}^{\alpha_i})$. Moreover, there is a one to one correspondence between points containing α and divisorial valuations of α given by $\beta \mapsto v := \mathrm{ord}_\beta$ ([23, pp. 389–391]). Note also that the ideals $\mathfrak{m}_\alpha = \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n = \mathfrak{a}$ corresponding to the points $\alpha_1 \subset \cdots \subset \alpha_n$ are the simple v-ideals containing $\mathfrak{a}$ ([23, pp. 392]).

A point β is said to be *proximate* to a point α, if $\beta \supsetneq \alpha$ and the valuation ring of ord_α contains β, in which case we write $\beta \succ \alpha$. The notion of proximity can be interpreted geometrically as follows. The sequence of quadratic transforms $\alpha = \alpha_1 \subset \cdots \subset \alpha_n$ corresponds to a sequence of regular surfaces

$$\pi : \mathcal{X} = \mathcal{X}_{n+1} \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_2} \mathcal{X}_2 \xrightarrow{\pi_1} \mathcal{X}_1 = \mathrm{Spec}\,\alpha, \tag{2}$$

where $\pi_i : \mathcal{X}_{i+1} \to \mathcal{X}_i$ is the blow up of $\mathcal{X}_i$ at a closed point $\varsigma_i \in \mathcal{X}_i$ with $\mathcal{O}_{\mathcal{X}_i, \varsigma_i} = \alpha_i$ for every $i = 1, \dots, n$, and $\pi = \pi_n \circ \cdots \circ \pi_1$. Set $E'_i = \pi_i^{-1}\{\varsigma_i\} \subset \mathcal{X}_{i+1}$. Write $E^*_i = (\pi_n \circ \cdots \circ \pi_{i+1})^* E'_i \subset \mathcal{X}$ for the total transform of E'_i and let $E_i^{(j)}$ denote the strict transform of E'_i on $\mathcal{X}_j$. Especially, set $E_i := E_i^{(n+1)}$. If $j > i$, then $\varsigma_j \in E_i^{(j)}$ if and only if $\alpha_j \succ \alpha_i$, which is occasionally abbreviated by the notation $j \succ i$.

Recall that α_i is always proximate to α_{i-1}, and there is at most one point α_j such that $\alpha_i \succ \alpha_j$ and $j \neq i-1$. In the case such a point α_j exists, α_i is said to be a *satellite* to α_j (or shorter i is satellite to j), and then also $\alpha_\nu \succ \alpha_j$ for every $j < \nu \leq i$. In the opposite case α_i (or just i) is said to be *free*. Note that α_2 is always free. Also α_1 is regarded as a free point.

The proximity relations between the base points $\alpha = \alpha_1 \subset \cdots \subset \alpha_n$ of a simple complete ideal $\mathfrak{a}$ of finite colength can be represented in the *proximity matrix* of $\mathfrak{a}$. Following [2, Definition-Lemma 1.5], it is

$$P := (p_{i,j})_{n \times n}, \text{ where } p_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ -1, & \text{if } i \succ j; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Note that this is the transpose of the proximity matrix Lipman gives in [15, p. 6]. Let $P^{-1} = (x_{i,j})_{n \times n}$ denote the inverse of the proximity matrix P, and write X_i for the i:th row of P^{-1}. Observe that by e.g. [15, Corollary 3.1] these rows correspond to the point basis vectors of the simple complete $\mathfrak{m}_\alpha$-primary v-ideals containing $\mathfrak{a}$, and that they are in the descending order, so that X_1 represents $\mathfrak{m}_\alpha$ and X_n is the point basis of $\mathfrak{a}$. The next well known proposition shows that the proximity matrix of a simple complete ideal is totally determined by the point basis of the ideal. Because it will be crucial for our arguments in the sequel, we shall give a proof for the convenience of the reader.

Proposition 2.1. *Let $\mathfrak{a}$ be a simple complete ideal of finite colength in a two-dimensional regular local ring α, and let $I = (a_1, \dots, a_n)$ be the point basis of $\mathfrak{a}$. For every index $i < k \leq n$, we have*

$$a_i = a_{i+1} + \cdots + a_k,$$

if and only if $\{\nu \mid \alpha_\nu \succ \alpha_i\} = \{i+1, \dots, k\}$. Then necessarily

$$a_i \geq a_{i+1} = \cdots = a_{k-1} \geq a_k,$$

and if $k > i+1$, then $a_{k-1} = a_k$ exactly when $k+1$ is free or $k = n$. Moreover, $a_{n-1} = a_n = 1$.

Proof. As noted above, the point basis vector of $\mathfrak{a}$ is the bottom row of the inverse of the proximity matrix $P = (p_{i,j})_{n \times n}$ of $\mathfrak{a}$. Since $P^{-1}P = 1$, we have

$$\sum_{\nu=1}^{n} a_\nu p_{\nu,i} = \delta_{n,i}. \tag{4}$$

for every $i = 1, \dots, n$. Suppose that $i < n$. If $\{\nu \mid \alpha_\nu \succ \alpha_i\} = \{i+1, \dots, k\}$, then $p_{i+1,i} = \cdots = p_{k,i} = -1$ and $p_{j,i} = 0$ for $j \notin \{i, \dots, k\}$, while $p_{i,i} = 1$. It follows from Equation (4) that $a_i = a_{i+1} + \cdots + a_k$.

Conversely, if $a_i = a_{i+1} + \cdots + a_k$ and $k' := \max\{\nu \mid \alpha_\nu \succ \alpha_i\}$, then $p_{\nu,i} = -1$ if and only if $\nu \in \{i+1, \dots k'\}$, but then by Equation (4) we obtain

$$a_{i+1} + \cdots + a_k = a_i = a_{i+1} + \cdots + a'_k.$$

This forces the equality $k = k'$. Especially, choosing $i = n$ we recover the fact that $a_n = 1$, and choosing $i = n-1$ yields $a_{n-1} = a_n$.

Let $i < j < k-1$. If $\alpha_\nu \succ \alpha_j$ for some $\nu > j+1$, then $\alpha_{j+2} \succ \alpha_j$. This is impossible, because α_{j+2} is already proximate to α_{j+1} and α_i. Therefore α_{j+1} is the only point proximate to α_j, which implies that $a_j = a_{j+1}$. Now the second assertion is clear.

Suppose that $k - 1 > i$. We already observed that $a_{n-1} = a_n$, so we may assume $k < n$. If $\alpha_{k+1} \succ \alpha_j$ for some $j < k$, then also $\alpha_k \succ \alpha_j$. Because α_k is already proximate to both α_{k-1} and α_i, we must have $j \in \{i, k-1\}$. Since $k = \max\{\nu \mid \alpha_\nu \succ \alpha_i\}$ and $\alpha_{k+1} \succ \alpha_j$, the only possibility is $j = k-1$. Then both $\alpha_k \succ \alpha_{k-1}$ and $\alpha_{k+1} \succ \alpha_{k-1}$, which implies $a_{k-1} \geq a_k + a_{k+1}$. Especially, $a_{k-1} > a_k$. On the other hand, as $a_{k-1} = a_k + a_{k+1} + \cdots + a_{k'}$ for some $k' \geq k$, we see that $a_{k-1} > a_k$ implies $\alpha_{k+1} \succ \alpha_{k-1}$. □

The lattice $\Lambda := \mathbb{Z}E_1 + \cdots + \mathbb{Z}E_n$ of the exceptional divisors on $\mathcal{X}$ has two other convenient bases besides $\{E_i \mid i = 1, \dots, n\}$, namely $\{E_i^* \mid i = 1, \dots, n\}$

and $\{\hat{E}_i \mid i = 1, \ldots, n\}$, where the $\hat{E}_i$:s are such that the intersection product with E_j is the negative Kronecker delta, i.e.,

$$E_i \cdot \hat{E}_j = -\delta_{i,j}$$

for $i, j = 1, \ldots, n$. Write

$$E := (E_1, \ldots, E_n)^{\mathrm{T}}, E^* := (E_1^*, \ldots, E_n^*)^{\mathrm{T}} \text{ and } \hat{E} := (\hat{E}_1, \ldots, \hat{E}_n)^{\mathrm{T}}, \tag{5}$$

where T stands for the transpose. The proximity matrix P is the base change matrix between E and E^*, more precisely $E = P^{\mathrm{T}} E^*$. Furthermore, $E_i^* \cdot E_j^* = -\delta_{i,j}$ (cf. [2, p. 174]). Then

$$(E_i \cdot E_j)_{n\times n} = EE^{\mathrm{T}} = P^{\mathrm{T}} E^* (P^{\mathrm{T}} E^*)^{\mathrm{T}} = P^{\mathrm{T}} (E_i^* \cdot E_j^*)_{n\times n} P = -P^{\mathrm{T}} P. \tag{6}$$

Assume that $E_i = \lambda_1 \hat{E}_1 + \cdots + \lambda_n \hat{E}_n$. Then $E_i \cdot E_j = -\lambda_j$, since $E_i \cdot \hat{E}_j = -\delta_{i,j}$. Therefore $E = P^{\mathrm{T}} P \hat{E}$. This further implies that $\hat{E} = P^{-1} E^*$. Let $D \in \Lambda$ and let d, d^* and $\hat{d}$ denote the row vectors in $\mathbb{Z}^n$ satisfying $D = dE = d^* E^* = \hat{d}\hat{E}$. Then

$$d^* = dP^{\mathrm{T}} \text{ and } \hat{d} = dP^{\mathrm{T}} P = d^* P. \tag{7}$$

Note that the intersection product of D and $F = fE \in \Lambda$ is

$$D \cdot F = dEE^{\mathrm{T}} f^{\mathrm{T}} = -dP^{\mathrm{T}} P f^{\mathrm{T}} = -d^* (f^*)^{\mathrm{T}} = -(d_1^* f_1^* + \cdots + d_n^* f_n^*). \tag{8}$$

Recall that a divisor $D \in \Lambda$ is *antinef*, if the intersection product $E_i \cdot D$ is non-positive for every i. According to Equation (8) this means that $\hat{d} = dP^{\mathrm{T}} P$ is non-negative at every entry, which can be abbreviated $\hat{d} = d^* P \geq 0$. This is to say that the row vector $d^* = (d_1^*, \ldots, d_n^*)$ satisfies the *proximity inequalities*

$$d_i^* \geq \sum_{j \succ i} d_j^*. \tag{9}$$

By [15, Theorem 2.1] the proximity inequalities guarantee that there exist a unique complete ideal $\mathfrak{d}$ of finite colength in α having the point basis $\mathbf{B}(\mathfrak{d}) = d^*$. Then $\mathfrak{d}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(-D)$ so that $\mathfrak{d} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-D))$, as $\mathfrak{d}$ is complete. Thus we recover the following proposition (cf. [11, §18, p. 238-239]):

Proposition 2.2. *There is a one to one correspondence between the antinef divisors in Λ and the complete ideals of finite colength in α generating invertible $\mathcal{O}_{\mathcal{X}}$-sheaves, given by $D \leftrightarrow \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-D))$.*

Remark 2.3. Let $\mathfrak{d}$ be a complete ideal of finite colength in α such that $\mathfrak{d}\mathcal{O}_{\mathcal{X}}$ is invertible, and let $D = dE \in \Lambda$ be the antinef divisor corresponding to $\mathfrak{d}$ so that $\mathfrak{d}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(-D)$. The vector $\mathbf{V}(\mathfrak{d}) := d$ is called the *valuation vector* of $\mathfrak{d}$. Observe that $d_i = \mathrm{ord}_{\alpha_i}(\mathfrak{d})$. Recall also that if $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the simple v-ideals containing $\mathfrak{a}$, then $\mathfrak{p}_i\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(-\hat{E}_i)$ so that $\mathfrak{p}_i = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-\hat{E}_i))$. Because $D = \hat{d}\hat{E}$, we see that $\mathfrak{d} = \mathfrak{p}_1^{\hat{d}_1} \cdots \mathfrak{p}_n^{\hat{d}_n}$. We say that $\mathbf{F}(\mathfrak{d}) := \hat{d}$ is the *factorization vector* of $\mathfrak{d}$.

For any divisor $D \in \Lambda$, there exists by e.g. [16, Lemma 1.2] a minimal one among the antinef divisors $D^{\sim}$ satisfying $D^{\sim} \geq D$, which is to say that $D^{\sim} - D$ is effective. This is called the *antinef closure* of D. According to [11, §18, p. 238] an antinef divisor is effective. We can construct $D^{\sim}$ by the so called *Laufer-algorithm* described in [5, Proposition 1]: Set $D_1 = D$. For $i \geq 1$, let $D_i = D^{\sim}$ when D_i is antinef. Otherwise there exsist $\nu_i \in \{1, \ldots, n\}$ such that $D_i \cdot E_{\nu_i} > 0$. In this case set $D_{i+1} = D_i + E_{\nu_i}$. We have

$$\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-D)) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-D^{\sim})) \tag{10}$$

for any divisor $D \in \Lambda$ by [16, Lemma 1.2].

3 Arithmetic of the point basis

In this section we want to concentrate on the structure of the point basis of a simple ideal. From now on, let $\alpha = \alpha_1 \subset \cdots \subset \alpha_n$ be the quadratic sequence of the base points of a simple complete ideal $\mathfrak{a}$ of finite colength in a two-dimensional regular local ring α, and let $I = (a_1, \ldots, a_n)$ denote the point basis of $\mathfrak{a}$.

Definition 3.1. *If α_γ is a satellite point and $\alpha_{\gamma+1}$ is not, then α_γ is a terminal satellite point. If α_τ is free and $\alpha_{\tau+1}$ is not, then we say α_τ is a terminal free point, and if α_υ is a free point but $\alpha_{\upsilon-1}$ is not, then α_υ is a leading free point.*

Remark 3.2. Note that α_n is either a terminal satellite or a terminal free point. The first point α_1 is considered as a leading free point, and if α_γ is a satellite point, then $2 < \gamma \leq n$. We say that the quadratic sequence $\alpha = \alpha_1 \subset \cdots \subset \alpha_n$ (or $\mathfrak{a}$) has a free point (a leading free point, a terminal free point, a satellite, a terminal satellite) at i, if α_i is a free point (a leading free point, a terminal free point, a satellite, a terminal satellite, respectively). We may also say that i is free (a satellite, a terminal satellite, resp.), if there is no confusion.

Clearly, every free point except α_1 belongs to a sequence of free points preceded by either α_1 or a terminal satellite. Moreover, except α_1, the leading free points are exactly the points immediately following a terminal satellite point. We use the following notation for terminal satellites and terminal free points.

Notation 3.3. *Let us write $\Gamma_{\mathfrak{a}}$, or just Γ if there is no confusion, for the set*

$$\{\gamma_1, \ldots, \gamma_g\} := \{\gamma \mid \alpha_\gamma \text{ is a terminal satellite point of } \mathfrak{a}\},$$

where $\gamma_1 < \cdots < \gamma_g$. Write also

$$\Gamma^* := \{\gamma \in \Gamma \mid \gamma < n\} = \{\gamma_1, \ldots, \gamma_{g^*}\}.$$

Moreover, let $\alpha_{\tau_1}, \ldots, \alpha_{\tau_g}$ denote the terminal free points of $\mathfrak{a}$ anterior to α_{γ_g}, so that

$$\tau_1 < \gamma_1 < \tau_2 < \gamma_2 < \cdots < \tau_g < \gamma_g. \tag{11}$$

Set $\gamma_0 := 1 =: \tau_0$ and $\gamma_{g+1} := n =: \tau_{g+1}$, and then define

$$\bar{\Gamma} := \Gamma \cup \{\gamma_{g+1}\}.$$

Note that the number g^* of elements in Γ^* is either $g-1$ or g depending on whether α_n is a satellite or not. Thus $\gamma_{g^*+1} = n$ and $\bar{\Gamma} = \Gamma^* \cup \{n\}$. If necessary, we may write $\gamma_\nu = \gamma_\nu^{\mathfrak{a}}$ or $\tau_\nu = \tau_\nu^{\mathfrak{a}}$ to specify the ideal in question.

Example 3.4. If $\Gamma^* = \emptyset$, then the ideal is monomial, i.e., there exist a regular system of parameters for α such that $\mathfrak{a}$ is monomial in these parameters.

Proposition 3.5. *The multiplicity at a non-terminal free point is equal to the multiplicity at the terminal satellite point preceding it or to the multiplicity at the first point, if there is no preceding satellite point. The multiplicity at the subsequent terminal free point is strictly less, or the terminal free point is the last point, in which case they both are equal to one. Moreover, the multiplicities at a terminal satellite point and at the point immediately preceding it are equal.*

Proof. It follows from Proposition 2.1 that $a_i \geq a_j$, if $j > i$. Moreover, $a_{i-1} > a_i$ if and only if $i+1$ is a satellite to $i-1$. By Definition 3.1 we see that $i+1$ is free for $\gamma_{\nu-1} < i < \tau_\nu$. If $1 \leq \nu \leq g$, then $\tau_\nu + 1$ must be a satellite to $\tau_\nu - 1$ as τ_ν is free. Note also that $\gamma_\nu + 1$ is not a satellite. Therefore

$$a_{\gamma_{\nu-1}} = \cdots = a_{\tau_\nu - 1} > a_{\tau_\nu} \geq a_{\gamma_\nu - 1} = a_{\gamma_\nu}$$

for $\nu = 1, \ldots, g$. Furthermore, because γ_g is the last satellite, we must have $a_{\gamma_g} = \cdots = a_n$ and $a_n = 1$, as observed in Proposition 2.1. ∎

For $\nu = 1, \ldots, g+1$, the sequence $a_{\gamma_{\nu-1}}, \ldots, a_{\gamma_\nu}$ of multiplicities may be written as

$$\overbrace{s_{\nu,1}, \ldots, s_{\nu,1}}^{r_{\nu,1} \text{ times}}, \overbrace{s_{\nu,2}, \ldots, s_{\nu,2}}^{r_{\nu,2} \text{ times}}, \ldots, \overbrace{s_{\nu,m_\nu}, \ldots, s_{\nu,m_\nu}}^{r_{\nu,m_\nu}+1 \text{ times}}, \tag{12}$$

where $s_{\nu,1} > \cdots > s_{\nu,m_\nu}$. Proposition 3.5 implies that $m_\nu > 1$ and $r_{\nu,m_\nu} \geq 1$ for $1 \leq \nu \leq g$, while $m_{g+1} = 1$ and $r_{g+1,1} := \gamma_{g+1} - \gamma_g \geq 0$. Since there are no terminal satellites between $\gamma_{\nu-1}$ and γ_ν, Proposition 2.1 yields

$$s_{\nu,j-1} = r_{\nu,j} s_{\nu,j} + s_{\nu,j+1} \tag{13}$$

for $1 < j \leq m_\nu$, where we set $s_{\nu,m_\nu+1} := a_{\gamma_\nu}$ for $1 \leq \nu \leq g+1$. Because of Proposition 3.5, the same holds also for $j = 1$, if we set $s_{\nu,0} := a_{\gamma_{\nu-1}} + \cdots + a_{\tau_\nu}$ for every $\nu = 1, \ldots, g+1$. When $j = m_\nu$, Equation (13) yields $s_{\nu,m_\nu} \mid s_{\nu,m_\nu-1}$. This shows that we may obtain the sequence $a_{\gamma_{\nu-1}}, \ldots, a_{\gamma_\nu}$ from $a_{\gamma_{\nu-1}}$ and $a_{\gamma_{\nu-1}} + \cdots + a_{\tau_\nu}$ by the Euclidian division algorithm.

Notation 3.6. *Let $\nu \in \{1, \ldots, g+1\}$. We write $\beta'_\nu(\mathfrak{a})$, or just β'_ν if the ideal is clear from the context, for the positive rational number*

$$\beta'_\nu := \frac{a_{\gamma_{\nu-1}} + \cdots + a_{\tau_\nu}}{a_{\gamma_{\nu-1}}}.$$

We also set $\beta'_\nu := 0$ for $\nu > g+1$.

Proposition 3.7. *The point basis of $\mathfrak{a}$ is totally determined by the rational numbers $\beta'_1, \ldots, \beta'_{g^*+1}$. In particular, the numbers $\beta'_1, \ldots, \beta'_g$ yield the multiplicities $a_1, \ldots, a_{\gamma_g}$. Moreover, for $\nu \in \{1, \ldots, g+1\}$*

$$\gcd\{a_1, \ldots, a_{\gamma_\nu}\} = \gcd\{a_{\gamma_{\nu-1}}, a_{\tau_\nu}\} = a_{\gamma_\nu}.$$

Proof. Using the Euclidian division algorithm described above we get

$$\gcd\{a_{\gamma_{\nu-1}}, a_{\tau_\nu}\} = \gcd\{s_{\nu,j}, s_{\nu,j+1}\} = a_{\gamma_\nu} \tag{14}$$

for every $0 \leq j \leq m_\nu$. Especially, a_{γ_ν} divides $a_{\gamma_{\nu-1}}$, and so a_{γ_ν} divides a_i for every $i \leq \gamma_\nu$. Hence $\gcd\{a_1, \ldots, a_{\gamma_\nu}\} = a_{\gamma_\nu}$. This shows the last claim.

To prove the first two claims, we observe that for every $\nu = 1, \ldots, g+1$

$$\beta'_\nu = \frac{s_{\nu,0}}{s_{\nu,1}} = \frac{\mathrm{N}_\nu}{\mathrm{D}_\nu}$$

where N_ν and D_ν are integers with $\gcd\{\mathrm{N}_\nu, \mathrm{D}_\nu\} = 1$. Suppose that we know the pair $(\beta'_\nu, a_{\gamma_\nu})$. By Equation (14) we have $s_{\nu,0} = a_{\gamma_\nu}\mathrm{N}_\nu$ and $s_{\nu,1} = a_{\gamma_{\nu-1}} =$

$a_{\gamma_\nu}\mathrm{D}_\nu$. Then we obtain the multiplicities $a_{\gamma_{\nu-1}}, \dots, a_{\gamma_\nu}$ by the Euclidian division algorithm. Recall that $a_{\gamma_g} = a_n = 1$ by Proposition 3.5. Starting from the pair (β'_{g^*+1}, a_n), or (β'_g, a_{γ_g}), we then get all the multiplicities

$$a_1, \dots, a_n, \text{ or } a_1, \dots, a_{\gamma_g},$$

respectively. □

Remark 3.8. It follows from the Euclidian division algorithm (see Formula (12), Equation (13) and Proposition 3.7), that each β'_ν can be obtained from the integers $r_{\nu,1}, \dots, r_{\nu,m_\nu}$ as a continued fraction

$$\beta'_\nu = r_{\nu,1} + \cfrac{1}{r_{\nu,2} + \cdots + \frac{1}{r_{\nu,m_\nu}+1}}.$$

Note that these numbers are the *Puiseux exponents* Spivakovsky defines in [20, Definition 6.4].

We now want to investigate the relationship between the point bases of the ideals $\mathfrak{p}_i$ to that of the ideal $\mathfrak{a}$. Let P be the proximity matrix of $\mathfrak{a}$, and write X_i for the i:th row of $(x_{i,j})_{n\times n} = P^{-1}$. Recall that $(x_{i,1}, \dots, x_{i,i})$ is the point basis of $\mathfrak{p}_i$ and that $X_n = I$.

We first observe the following.

Proposition 3.9. *Let $i, j \in \{1, \dots, n\}$, with $i \geq j$. Then $\mathfrak{p}_i$ has a satellite or a free point at j, if and only if $\mathfrak{a}$ has a satellite or a free point at j, respectively. Moreover, this point is terminal for $\mathfrak{p}_i$ if and only if it is terminal for $\mathfrak{a}$ or $j = i$. If $i > j$, then $x_{i,j} = x_{i,j+1}$ if and only if $a_j = a_{j+1}$ or $i = j + 1$.*

Proof. The first two claims are obvious, since the base points of $\mathfrak{p}_i$ are $\alpha_1 \supset \cdots \supset \alpha_i$, which are base points of $\mathfrak{a}$. Suppose then that $i > j$. By Proposition 2.1 we have $x_{i,j} = x_{i,j+1}$ if and only if α_{j+1} is the only point among the base points of $\mathfrak{p}_i$ proximate to α_j. An equivalent condition to this is that either α_{j+2} is not proximate to α_j or $i = j + 1$, in other words, either $a_j = a_{j+1}$ or $i = j + 1$. □

As a corollary to Proposition 3.5 we now get

Proposition 3.10. *Let $\nu \in \{0, \dots, g\}$. Then $x_{i,j} = 1$ for $\gamma_\nu \leq j \leq i \leq \tau_{\nu+1}$.*

Proof. Consider the ideal $\mathfrak{p}_i$. Because $\mathfrak{a}$ has a satellite point at $\gamma_\nu \leq i$ and a free point at every j satisfying $\gamma_\nu < j \leq i$, Proposition 3.9 shows that $\gamma_\nu^{\mathfrak{p}_i} = \gamma_\nu$ and $\tau_{\nu+1}^{\mathfrak{p}_i} = i$. In particular, $\mathfrak{p}_i$ has the last satellite point at γ_ν and $\gamma_{\nu+1}^{\mathfrak{p}_i} = i$. It follows from Proposition 3.5 that $x_{i,\gamma_\nu} = \cdots = x_{i,i} = 1$. □

Note that the base points of the transform $\mathfrak{p}_m^{\alpha_k}$ are $\alpha_k \subset \cdots \subset \alpha_m$ and the transform of $\mathfrak{p}_m^{\alpha_k}$ at α_m is the maximal ideal, i.e., $\mathfrak{p}_m^{\alpha_k}$ is the simple complete ideal in α_k corresponding to the point α_m. It follows that $(x_{m,k}, \ldots, x_{m,m})$ is the point basis of $\mathfrak{p}_m^{\alpha_k}$. Observe also that the sequence of multiplicities $(a_k, \ldots, a_m)$, where $1 \leq k \leq m \leq n$, is a point basis of a complete ideal, though not necessarily simple, because this sequence satisfies the proximity inequalities (9). Indeed, we have

$$a_i - \sum_{m \geq j \succ i} a_j \geq a_i - \sum_{j \succ i} a_j \geq \delta_{i,n} \geq 0$$

for every $k \leq i \leq m$. This motivates the following notation.

Notation 3.11. *For $X = (x_1, \ldots, x_n) \in \mathbb{N}^n$ and for $i, j \in \{1, \ldots, n\}$ we write*

$$X^{[i,j]} := (\overset{(1)}{0}, \ldots, \overset{(i-1)}{0}, x_i, \ldots, x_j, \overset{(j+1)}{0}, \ldots, \overset{(n)}{0})$$

Moreover, we set

$$X^{[i,j)} := X^{[i,j-1]} \text{ and } X^{(i,j]} := X^{[i+1,j]}.$$

In addition to that, we write

$$X^{\leq i} := X^{[1,i]}, X^{<i} := X^{[1,i)}, X^{\geq i} := X^{[i,n]} \text{ and } X^{>i} := X^{(i,n]}.$$

For the truncated rows of P^{-1} we obtain the following result.

Proposition 3.12. *Let $i, k \in \{1, \ldots, n\}$.*

$$X_i^{\leq k} = \begin{cases} X_i, & \text{if } k \geq i; \\ x_{i,k} X_k, & \text{if } k < j \text{ and } k+1 \text{ is free;} \\ x_{i,k} X_k + \varrho_{i,k} X_h, & \text{if } k < j \text{ and } k+1 \text{ is a satellite to } h, \end{cases}$$

where $\varrho_{i,k} := x_{i,h} - (x_{i,h+1} + \cdots + x_{i,k})$. In particular,

$$X_i^{\leq \gamma_\nu} = x_{i,\gamma_\nu} X_{\gamma_\nu},$$

where ν is such that $i \geq \gamma_\nu$.

Proof. The case $k \geq i$ being trivial, we may restrict ourselves to the case $k < i$. Proposition 3.9 now implies that we may replace $\mathfrak{a}$ by $\mathfrak{p}_i$, and so we may assume that $i = n$. Consider the factorization vector $F := X_n^{\leq k} P$ so that

$$X_n^{\leq k} = FP^{-1} = \sum_{j=1}^{n} F_j X_j.$$

Now $F_j = 0$ for $k < j \leq n$, and if $1 \leq j \leq k$, then

$$F_j = a_j - \sum_{k \geq \nu \succ j} a_\nu.$$

Because $a_j = \sum_{\nu \succ j} a_\nu + \delta_{j,n}$ by Proposition 2.1, we see that $F_j > 0$, if and only if $k+1 \succ j$ or $j = k = n$. Clearly, $F_k = a_k$.

It now follows that if $k+1$ is free or $k = n$, then $F_j = 0$ for every $j \neq k$, implying $X_n^{\leq k} = a_k X_k$. Assume then that $k+1$ is a satellite to some (unique) $h < k$. Now $F_h = a_h - (a_{h+1} + \cdots + a_k)$, while $F_j = 0$ for every $j \notin \{h, k\}$. Then

$$X_n^{\leq k} = a_k X_k + (a_h - \cdots - a_k) X_h,$$

as wanted. □

For certain computations, it is convenient to introduce the following notation.

Notation 3.13. *Let $P^{-1} = (x_{i,j})_{n \times n}$ be the inverse of the proximity matrix of $\mathfrak{a}$. For any $\nu \in \{0, \ldots, g\}$ and $\gamma_\nu < i \leq n$, write*

$$\rho_{i,\gamma_\nu} = \rho^{\mathfrak{a}}_{i,\gamma_\nu} := x_{i,\gamma_\nu+1} + \cdots + x_{i,\tau_{\nu+1}} \text{ and } \rho_\nu := \rho_{\gamma_{\nu+1},\gamma_\nu}.$$

In the case $i \leq \gamma_\nu$ we set $\rho_{i,\gamma_\nu} = 0$.

Remark 3.14. Observe that $\beta'_\nu = 1 + \rho_{n,\gamma_{\nu-1}}/a_{\gamma_{\nu-1}}$ for $\nu = 1, \ldots, g+1$.

Corollary 3.15. *Let $\gamma \in \{\gamma_0, \ldots, \gamma_{g+1}\}$, and take an element $x_{i,j}$ of P^{-1} with $j \leq \gamma \leq i$. Then $x_{i,j} = x_{i,\gamma} x_{\gamma,j}$. It follows that*

$$a_{\gamma_\nu} = a_{\gamma_{\nu+1}} x_{\gamma_{\nu+1},\gamma_\nu} \text{ and } a_{\gamma_\nu} = x_{\gamma_{\nu+1},\gamma_\nu} \cdots x_{\gamma_{g+1},\gamma_g}$$

for $\nu = 0, \ldots, g$. Moreover, if $\gamma_{\nu+1} \leq i$, then we have $\rho_{i,\gamma_\nu} = x_{i,\gamma_{\nu+1}} \rho_\nu$.

Proof. For the first claim we observe that either $\gamma + 1$ is free or $\gamma = n$. The latter case is trivial, while the former implies by Proposition 3.12 that $X_i^{\leq \gamma} = x_{i,\gamma} X_\gamma$. Especially, this gives $x_{i,j} = x_{i,\gamma} x_{\gamma,j}$. Thus the first claim holds. Choosing $j = \gamma_\nu$, $\gamma = \gamma_{\nu+1}$ and $i = n$ we see that $a_{\gamma_\nu} = x_{\gamma_{\nu+1},\gamma_\nu} a_{\gamma_{\nu+1}}$, and then the second equality follows by induction. As $\tau_{\nu+1} \leq \gamma_{\nu+1}$ we see that $x_{i,j} = x_{i,\gamma_\nu+1} x_{\gamma_\nu+1,j}$ for every $j = \gamma_\nu + 1, \ldots, \tau_{\nu+1}$ and $i \geq \gamma_{\nu+1}$, which proves the last claim. □

Proposition 3.16. *Let $\mathfrak{p}_i$ be the simple complete v-ideal corresponding to the row X_i and let μ be the integer satisfying $\gamma_\mu < i \leq \gamma_{\mu+1}$ unless $i = 1$, in which case we set $\mu := 0$. For every $\nu \in \{0, \ldots, \mu+1\}$, we have*

$$\rho^{\mathfrak{p}_i}_{i,\gamma_\nu^{\mathfrak{p}_i}} = \rho_{i,\gamma_\nu}.$$

Proof. Suppose first that $0 \le \nu < \mu$. Then $\gamma_\nu < i$. It follows from Proposition 3.9 that $\gamma_\nu^{\mathfrak{p}_i} = \gamma_\nu$. Moreover, $\tau_{\nu+1}^{\mathfrak{p}_i} = \min\{i, \tau_{\nu+1}\}$. As $x_{i,j} = 0$ for $j > i$ we then get

$$\rho^{\mathfrak{p}_i}_{i,\gamma_\nu^{\mathfrak{p}_i}} = x_{i,\gamma_\nu^{\mathfrak{p}_i}+1} + \cdots + x_{i,\tau^{\mathfrak{p}_i}_{\nu+1}} = x_{i,\gamma_\nu+1} + \cdots + x_{i,\tau_{\nu+1}} = \rho_{i,\gamma_\nu}. \tag{15}$$

If $\nu = \mu$, then $\gamma_\nu < i$, but it may happen that $\tau_\nu \ge i$. If this is the case, then $\mathfrak{a}$ has a free point at i. By Proposition 3.9 this means that $\tau^{\mathfrak{p}_i}_{\nu+1} = i$, and since $x_{i,j} = 0$ for $j > i$, we observe that Equation (15) holds. If $\nu = \mu + 1$, then $\gamma_\nu \ge \gamma_\nu^{\mathfrak{p}_i} = i$ and by definition $\rho^{\mathfrak{p}_i}_{i,\gamma_\nu^{\mathfrak{p}_i}} = \rho_{i,\gamma_\nu} = 0$. □

Corollary 3.17. *if $\gamma_\nu \le i$, then $\beta'_\nu(\mathfrak{p}_i)$ doesn't depend on i.*

Proof. Assuming $\gamma_\nu \le i$ we get by using Corollary 3.15

$$\frac{\rho_{i,\gamma_{\nu-1}}}{x_{i,\gamma_{\nu-1}}} = \frac{x_{i,\gamma_\nu}\rho_{\nu-1}}{x_{i,\gamma_\nu}x_{\gamma_\nu,\gamma_{\nu-1}}} = \frac{a_{\gamma_\nu}\rho_{\nu-1}}{a_{\gamma_\nu}x_{\gamma_\nu,\gamma_{\nu-1}}} = \frac{\rho_{n,\gamma_{\nu-1}}}{a_{\gamma_{\nu-1}}}.$$

Since $\gamma_{\nu-1} \le i$, it follows from Proposition 3.16 that $\rho^{\mathfrak{p}_i}_{i,\gamma^{\mathfrak{p}_i}_{\nu-1}} = \rho_{i,\gamma_{\nu-1}}$. Because $\beta'_\nu(\mathfrak{p}_i) = 1 + \rho^{\mathfrak{p}_i}_{i,\gamma^{\mathfrak{p}_i}_{\nu-1}}/x_{i,\gamma_{\nu-1}}$ and $\beta'_\nu = 1 + \rho_{n,\gamma_{\nu-1}}/a_{\gamma_{\nu-1}}$, we then observe that $\beta'_\nu(\mathfrak{p}_i) = \beta'_\nu$. Thus we get the claim. □

4 The Dual graph

Let $\mathfrak{a}$ be a simple complete ideal of finite colength in a two-dimensional regular local ring α. Consider the resolution (2) of $\mathfrak{a}$. The configuration of the exceptional divisors $E_1, \ldots, E_n$ on $\mathcal{X}$ arising from (2) can be represented by a weighted graph (see, e.g., [19, p. 111, 5. pp. 124 – 129]). This graph is called the *dual graph* of $\mathfrak{a}$. Its vertices are $\epsilon_1, \ldots, \epsilon_n$, where ϵ_i corresponds to E_i for every i. The *weight* of a vertex ϵ_i is $w_i := -E_i \cdot E_i$. Two vertices are *adjacent*, if they are joined by an edge. This takes place, if and only if the intersection of the corresponding divisors on $\mathcal{X}$ is nonempty. A vertex is called an *end*, if it is adjacent to at most one vertex. If it has more than two adjacent vertices, it is said to be a *star*.

Proposition 4.1. *Let ϵ_i and ϵ_j be vertices with $i < j$ in the dual graph of $\mathfrak{a}$. They are adjacent, if and only if $j = \max\{\nu \mid \nu \succ i\}$. In particular, if this is the case, then j is uniquely determined.*

Proof. Vertices ϵ_i and ϵ_j are adjacent, if and only if $E_i \cdot E_j \ne 0$. By Equation (6) and by the definition of the proximity matrix we have

$$-E_i \cdot E_j = \sum_{\nu=j}^{\mu} p_{\nu,i}p_{\nu,j},$$

where $\mu := \max\{\nu \mid \nu \succ i\}$. Clearly, $j > \mu$ implies $E_i \cdot E_j = 0$. If $j < \mu$, then

$$-E_i \cdot E_j = p_{j+2,i}p_{j+2,j} + \cdots + p_{\mu,i}p_{\mu,j}.$$

Observe that this is nonzero, if and only if $p_{j+2,i}p_{j+2,j} \neq 0$, but this is impossibile, because then $j+2$ would be proximate to i, j and $j+1$. Then we see that $E_i \cdot E_j \neq 0$ with $i < j$ if and only if $j = \mu$, in which case $E_i \cdot E_j = 1$. □

Remark 4.2. Observe that $w_i = -E_i \cdot E_i = 1 + \mu - i$, where we write $\mu := \max\{\nu \mid \nu \succ i\}$ so that $\mu - i$ is the number of points proximate to i. Thus vertices ϵ_i and ϵ_j with $i < j$ are adjacent, if and only if

$$j = i + w_i - 1.$$

Furthermore, if $i \neq j$, then $E_i \cdot E_j = \delta_{i,i+w_i-1}$. Thereby we observe that the matrix $P^{\mathrm{T}}P$, which represents the dual graph of $\mathfrak{a}$ by Equation (6), is totally determined by the sequence of the weights $(w_1, \ldots, w_n)$. Moreover, we obtain the proximity matrix $P = (p_{i,j})_{n\times n}$ from the w_i:s, since $p_{i,j} = -1$, if and only if $i < j < i + w_i$. Thus the sequence $(w_1 \ldots, w_n)$ gives equivalent data to the point basis of $\mathfrak{a}$.

Proposition 4.3. *The stars of the dual graph of $\mathfrak{a}$ are precisely the vertices ϵ_γ such that $\gamma < n$ and $\mathfrak{a}$ has a terminal satellite at γ. The ends of the dual graph are exactly the vertices ϵ_τ such that $\tau = 1$ or $\mathfrak{a}$ has a terminal free point at τ.*

Proof. Suppose that $\gamma < n$ is a terminal satellite. By Proposition 4.1 we know that ϵ_μ is adjacent to ϵ_γ for $\mu = \max\{i \mid i \succ \gamma\}$. Because γ is a terminal satellite, we get for some $\nu < \gamma - 1$

$$\gamma = \max\{i \mid i \succ \gamma - 1\} = \max\{i \mid i \succ \nu\}.$$

By using Proposition 4.1 again, we see that ϵ_ν, $\epsilon_{\gamma-1}$ and ϵ_μ are adjacent to ϵ_γ. Thus ϵ_γ is a star.

Conversely, suppose that ϵ_γ is a star. Then there are three vertices adjacent to ϵ_γ, say ϵ_i, ϵ_j and ϵ_k. Noting that a point cannot be proximate to three different points, it follows from Proposition 4.1 that exactly one of the indices is greater that γ. Therefore we may suppose that $j < k < \gamma < i$. In particular, $\gamma < n$. If $\epsilon_{\gamma+1}$ is a satellite to some m, then also $\gamma \succ m$, which implies that $m \in \{j, k\}$. On the other hand, as ϵ_j and ϵ_k are adjacent to ϵ_γ we see by Proposition 4.1 that $\gamma + 1$ is proximate to neither j nor k, which is a contradiction. Therefore $\gamma + 1$ is free, and so γ is a terminal satellite.

Suppose that $\mathfrak{a}$ has a terminal free point at $\tau > 1$. Suppose also that ϵ_i is adjacent to ϵ_τ. If $i < \tau$, then $\tau \succ i$ by Proposition 4.1, and because τ is not a satellite to any $i < \tau - 1$, it follows that $i = \tau - 1$, but since ϵ_i is adjacent to ϵ_τ, Proposition 4.1 implies that $\tau + 1 \not\succ \tau - 1$. On the other hand, we know that $\tau + 1$ is not free as τ is terminal. This means that $\tau = n$, and $\epsilon_{\tau-1}$ is the only vertex adjacent to ϵ_τ, i.e., τ is an end. Furthermore, we see that $\tau < n$ implies $i > \tau$, but then i is uniquely determined by Proposition 4.1. Hence ϵ_τ is an end. For the same reason ϵ_1 is an end.

Let us then prove the converse. As we just saw, ϵ_1 is always an end. Proposition 4.1 yields that ϵ_j adjacent to ϵ_n whenever $n \succ j$. Thus ϵ_n is an end, if and only if n is free. Suppose then that ϵ_i is an end with $1 < i < n$. Then Proposition 4.1 implies that the only vertex adjacent to ϵ_i is ϵ_μ, where $\mu := \max\{\nu \mid \nu \succ i\}$ is greater than i. Moreover, if $i \succ j$ for some $j < i$, then $i + 1 \succ j$, too. Especially, this shows that $i + 1 \succ i - 1$, and since $i + 1 \succ i$, it follows that $i \not\succ j$ for any $j < i - 1$. Therefore i is free, while $i + 1$ is a satellite. So, if ϵ_i in an end for some $i \in \{1, \ldots, n\}$, then $i = 1$ or $\mathfrak{a}$ has a terminal free point at i. $\square$

Remark 4.4. By Proposition 4.3 we observe that the vertex ϵ_i is a star exactly, when $i \in \Gamma^*$, and g^* is the number of the star vertices of the dual graph.

We will now recall how the dual graph can be constructed from the point basis $(a_1, \ldots, a_n)$ of $\mathfrak{a}$. Let $\{\gamma_0, \ldots, \gamma_{g+1}\}$ be as given in Notation 3.3. Let also the integers $s_{\nu,\mu}$, $r_{\nu,\mu}$ and m_ν be as in Formula (12), where $\nu = 1, \ldots, g + 1$ and $\mu = 1, \ldots, m_\nu$. Set

$$\kappa_{\nu,\mu} := \sum_{i=1}^{\nu-1} \sum_{j=1}^{m_i} r_{i,j} + \sum_{j=1}^{\mu} r_{\nu,j} \tag{16}$$

for $\nu = 1, \ldots, g + 1$ and $\mu = 0, \ldots, m_\nu$. It follows that for $\mu > 0$

$$\kappa_{\nu,\mu} = \kappa_{\nu,\mu-1} + r_{\nu,\mu}, \text{ and } \kappa_{\nu,\mu} = \kappa_{\nu-1,m_{\nu-1}} + \sum_{j=1}^{\mu} r_{\nu,j}$$

when $\nu > 1$. Moreover, for every $\nu = 1, \ldots, g + 1$ we have

$$\kappa_{\nu,0} = \gamma_{\nu-1} - 1 \text{ and } \kappa_{\nu,m_\nu} = \gamma_\nu - 1.$$

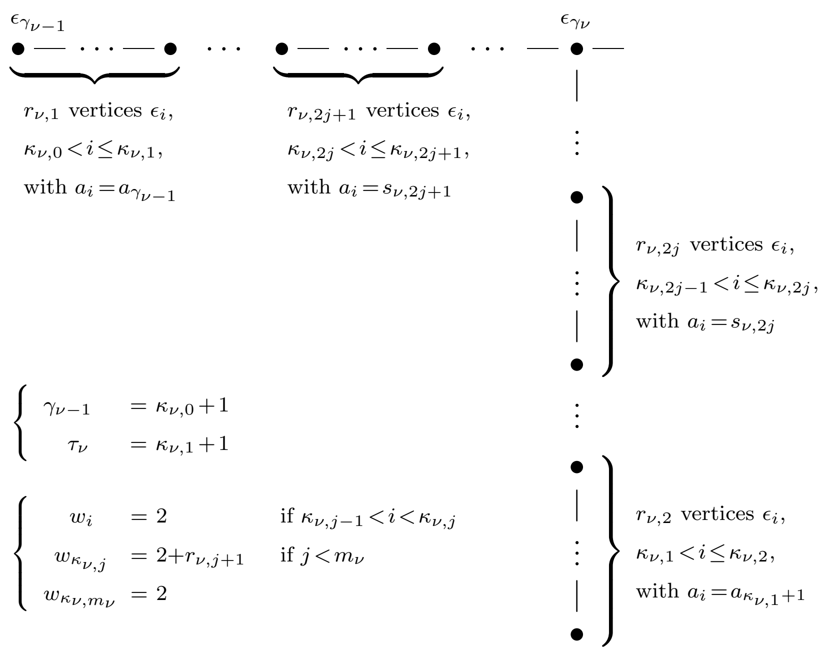
Note that $a_i = s_{\nu,\mu}$ for $\kappa_{\nu,\mu-1} < i \leq \kappa_{\nu,\mu}$, and $a_{\kappa_{\nu,m_\nu}} = a_{\gamma_\nu}$.

Lemma 4.5. *Let $\epsilon_1, \ldots, \epsilon_n$ be the vertices of the dual graph of $\mathfrak{a}$. Assume that $\kappa_{\nu,\mu-1} < i < \kappa_{\nu,\mu}$ or $i = \kappa_{\nu,m_\nu}$ for $\nu \in \{1, \ldots, g+1\}$ and $\mu \in \{1, \ldots, m_\nu\}$. If $j > i$, then ϵ_j is adjacent to ϵ_i, if and only if $j = i + 1$. Moreover, $\epsilon_{\kappa_{\nu,\mu}}$ is adjacent to $\epsilon_{\kappa_{\nu,\mu+1}+1}$ for $\mu \in \{1, \ldots, m_\nu - 1\}$*

Proof. If $\kappa_{\nu,\mu-1} < i < \kappa_{\nu,\mu}$ or $i = \kappa_{\nu,m_\nu}$, then we have $a_i = a_{i+1}$. It follows from Proposition 2.1 that $j \succ i$ if and only if $j = i+1$, which is to say that i has no satellites. By Proposition 4.1 this takes place exactly when ϵ_i is adjacent ϵ_{i+1}.

Assume that $\mu \in \{1, \ldots, m_\nu - 1\}$. An application of Equation (13) then shows that and $i = \kappa_{\nu,\mu}$. As we observed above, $a_{\kappa_{\nu,\mu}} = a_{\kappa_{\nu,\mu}+1} + \cdots + a_{\kappa_{\nu,\mu+1}+1}$. So $\kappa_{\nu,\mu+1} + 1 = \max\{j \mid j \succ \kappa_{\nu,\mu}\}$ by Proposition 2.1, and the claim follows from Proposition 4.1. $\square$

Using Lemma 4.5 together with Proposition 4.3 and Remark 4.2, we are able to construct the dual graph of $\mathfrak{a}$ from the point basis. The figure below describes a fragment of the dual graph of $\mathfrak{a}$. It illustrates the organization of the vertices $\epsilon_{\gamma_{\nu-1}}, \ldots, \epsilon_{\gamma_\nu}$ and the corresponding multiplicities $a_{\gamma_{\nu-1}}, \ldots, a_{\gamma_\nu}$ in the point basis of $\mathfrak{a}$.



Observe that the vertices ϵ_i for $\kappa_{\nu,m_\nu-1} < i \leq \kappa_{\nu,m_\nu}$ with the multiplicity $a_i = a_{\gamma_\nu}$ lie at the horizontal or vertical branch, depending on whether m_ν is odd or even, respectively. Note also that for $\nu = 1, \ldots, g$ the vertex ϵ_{γ_ν} belongs to the next segment of the dual graph.

Example 4.6. Let $\mathfrak{a}$ be a simple complete ideal in a two-dimensional regular local ring α having the resolution (2). and let $n = 8$ in this resolution.

Suppose that $i \succ j$ for $i, j \in \{1, \ldots, 8\}$ with $i > j$, if and only if $i = j + 1$ or $(i, j) \in \{(3, 1), (6, 4), (7, 4)\}$. The proximity matrix of $\mathfrak{a}$ is then

$$P = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & -1 & 1 & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 \end{bmatrix},$$

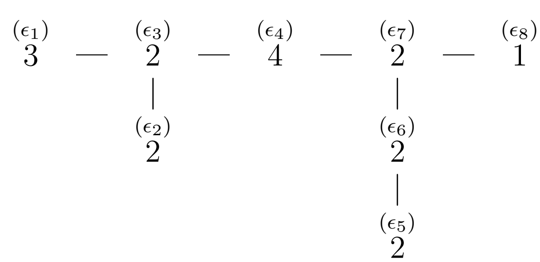
and the inverse of P is

$$P^{-1} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 4 & 2 & 2 & 2 & 1 & 1 & \cdot & \cdot \\ 6 & 3 & 3 & 3 & 1 & 1 & 1 & \cdot \\ 6 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

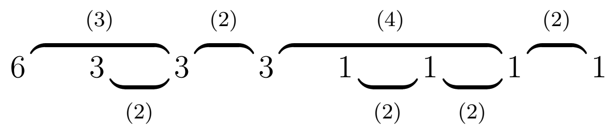
Thus the point basis of $\mathfrak{a}$ is $I = (a_1, \ldots, a_8) = (6, 3, 3, 3, 1, 1, 1, 1)$. The dual graph of $\mathfrak{a}$ is presented in the matrix

$$P^{\mathrm{T}}P = \begin{bmatrix} 3 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & 2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & 2 & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & 4 & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & -1 & 2 & -1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 \end{bmatrix},$$

We may draw the dual graph as follows:



The stars of the dual graph are ϵ_3 and ϵ_7, while the ends are $\epsilon_1, \epsilon_2, \epsilon_5$ and ϵ_8. Thus $g = 2$, $\Gamma = \{3, 7\}$ and $\{\tau_0, \ldots, \tau_3\} = \{1, 2, 5, 8\}$. Now $\beta'_1 = 9/6$, $\beta'_2 = 7/3$ and $\beta'_3 = 2$. On the other hand, starting from $(\beta'_1, \ldots, \beta'_{g+1}) = (3/2, 7/3, 2)$ we get first $(a_{\gamma_0}, a_{\gamma_1}, a_{\gamma_2}) = (6, 3, 1)$ and then $(a_{\gamma_0}\beta'_1, \ldots, a_{\gamma_2}\beta'_3) = (9, 7, 2)$. By using the Euclidian algorithm as above we get $(a_1, \ldots, a_{\gamma_1}) = (6, 3, 3)$, $(a_{\gamma_1}, \ldots, a_{\gamma_2}) = (3, 3, 1, 1, 1)$ and $(a_{\gamma_2}, \ldots, a_n) = (1, 1)$, and so we may reconstruct the point basis, which yields the proximity matrix by Proposition 2.1. Note that the dual graph can be obtained directly from the point basis by the following manner: From every entry $a_1, \ldots, a_{n-1}$ in the point basis, draw an arc extending to a_k such that $a_i = a_{i+1} + \cdots + a_k$, i.e.,



The entries and the arcs correspond the vertices and the edges of the dual graph. Moreover, the lengt $k - i + 1$ of the arc starting from a_i, indicated in parenthesis, is exactly the weight w_i, while $w_n = 1$.

5 On certain intersection products

Let $\mathfrak{a}$ be a simple complete ideal of finite colength in a regular local ring α with the base points $\alpha = \alpha_1 \subset \cdots \subset \alpha_n$. Let X_i denote the i:th row of the inverse $(x_{i,j})_{n\times n} := P^{-1}$ of the proximity matrix. For any rows X_i and X_j of $P^{-1} = (x_{i,j})_{n\times n}$, write $X_i \cdot X_j$ for the dot product, i.e.,

$$X_i \cdot X_j := x_{i,1}x_{j,1} + \cdots + x_{i,n}x_{j,n},$$

and for any $\nu \in \{0, \ldots, g\}$, write

$$[X_i \cdot X_j]_\nu := X_i^{(\gamma_\nu,\gamma_{\nu+1}]} \cdot X_j^{(\gamma_\nu,\gamma_{\nu+1}]},$$

so that

$$X_i \cdot X_j = x_{i,1}x_{j,1} + [X_i \cdot X_j]_0 + \cdots + [X_i \cdot X_j]_g.$$

Assume that the integers $\kappa_{\nu,\mu}$ attached to the ideal $\mathfrak{a}$ are as in Equation (16). Then define

$$U := \{i \mid \kappa_{\nu,\mu-1} < i \leq \kappa_{\nu,\mu} \text{ for any } \nu \text{ and } \mu \notin 2\mathbb{N}, \text{ or } i = n\}. \quad (17)$$

Remark 5.1. Note that since $\gamma_\nu = \kappa_{\nu+1,0} + 1$ for $\nu \in \{0, \ldots, g+1\}$, we have

$$\gamma_\nu \in U \text{ for } \nu \in \{0, \ldots, g+1\}.$$

Obviously, this gives $\tau_0, \tau_{g+1} \in U$. If $\nu \in \{1, \dots, g\}$, then $\tau_\nu = \kappa_{\nu,1} + 1$ by Proposition 3.5, which implies that

$$\tau_\nu \notin U \text{ for } \nu \in \{1, \dots, g\}.$$

Proposition 5.2. *Let $1 \le i \le j \le n$ and let $\nu \in \{0, \dots, g\}$. Then*

$$[X_i \cdot X_j]_\nu = \min\{x_{i,\gamma_\nu}\rho_{j,\gamma_\nu}, x_{j,\gamma_\nu}\rho_{i,\gamma_\nu}\} = \begin{cases} x_{j,\gamma_\nu}\rho_{i,\gamma_\nu}, & \text{if } i \in U; \\ x_{i,\gamma_\nu}\rho_{j,\gamma_\nu}, & \text{if } i \notin U. \end{cases}$$

Moreover, if $\gamma_{\nu+1} \le i, j$, then $[X_i \cdot X_j]_\nu = x_{i,\gamma_\nu}\rho_{j,\gamma_\nu} = x_{j,\gamma_\nu}\rho_{i,\gamma_\nu}$.

Proof. Let us show that it is enough to consider the case $\gamma_\nu \le i \le j \le \gamma_{\nu+1}$. Indeed, suppose first that $i < \gamma_\nu$. Then $x_{i,k} = 0$ for $\gamma_\nu \le k \le n$. Subsequently, $x_{i,\gamma_\nu} = 0 = \rho_{i,\gamma_\nu}$, and

$$[X_i \cdot X_j]_\nu = X_i^{(\gamma_\nu,\gamma_{\nu+1}]} \cdot X_j^{(\gamma_\nu,\gamma_{\nu+1}]} = 0 = x_{i,\gamma_\nu}\rho_{j,\gamma_\nu} = x_{j,\gamma_\nu}\rho_{i,\gamma_\nu}.$$

So the claim is clear in this case, and we may assume $\gamma_\nu \le i$.

If $\gamma_{\nu+1} \le j$, then we obtain by using Proposition 3.12

$$[X_i \cdot X_j]_\nu = [X_i \cdot X_j^{\le \gamma_{\nu+1}}]_\nu = x_{j,\gamma_{\nu+1}}[X_i \cdot X_{\gamma_{\nu+1}}]_\nu.$$

It follows from Corollary 3.15 that

$$\begin{aligned} x_{i,\gamma_\nu}\rho_{j,\gamma_\nu} &= x_{j,\gamma_{\nu+1}}x_{i,\gamma_\nu}\rho_{\gamma_{\nu+1},\gamma_\nu} \\ x_{j,\gamma_\nu}\rho_{i,\gamma_\nu} &= x_{j,\gamma_{\nu+1}}x_{\gamma_{\nu+1},\gamma_\nu}\rho_{i,\gamma_\nu}, \end{aligned}$$

and so we are reduced to the case $j = \gamma_{\nu+1}$. Similarily, if $\gamma_{\nu+1} < i$ (so that $\gamma_{\nu+1} < j$), then

$$[X_i \cdot X_j]_\nu = [X_i^{\le \gamma_{\nu+1}} \cdot X_j^{\le \gamma_{\nu+1}}]_\nu = x_{i,\gamma_{\nu+1}}x_{j,\gamma_{\nu+1}}[X^2_{\gamma_{\nu+1}}]_\nu.$$

and

$$\begin{aligned} x_{j,\gamma_\nu}\rho_{i,\gamma_\nu} &= x_{i,\gamma_{\nu+1}}x_{j,\gamma_{\nu+1}}(x_{\gamma_{\nu+1},\gamma_\nu}\rho_{\gamma_{\nu+1},\gamma_\nu}) \\ x_{i,\gamma_\nu}\rho_{j,\gamma_\nu} &= x_{i,\gamma_{\nu+1}}x_{j,\gamma_{\nu+1}}(x_{\gamma_{\nu+1},\gamma_\nu}\rho_{\gamma_{\nu+1},\gamma_\nu}). \end{aligned}$$

Then we are reduced to the case $i = j = \gamma_{\nu+1}$.

Let $(a_1, \dots, a_n)$ denote the point basis of $\mathfrak{a}$ and let $\gamma_\nu \le i \le j \le \gamma_{\nu+1}$. It follows from Proposition 3.9 that if $k < u$, then $a_{k-1} > a_k$ exactly when $x_{u,k-1} > x_{u,k}$. We may rewrite the sequence $x_{u,\gamma_\nu}, \dots, x_{u,i}$ of multiplicities for any $u \ge i$ as

$$\overbrace{s_1^u, \dots, s_1^u}^{r_1 \text{ times}}, \overbrace{s_2^u, \dots, s_2^u}^{r_2 \text{ times}}, \dots, \overbrace{s_\lambda^u, \dots, s_\lambda^u}^{r_\lambda \text{ times}}, \overbrace{s_{\lambda+1}^u, \dots, s_{\lambda+1}^u}^{r' \text{ times}}, \tag{18}$$

where $s^u_1 > \cdots > s^u_\lambda \geq s^u_{\lambda+1}$ and $s^u_\lambda > s^u_{\lambda+1}$ whenever $u > i$. Note that the equality $s^i_\lambda = s^i_{\lambda+1}$ can take place if and only if $r' = 1$. As in Equation (13), we get for any $1 \leq \mu \leq \lambda$

$$r_\mu s^u_\mu = s^u_{\mu-1} - s^u_{\mu+1}, \tag{19}$$

where

$$s^u_0 = x_{u,\gamma_\nu} + \cdots + x_{u,\tau^{\mathfrak{p}_u}_{\nu+1}}.$$

Because $\gamma_\nu \leq i \leq u$, Proposition 3.9 yields $\tau^{\mathfrak{p}_u}_{\nu+1} = \min\{u, \tau_{\nu+1}\}$ so that

$$s^u_0 = x_{u,\gamma_\nu} + \cdots + x_{u,\tau_{\nu+1}} = x_{u,\gamma_\nu} + \rho_{u,\gamma_\nu}.$$

It follows from Proposition 2.1 that for $\lambda > 0$ and $u \geq i$

$$s^i_\lambda = r' s^i_{\lambda+1}, \text{ while } s^u_\lambda \geq r' s^u_{\lambda+1} \tag{20}$$

This holds true also in the case $\lambda = 0$. Indeed, if $\lambda = 0$, then this is a direct consequence of the definition of s^u_0.

Grouping the terms in $[X_i \cdot X_j]_\nu$ by using Equation (18) gives now

$$\begin{aligned} [X_i \cdot X_j]_\nu &= \sum_{k=\gamma_\nu}^{i} x_{i,k} x_{j,k} - x_{i,\gamma_\nu} x_{j,\gamma_\nu} \\ &= \sum_{\mu=1}^{\lambda} r_\mu s^i_\mu s^j_\mu + r' s^i_{\lambda+1} s^j_{\lambda+1} - s^i_1 s^j_1. \end{aligned} \tag{21}$$

Assume that $i \in U$ in which case λ is even. Using Equation (19) yields

$$\begin{aligned} \sum_{\mu=1}^{\lambda} r_\mu s^i_\mu s^j_\mu &= (s^i_0 - s^i_2) s^j_1 + s^i_2 (s^j_1 - s^j_3) + \cdots \\ &\qquad + (s^i_{\lambda-2} - s^i_\lambda) s^j_{\lambda-1} + s^i_\lambda (s^j_{\lambda-1} - s^j_{\lambda+1}) \\ &= s^i_0 s^j_1 - s^i_\lambda s^j_{\lambda+1} \end{aligned}$$

and

$$\begin{aligned} \sum_{\mu=1}^{\lambda} r_\mu s^i_\mu s^j_\mu &= (s^j_0 - s^j_2) s^i_1 + s^j_2 (s^i_1 - s^i_3) + \cdots \\ &\qquad + (s^j_{\lambda-2} - s^j_\lambda) s^i_{\lambda-1} + s^j_\lambda (s^i_{\lambda-1} - s^i_{\lambda+1}) \\ &= s^j_0 s^i_1 - s^j_\lambda s^i_{\lambda+1}. \end{aligned}$$

By Equation (21) we then obtain

$$\begin{aligned}[X_i \cdot X_j]_\nu &= s_0^i s_1^j - s_\lambda^i s_{\lambda+1}^j + r' s_{\lambda+1}^i s_{\lambda+1}^j - s_1^i s_1^j \\ &= s_0^j s_1^i - s_\lambda^j s_{\lambda+1}^i + r' s_{\lambda+1}^i s_{\lambda+1}^j - s_1^i s_1^j.\end{aligned}$$

Furthermore, by Equation (20)

$$r' s_{\lambda+1}^i s_{\lambda+1}^j - s_\lambda^i s_{\lambda+1}^j = 0 \text{ and } r' s_{\lambda+1}^i s_{\lambda+1}^j - s_{\lambda+1}^i s_\lambda^j \leq 0.$$

Therefore

$$[X_i \cdot X_j]_\nu = (s_0^i - s_1^i) s_1^j \leq (s_0^j - s_1^j) s_1^i. \tag{22}$$

Similarily, if $i \notin U$, then λ is odd and

$$\begin{aligned}\sum_{\mu=1}^{\lambda} r_\mu s_\mu^i s_\mu^j &= (s_0^i - s_2^i) s_1^j + s_2^i (s_1^j - s_3^j) + \cdots \\ &\quad + s_{\lambda-1}^i (s_{\lambda-2}^j - s_\lambda^j) + (s_{\lambda-1}^i - s_{\lambda+1}^i) s_\lambda^j \\ &= s_0^i s_1^j - s_{\lambda+1}^i s_\lambda^j.\end{aligned}$$

and

$$\begin{aligned}\sum_{\mu=1}^{\lambda} r_\mu s_\mu^i s_\mu^j &= (s_0^j - s_2^j) s_1^i + s_2^j (s_1^i - s_3^i) + \cdots \\ &\quad + s_{\lambda-1}^j (s_{\lambda-2}^i - s_\lambda^i) + (s_{\lambda-1}^j - s_{\lambda+1}^j) s_\lambda^i \\ &= s_0^j s_1^i - s_{\lambda+1}^j s_\lambda^i.\end{aligned}$$

As above by using Equations (20) and (21) this yields that

$$[X_i \cdot X_j]_\nu = (s_0^j - s_1^j) s_1^i \leq s_1^i (s_0^i - s_1^j).$$

Because $(s_0^u, s_1^u) = (x_{u,\gamma_\nu} + \rho_{u,\gamma_\nu}, x_{u,\gamma_\nu})$ for $u = i, j$, this together with Equation (22) gives the claim. □

Corollary 5.3. *Assume $1 \leq i \leq j \leq n$. Set $\eta := \gamma_\nu$ and $\gamma := \gamma_{\nu+1}$, where ν is such that $\eta < i \leq \gamma$ whenever $i > 1$, and $\nu = 0$ if $i = 1$. With the notation above,*

$$X_i \cdot X_j = x_{i,\eta} x_{j,\eta} X_\eta^2 + [X_i \cdot X_j]_\nu.$$

Especially, if $i = \gamma \leq j$, then

$$X_\gamma \cdot X_j = x_{\gamma,\eta}(x_{j,\eta} X_\eta^2 + \rho_{j,\eta}) = x_{j,\eta}(x_{\gamma,\eta} X_\eta^2 + \rho_\nu).$$

Proof. Obviously, $X_i \cdot X_j = X_i \cdot X_j^{\leq\eta} + X_i \cdot X_j^{>\eta}$. Because $i \leq \gamma$, we get $X_i \cdot X_j^{>\eta} = [X_i \cdot X_j]_\nu$, and by Proposition 3.12

$$X_i \cdot X_j^{\leq\eta} = X_i^{\leq\eta} \cdot X_j^{\leq\eta} = x_{i,\eta} x_{j,\eta} X_\eta^2.$$

Thus the first claim is clear, and the second claim follows from Proposition 5.2. ☐

Corollary 5.4. *Suppose that* $\eta := \gamma_\nu \leq j \leq k \leq \gamma_{\nu+1} =: \gamma$ *for some* $\nu \in \{0, \dots, g\}$. *Set*

$$\sigma_1(j,k) := \frac{x_{j,\eta}}{x_{k,\eta}} \text{ and } \sigma_2(j,k) := \frac{x_{j,\eta} X_\eta^2 + \rho_{j,\eta}}{x_{k,\eta} X_\eta^2 + \rho_{k,\eta}}.$$

For any $i \in \{1, \dots, n\}$ *the following equalities hold:*

i) If $i \leq j$, *then*

$$\frac{X_i \cdot X_j}{X_i \cdot X_k} = \begin{cases} \sigma_1(j,k) & \text{if } i \in U \text{ or } i \leq \eta; \\ \sigma_2(j,k) & \text{if } i \notin U \text{ and } i > \eta. \end{cases}$$

ii) If $j < i$, *then*

$$\frac{X_i \cdot X_j}{X_i \cdot X_k} = \begin{cases} \sigma_1(j,k) & \text{if } j \notin U, \text{ and } k \geq i \in U \text{ or } i > k \notin U; \\ \sigma_2(j,i)\sigma_1(i,k) & \text{if } j \in U, \text{ and } k \geq i \in U \text{ or } i > k \notin U; \\ \sigma_1(j,i)\sigma_2(i,k) & \text{if } j \notin U, \text{ and } k \geq i \notin U \text{ or } i > k \in U; \\ \sigma_2(j,k) & \text{if } j \in U, \text{ and } k \geq i \notin U \text{ or } i > k \in U. \end{cases}$$

iii) If $k' + 1 > k$ *is free, then* $X_i \cdot X_j / X_i \cdot X_k$ *is constant for* $i \geq k'$

iv) For every $1 \leq i \leq n$

$$\sigma_{3-\upsilon}(j,k) \geq \frac{X_i \cdot X_j}{X_i \cdot X_k} \geq \sigma_\upsilon(j,k) = \frac{X_k \cdot X_j}{X_k^2},$$

where $\upsilon = 1$ *for* $j \notin U$ *and* $\upsilon = 2$ *for* $j \in U$.

Proof. Before embarking the proof we first make two observations. Assume that $i \leq \eta$. As $\eta \leq j \leq k$ we then get by using Proposition 3.12 that

$$\frac{X_i \cdot X_j}{X_i \cdot X_k} = \frac{X_i \cdot X_j^{\leq\eta}}{X_i \cdot X_k^{\leq\eta}} = \frac{x_{j,\eta} X_i \cdot X_\eta}{x_{k,\eta} X_i \cdot X_\eta} = \frac{x_{j,\eta}}{x_{k,\eta}} = \sigma_1(j,k). \tag{23}$$

Suppose then that $i \geq \eta$. By using Corollary 5.3 we get

$$\frac{X_i \cdot X_j}{X_i \cdot X_k} = \frac{x_{i,\eta} x_{j,\eta} X_\eta^2 + [X_i \cdot X_j]_\nu}{x_{i,\eta} x_{k,\eta} X_\eta^2 + [X_i \cdot X_k]_\nu}. \tag{24}$$

(i) Proposition 5.2 implies that if $j \geq i \in U$, then $[X_i \cdot X_j]_\nu = x_{j,\eta}\rho_{i,\eta}$ and $[X_i \cdot X_k]_\nu = x_{k,\eta}\rho_{i,\eta}$, Furthermore, if $j \geq i \notin U$, then $[X_i \cdot X_j]_\nu = x_{i,\eta}\rho_{j,\eta}$ and $[X_i \cdot X_k]_\nu = x_{i,\eta}\rho_{k,\eta}$. This together with Equations (23) and (24) proves (i).

(ii) We have $i \geq \eta$, and again, we make use of Equation (24). Now $i > j$, and thus by Proposition 5.2 $[X_i \cdot X_j]_\nu$ is either $x_{j,\eta}\rho_{i,\eta}$ or $x_{i,\eta}\rho_{j,\eta}$, depending on whether $j \notin U$ or $j \in U$. Proposition 5.2 shows also that $[X_i \cdot X_k]_\nu = x_{k,\eta}\rho_{i,\eta}$ when $k \geq i \in U$ or $i > k \notin U$, and $[X_i \cdot X_k]_\nu = x_{i,\eta}\rho_{k,\eta}$ when $k \geq i \notin U$ or $i > k \in U$. Putting all this together we obtain (ii).

(iii) If $k' + 1 > k$ and $k' + 1$ is free, then we get by Proposition 3.12 for every $i \geq k'$

$$\frac{X_i \cdot X_j}{X_i \cdot X_k} = \frac{X_i \cdot X_j^{\leq k'}}{X_i \cdot X_k^{\leq k'}} = \frac{X_i^{\leq k'} \cdot X_j}{X_i^{\leq k'} \cdot X_k} = \frac{x_{i,k'} X_{k'} \cdot X_j}{x_{i,k'} X_{k'} \cdot X_k} = \frac{X_{k'} \cdot X_j}{X_{k'} \cdot X_k}.$$

(iv). We may assume $j < k$, since the case $j = k$ is trivial. Choosing $i = k$ in Equation (24) we see that

$$\frac{X_k \cdot X_j}{X_k^2} = \frac{x_{j,\eta} x_{k,\eta} X_\eta^2 + [X_k \cdot X_j]_\nu}{x_{k,\eta} x_{k,\eta} X_\eta^2 + x_{k,\eta}\rho_{k,\eta}},$$

where $[X_k \cdot X_j]_\nu = \min\{x_{k,\eta}\rho_{j,\eta}, x_{j,\eta}\rho_{k,\eta}\}$ by Proposition 5.2. Thereby

$$\frac{X_k \cdot X_j}{X_k^2} = \sigma_\upsilon(j,k) \leq \sigma_{3-\upsilon}(j,k), \tag{25}$$

where $\upsilon = 1$ if $j \notin U$ and $\upsilon = 2$ if $j \in U$. Clearly, the claim holds, if i is such that

$$\frac{X_i \cdot X_j}{X_i \cdot X_k} \in \{\sigma_1(j,k), \sigma_2(j,k)\},$$

Let us check the remaining cases. Suppose first that $i > j \in U$, and $k \geq i \in U$ or $i > k \notin U$. By Proposition 5.2 we know that $[X_i \cdot X_j]_\nu = x_{i,\eta}\rho_{j,\eta} \leq x_{j,\eta}\rho_{i,\eta}$ and $[X_i \cdot X_k]_\nu = x_{k,\eta}\rho_{i,\eta} \leq x_{i,\eta}\rho_{k,\eta}$. Then

$$\frac{x_{i,\eta} x_{j,\eta} X_\eta^2 + x_{i,\eta}\rho_{j,\eta}}{x_{i,\eta} x_{k,\eta} X_\eta^2 + x_{i,\eta}\rho_{k,\eta}} \leq \frac{x_{i,\eta} x_{j,\eta} X_\eta^2 + x_{i,\eta}\rho_{j,\eta}}{x_{i,\eta} x_{k,\eta} X_\eta^2 + x_{k,\eta}\rho_{i,\eta}} \leq \frac{x_{i,\eta} x_{j,\eta} X_\eta^2 + x_{j,\eta}\rho_{i,\eta}}{x_{i,\eta} x_{k,\eta} X_\eta^2 + x_{k,\eta}\rho_{i,\eta}},$$

in other words,

$$\sigma_2(j,k)\sigma_1(i,i) \leq \sigma_2(j,i)\sigma_1(i,k) \leq \sigma_2(i,i)\sigma_1(j,k).$$

Note that $\sigma_\upsilon(i,i) = 1$ for $\upsilon \in \{1,2\}$ and $X_i \cdot X_j / X_i \cdot X_k = \sigma_2(j,i)\sigma_1(i,k)$ by (ii). This together with Equation (25) gives the claim.

Suppose next that $i > j \notin U$, and $k \geq i \notin U$ or $i > k \in U$. According to Proposition 5.2 we have $[X_i \cdot X_j]_\nu = x_{j,\eta}\rho_{i,\eta} \leq x_{i,\eta}\rho_{j,\eta}$ and $[X_i \cdot X_k]_\nu = x_{i,\eta}\rho_{k,\eta} \leq x_{k,\eta}\rho_{i,\eta}$. This gives

$$\frac{x_{i,\eta}x_{j,\eta}X_\eta^2 + x_{j,\eta}\rho_{i,\eta}}{x_{i,\eta}x_{k,\eta}X_\eta^2 + x_{k,\eta}\rho_{i,\eta}} \leq \frac{x_{i,\eta}x_{j,\eta}X_\eta^2 + x_{j,\eta}\rho_{i,\eta}}{x_{i,\eta}x_{k,\eta}X_\eta^2 + x_{i,\eta}\rho_{k,\eta}} \leq \frac{x_{i,\eta}x_{j,\eta}X_\eta^2 + x_{i,\eta}\rho_{j,\eta}}{x_{i,\eta}x_{k,\eta}X_\eta^2 + x_{i,\eta}\rho_{k,\eta}},$$

which says

$$\sigma_1(j,k)\sigma_2(i,i) \leq \sigma_1(j,i)\sigma_2(i,k) \leq \sigma_1(i,i)\sigma_2(j,k).$$

Again, (ii) shows that $X_i \cdot X_j / X_i \cdot X_k = \sigma_1(j,i)\sigma_2(i,k)$, which together with Equation (25) gives the claim. Thus the proof is complete. □

Proposition 5.5. *Let $0 \neq Z = r_1X_1 + \cdots + r_nX_n$, where $r_1, \ldots, r_n \in \mathbb{N}$. We then have for any $j \leq k$.*

$$\frac{Z \cdot X_j}{Z \cdot X_k} \geq \frac{X_j \cdot X_k}{X_k^2}.$$

Proof. Assume first that $Z = X_i$ is a row of P^{-1}. If $i > k$, then

$$\frac{Z \cdot X_j}{Z \cdot X_k} = \frac{X_i^{\leq k} \cdot X_j}{X_i^{\leq k} \cdot X_k}.$$

Proposition 3.12 yields $X_i^{\leq k} = x_{i,k}X_k + \varrho X_h$, where $\varrho = x_{i,h} - (x_{i,h+1} + \cdots + x_{i,k})$ in the case $k+1$ is a satellite to h and otherwise $\varrho = 0$. Let us recall the following elementary fact: If $a,b,c,d,e,f \in \mathbb{N}$ such that $bdf \neq 0$, then

$$\frac{a}{b} \sim \frac{e}{f}, \frac{c}{d} \sim \frac{e}{f} \Rightarrow \frac{a+c}{b+d} \sim \frac{e}{f}, \tag{26}$$

where $\sim$ is one of the relations $=$, $<$ or $>$. Applying this gives

$$\frac{X_h \cdot X_j}{X_h \cdot X_k} \geq \frac{X_k \cdot X_j}{X_k^2} \Longrightarrow \frac{Z \cdot X_j}{Z \cdot X_k} = \frac{x_{i,k}X_k \cdot X_j + \varrho X_h \cdot X_j}{x_{i,k}X_k^2 + \varrho X_h \cdot X_k} \geq \frac{X_k \cdot X_j}{X_k^2}.$$

This shows that it is enough to consider the case $i \leq k$. Moreover, we may suppose $i \leq j$, because

$$\frac{X_i \cdot X_j}{X_i \cdot X_k} \geq \frac{X_k \cdot X_j}{X_k^2} \Longleftrightarrow \frac{X_i \cdot X_j}{X_k \cdot X_j} = \frac{X_i \cdot X_k}{X_k^2},$$

and so, if $j < i \leq k$, then we may simply switch the roles of X_i and X_j.

By the above, we may restrict ourselves to the case $i \leq j \leq k$. Let $\nu \in \{0, \dots, g\}$ be such that $\gamma_\nu \leq j \leq \gamma_{\nu+1}$ and write $\gamma := \gamma_{\nu+1}$. If $k \leq \gamma$, then the claim is clear by Corollary 5.4 (iv). Especially,

$$\frac{X_i \cdot X_j}{X_i \cdot X_\gamma} \geq \frac{X_\gamma \cdot X_j}{X_\gamma^2}. \tag{27}$$

If $k > \gamma$, then according to Proposition 3.12 we have

$$X_i \cdot X_k = X_i \cdot X_k^{\leq \gamma} = x_{k,\gamma} X_i \cdot X_\gamma \text{ and } X_k \cdot X_j = X_k^{\leq \gamma} \cdot X_j = x_{k,\gamma} X_\gamma \cdot X_j.$$

By Proposition 3.12 we know that

$$x_{k,\gamma}^2 X_\gamma^2 = (X_k^{\leq \gamma})^2 < X_k^2.$$

Then we get by using these and Equation (27)

$$\frac{Z \cdot X_j}{Z \cdot X_k} = \frac{X_i \cdot X_j}{x_{k,\gamma} X_i \cdot X_\gamma} \geq \frac{X_\gamma \cdot X_j}{x_{k,\gamma} X_\gamma^2} = \frac{x_{k,\gamma} X_\gamma \cdot X_j}{x_{k,\gamma}^2 X_\gamma^2} \geq \frac{X_k \cdot X_j}{X_k^2}.$$

Thereby the claim holds for any row Z of P^{-1}.

Suppose then that $Z = r_1 X_1 + \dots + r_n X_n$ for some $(r_1, \dots, r_n) \in \mathbb{N}^n \setminus \{0\}$. By the above

$$\frac{X_i \cdot X_j}{X_i \cdot X_k} \geq \frac{X_k \cdot X_j}{X_k^2}$$

for every $i = 1, \dots, n$. Applying Equation (26) we obtain

$$\frac{Z \cdot X_j}{Z \cdot X_k} \geq \frac{X_k \cdot X_j}{X_k^2}$$

as desired. □

Remark 5.6. Note that by Equation (8) we have $X_i \cdot X_j = -\hat{E}_i \cdot \hat{E}_j$. Moreover, by setting $v_i := \mathrm{ord}_{\alpha_i}$ we may write

$$\hat{E}_i = \sum_{k=1}^{n} v_k(\mathfrak{p}_i) E_k = \mathbf{V}(\mathfrak{p}_i) E,$$

where $\mathfrak{p}_i$ is the simple complete ideal in α containing $\mathfrak{a}$ and having the point basis X_i (cf. Remark 2.3). Because $E_k \cdot \hat{E}_j = -\delta_{k,j}$, we obtain

$$X_i \cdot X_j = -\hat{E}_i \cdot \hat{E}_j = v_j(\mathfrak{p}_i) = v_i(\mathfrak{p}_j),$$

where the last equality (known as *reciprocity*, see [11, p. 247, Proposition 21.4]) is now obvious. Therefore Proposition 5.5 says especially for any $j < i$ and for any k that

$$\frac{v_k(\mathfrak{p}_j)}{v_k(\mathfrak{p}_i)} \geq \frac{v_i(\mathfrak{p}_j)}{v_i(\mathfrak{p}_i)}.$$

In the sequel we write for any vector $X = (x_1, \ldots, x_n)$

$$\Sigma X := x_1 + \cdots + x_n. \tag{28}$$

Remark 5.7. Observe that if X_i is a row of the inverse of the proximity matrix of $\mathfrak{a}$, then $\Sigma X_i = -K \cdot \hat{E}_i = -K \cdot X_i E^*$, where $K = E_1^* + \cdots + E_n^*$ is the canonical divisor (see page 29).

Proposition 5.8. *Let X_i be a row of the inverse of the proximity matrix of $\mathfrak{a}$. Let μ be such that $\gamma_\mu < i \leq \gamma_{\mu+1}$ unless $i = 1$ in which case we set $\mu := 0$. Then*

$$\Sigma X_i^{>\gamma_k} + 1 - x_{i,\gamma_k} = \rho_{i,\gamma_k} + \cdots + \rho_{i,\gamma_\mu}.$$

for every $k = 0, \ldots, \mu$

Proof. Clearly,

$$\Sigma X_i^{>\gamma_k} = \Sigma X_i^{(\gamma_k,\gamma_{k+1}]} + \cdots + \Sigma X_i^{(\gamma_g,\gamma_{g+1}]}.$$

If $\mu < \nu$, then $i \leq \gamma_\nu$, and we see that $\Sigma X_i^{(\gamma_\nu,\gamma_{\nu+1}]} = 0$. Thus

$$\Sigma X_i^{>\gamma_k} = \Sigma X_i^{(\gamma_k,\gamma_{k+1}]} + \cdots + \Sigma X_i^{(\gamma_\mu,\gamma_{\mu+1}]}.$$

We observe that the claim holds if

$$\Sigma X_i^{(\gamma_\nu,\gamma_{\nu+1}]} = \begin{cases} x_{i,\gamma_\nu} + \rho_{i,\gamma_\nu} - x_{i,\gamma_{\nu+1}} & \text{for } \nu < \mu; \\ x_{i,\gamma_\mu} + \rho_{i,\gamma_\mu} - 1 & \text{for } \nu = \mu. \end{cases}$$

It follows from Proposition 3.12 that $\Sigma X_i^{(\gamma_\nu,\gamma_{\nu+1}]} = x_{i,\gamma_{\nu+1}} \Sigma X_{\gamma_{\nu+1}}^{(\gamma_\nu,\gamma_{\nu+1}]}$, while Corollary 3.15 yields $x_{i,\gamma_\nu} + \rho_{i,\gamma_\nu} - x_{i,\gamma_{\nu+1}} = x_{i,\gamma_{\nu+1}}(x_{\gamma_{\nu+1},\gamma_\nu} + \rho_\nu - 1)$. Subsequently, it is enough to verify that

$$\Sigma X_i^{(\gamma_\mu,\gamma_{\mu+1}]} = x_{i,\gamma_\mu} + \rho_{i,\gamma_\mu} - 1$$

for every i (and then especially in the case $i = \gamma_{\mu+1}$).

Consider the transform $\mathfrak{b} := \mathfrak{p}_i^{\alpha_{\gamma_\mu}}$. Recall that the point basis of $\mathfrak{b}$ corresponds to $X_i^{[\gamma_\mu,i]}$. Moreover, it follows from Proposition 3.9 that there are no terminal satellites between γ_μ and $i \leq \gamma_{\mu+1}$, and therefore $\bar{\Gamma}_{\mathfrak{b}} = \{i\}$. By Proposition 3.16 we may reduce to the situation $i = n$ and $\bar{\Gamma} = \{n\}$.

As in Formula (12), the sequence of the multiplicities $a_1, \ldots, a_n$ is represented by some positive integers $m = m_1$, $r_j = r_{1,j}$ and $s_j = s_{1,j}$ for $1 \leq j \leq m$, where $a_1 = s_1 > \cdots > s_m = a_n$. By Equation (13) we have $r_j s_j = s_{j-1} - s_{j+1}$ for every $j = 1, \ldots, m$. Therefore

$$a_1 + \cdots + a_n = r_1 s_1 + \cdots + r_m s_m + s_m = s_0 + s_1 - s_m - s_{m+1} + s_m,$$

where $s_0 = a_1 + \cdots + a_{\tau_1}$ and $s_{m+1} = s_m = 1$. Then

$$a_2 + \cdots + a_n = s_0 - s_m = a_1 + \rho_{n,\gamma_0} - 1.$$

□

6 Multiplier ideals and jumping numbers

Let $\mathfrak{a}$ be an ideal in a two-dimensional regular local ring α. Let $\mathcal{X} \to \mathrm{Spec}(\alpha)$ be a *log-resolution* of $\mathfrak{a}$, i.e., a projective birational morphism $\pi : \mathcal{X} \to \mathrm{Spec}(\alpha)$ such that $\mathcal{X}$ regular and $\mathfrak{a}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(-D)$ for an effective Cartier divisor D on $\mathcal{X}$ with the property that $D + \mathrm{Exc}(\pi)$ has simple normal crossing support. Here $\mathrm{Exc}(\pi)$ denotes the sum of the exceptional divisors of π. Note that (2) is always a log-resolution. Recall that the relative canonical sheaf $\omega_{\mathcal{X}}$ can be defined as the dual of the relative Jacobian sheaf $\mathcal{J}_{\mathcal{X}}$ (cf. [12, p. 203, (2.3)]). The *canonical divisor* $K := K_{\mathcal{X}}$ of $\mathcal{X}$ is the unique exceptional divisor on $\mathcal{X}$ for which $\mathcal{O}_{\mathcal{X}}(K) = \omega_{\mathcal{X}}$.

Definition 6.1. *For a non-negative rational number c, the multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ is defined to be the ideal*

$$\mathcal{J}(\mathfrak{a}^c) := \Gamma\left(\mathcal{X}, \mathcal{O}_{\mathcal{X}}\left(K - \lfloor cD \rfloor\right)\right) \subset \alpha,$$

where $D \in \Lambda$ is the effective divisor satisfying $\mathfrak{a}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(-D)$ and $\lfloor cD \rfloor$ denotes the integer part of cD.

Lemma 6.2. *Let $\mathfrak{a}$ be a simple complete ideal of finite colength in a two-dimensional regular local ring α. Then the base points of the multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ are among the base points of $\mathfrak{a}$ for every non-negative rational exponent c.*

Proof. Let $\mathcal{X} \to \mathrm{Spec}(\alpha)$ be the resolution of $\mathfrak{a}$ as in (2), and let D be the antinef divisor satisfying $\mathfrak{a} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-D))$. Then, by (10)

$$\mathcal{J}(\mathfrak{a}^c) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-(\lfloor cD \rfloor - K))) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-(\lfloor cD \rfloor - K)^{\sim})).$$

Since the antinef closure is antinef, we observe by Proposition 2.2 that $\mathcal{J}(\mathfrak{a}^c)$ generates an invertible $\mathcal{O}_{\mathcal{X}}$-ideal. □

Definition 6.3. *Let $\mathfrak{a}$ be an ideal in a two-dimensional regular local ring α. By [6, Lemma 1.3] there is an increasing discrete sequence $0 = \xi_0 < \xi_1 < \xi_2 < \cdots$ of rational numbers ξ_i characterized by the properties that $\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(\mathfrak{a}^{\xi_i})$ for $c \in [\xi_i, \xi_{i+1})$, while $\mathcal{J}(\mathfrak{a}^{\xi_{i+1}}) \subsetneq \mathcal{J}(\mathfrak{a}^{\xi_i})$ for every i. The numbers $\xi_1, \xi_2, \ldots,$ are called the jumping numbers of $\mathfrak{a}$. We set*

$$\mathcal{H}_{\mathfrak{a}} = \{\xi_i \mid i = 1, 2, \ldots\}.$$

Remark 6.4. For practical reasons we don't consider 0 as a jumping number in contrary to [6, Definition 1.4]. Clearly, this is no restriction. Note that if $\mathfrak{a} = \alpha$, then $\mathcal{J}(\mathfrak{a}^c) = \alpha$ for every c, which means that the set of the jumping numbers is empty.

Definition 6.5. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of finite colength in a two-dimensional regular local ring α. We define the log-canonical threshold of $\mathfrak{a}$ with respect to $\mathfrak{b}$ to be*

$$c_{\mathfrak{b}} = c_{\mathfrak{b}}^{\mathfrak{a}} := \inf\{c \in \mathbb{Q}_{>0} \mid \mathcal{J}(\mathfrak{a}^c) \not\supseteq \mathfrak{b}\}.$$

Note that if $\mathfrak{b} = \alpha$, then $c_{\mathfrak{b}}$ is the usual log-canonical threshold.

Remark 6.6. By [10, Theorem 11.1.1] $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{c-1})$, when $c \geq 2$. If $\mathfrak{a}$ is proper, then we may find $c \gg 0$ such that $\mathcal{J}(\mathfrak{a}^c) \not\supseteq \mathfrak{b}_R$, and so

$$\{c \in \mathbb{Q}_{>0} \mid \mathcal{J}(\mathfrak{a}^c) \not\supseteq \mathfrak{b}\} \neq \emptyset.$$

It follows from Definition 6.3 that $c_{\mathfrak{b}} = \xi_i \in \mathbb{Q}$ for some $i = 1, 2 \ldots,$ provided that $\mathfrak{a}$ is proper. If $\mathfrak{a} = \alpha$, then the above set is empty and $c_{\mathfrak{b}} = \infty$ for any $\mathfrak{b}$.

Let I denote the point basis of $\mathfrak{a}$, and let P^{-1} be the inverse of the proximity matrix of $\mathfrak{a}$ with the rows $X_1, \ldots, X_n$. For an arbitrary vector $R = (r_1, \ldots, r_n) \in \mathbb{N}^n$, we set

$$\mathfrak{b}_R := \prod_{i=1}^{n} \mathfrak{p}_i^{r_i},$$

where $\mathfrak{p}_i$ is the simple complete v-ideal containing $\mathfrak{a}$ and having the point basis X_i. We write

$$c_R = c_R^{\mathfrak{a}} := c_{\mathfrak{b}_R}^{\mathfrak{a}}.$$

Proposition 6.7. *Let α be a two-dimensional regular local ring and let $\mathfrak{a} \subset \alpha$ be a simple complete ideal of finite colength. Let $\mathcal{H}_{\mathfrak{a}}$ denote the set of all jumping numbers of the ideal $\mathfrak{a}$. Then*

$$\mathcal{H}_{\mathfrak{a}} = \{c_R \in \mathbb{Q}_{>0} \mid R \in \mathbb{N}^n\}.$$

Proof. We may assume $\mathfrak{a} \neq \alpha$. Take $R \in \mathbb{N}^n$, and let $\mathfrak{b}_R$ be as above. As we observed in Remark 6.6, $c_R = \xi_i$ for some positive integer i. Hence

$$\mathcal{H}_{\mathfrak{a}} \supset \{c_R \in \mathbb{Q}_{\geq 0} \mid R \in \mathbb{N}^n\}.$$

To show the opposite inclusion, take a jumping number ξ_i $(i > 0)$. By Definition 6.3 we get $\mathcal{J}(\mathfrak{a}^c) = \mathcal{J}(\mathfrak{a}^{\xi_{i-1}}) \nsubseteq \mathcal{J}(\mathfrak{a}^{\xi_i})$ for any $c \in [\xi_{i-1}, \xi_i)$. By Lemma 6.2 the base points of $\mathcal{J}(\mathfrak{a}^c)$ are among the base points of $\mathfrak{a}$. Then $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{b}_R$ for some $R \in \mathbb{N}^n$, which means that $\xi_i = c_R$, i.e.,

$$\mathcal{H}_{\mathfrak{a}} \subset \{c_R \in \mathbb{Q}_{\geq 0} \mid R \in \mathbb{N}^n\}.$$

□

Corollary 6.8. *Let $\mathfrak{a}$ be a simple complete ideal of finite colength in a two-dimensional regular local ring α and take $R \in \mathbb{N}^n$. Then the set $\{\mathfrak{b}_\nu \mid \nu \in \mathbb{N}^n, c_\nu = c_R\}$ has the largest element containing all the others. Furthermore, the set $\{\mathcal{J}(\mathfrak{a}^c) \mid c < c_R\}$ has the least element, and these two coincide.*

Proof. By Proposition 6.7 $c_R = \xi_i$ for some $i \in \mathbb{Z}_+$. Then by Definition 6.3 we obtain $\mathcal{J}(\mathfrak{a}^{\xi_{i-1}}) = \min\{\mathcal{J}(\mathfrak{a}^c) \mid c < c_R\}$. On the other hand, as we observed above $\mathcal{J}(\mathfrak{a}^{\xi_{i-1}}) = \mathfrak{b}_\mu$ for some $\mu \in \mathbb{N}^n$, and clearly $c_\mu = \xi_i = c_R$. If $\nu \in \mathbb{N}^n$ is such that $c_\nu = c_R$, then $\mathcal{J}(\mathfrak{a}^c) \supset \mathfrak{b}_\nu$ for every $c < c_R$. Especially $\mathcal{J}(\mathfrak{a}^{\xi_{i-1}}) = \mathfrak{b}_\mu \supset \mathfrak{b}_\nu$, and therefore $\mathfrak{b}_\mu = \max\{\mathfrak{b}_\nu \mid c_\nu = c_R\}$. □

7 Key lemmas

In order to determine the set of the jumping numbers, we make use of Proposition 6.7. For the main proofs we shall need a few technical results which are mostly gathered in this section. As above, $\mathfrak{a}$ is a simple complete ideal of finite colength in a two-dimensional regular local ring α having the resolution (2) and the base points $\alpha = \alpha_1 \subset \cdots \subset \alpha_n$. Let P denote the proximity matrix and $I = (a_1 \ldots, a_n)$ the point basis of $\mathfrak{a}$ and let $\mathfrak{b}_R$ and c_R be as in Definition 6.5. Recall that $\mathcal{H}_{\mathfrak{a}} = \{c_R \mid R \in \mathbb{N}^n\}$ according to Proposition 6.7.

Notation 7.1. *Let X and Y be row vectors of P^{-1}. For any $R \in \mathbb{N}^n$ set $\tilde{R} := RP^{-1}$ and write*

$$R_Y[X] := \frac{\tilde{R} \cdot X + \Sigma X + 1}{X \cdot Y},$$

where ΣX is as defined in Equation (28). In the following, we usually write $R[X] := R_I[X]$.

Proposition 7.2. *Let $X_1, \dots, X_n$ be the rows of P^{-1}. Then for any $R \in \mathbb{N}^n$*

$$c_R = \min \{ R[X_i] \mid i = 1, \dots, n \}.$$

Proof. Set $D = \hat{E}_n$ so that $\mathfrak{a}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(-D)$. We have by Definition 6.1 and Equation (10)

$$\mathcal{J}(\mathfrak{a}^c) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-(\lfloor cD \rfloor - K))) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-(\lfloor cD \rfloor - K)^{\sim})),$$

where K denotes the canonical divisor and $(\lfloor cD \rfloor - K)^{\sim}$ stands for the antinef closure of $\lfloor cD \rfloor - K$. By Proposition 2.2 $\mathcal{J}(\mathfrak{a}^c)\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(-(\lfloor cD \rfloor - K)^{\sim})$ is invertible. Also $\mathfrak{b}_R \subset \alpha$ is invertible. Since $\mathcal{J}(\mathfrak{a}^c)$ and $\mathfrak{b}_R$ are both complete, we have $\mathcal{J}(\mathfrak{a}^c) \supset \mathfrak{b}_R$ exactly when $\mathcal{J}(\mathfrak{a}^c)\mathcal{O}_{\mathcal{X}} \supset \mathfrak{b}_R\mathcal{O}_{\mathcal{X}}$, which is equivalent to

$$\mathcal{O}_{\mathcal{X}}(-(\lfloor cD \rfloor - K)^{\sim}) \supset \mathcal{O}_{\mathcal{X}}(-\mathbf{V}(\mathfrak{b}_R)E),$$

where E is as in Equation (5). This means that

$$(\lfloor cD \rfloor - K)^{\sim} \leq \mathbf{V}(\mathfrak{b}_R)E.$$

Because $(\lfloor cD \rfloor - K)^{\sim}$ is by definition the minimal antinef divisor satisfying $(\lfloor cD \rfloor - K) \leq (\lfloor cD \rfloor - K)^{\sim}$, we see that this holds if and only if

$$\lfloor cD \rfloor - K \leq \mathbf{V}(\mathfrak{b}_R)E.$$

Recall that $\mathbf{V}(\mathfrak{b}_R) = (\tilde{R} \cdot X_1, \dots, \tilde{R} \cdot X_n)$ and $\mathbf{V}(\mathfrak{a}) = (I \cdot X_1, \dots, I \cdot X_n)$ by Equation (7). Similarly, $K = E_1^* + \cdots + E_n^* = (\Sigma X_1, \cdots, \Sigma X_n)E$. Therefore the inequality above is equivalent to

$$\lfloor cX_i \cdot I \rfloor - \Sigma X_i \leq \tilde{R} \cdot X_i$$

for every $i = 1, \dots, n$. So $\mathcal{J}(\mathfrak{a}^c) \not\supset \mathfrak{b}_R$ exactly if $\lfloor cX_i \cdot I \rfloor > \tilde{R} \cdot X_i + \Sigma X_i$ for some $i = 1, \dots, n$, or equivalently,

$$cX_i \cdot I \geq \tilde{R} \cdot X_i + \Sigma X_i + 1.$$

for some $i = 1, \dots, n$. This means that $c \geq R[X_i]$ for some $i = 1, \dots, n$. Now c_R is by Definition 6.5 the smallest rational number c, for which $\mathcal{J}(\mathfrak{a}^c) \not\supset \mathfrak{b}_R$. Thus we get the claim. □

The remaining problem is to tell for which X_i we reach the minimum of the $R[X_i]$:s. Let $\gamma_0, \dots, \gamma_{g+1}$ be as given in Notation 3.3. We shall now present a few useful equalities and equivalences which will be needed in calculating and comparing the $R[X_i]$:s.

Lemma 7.3. *Let U be as in Equation (17) and suppose that $u \in U$. When $u > 1$ let ν satisfy $\gamma_\nu < u \le \gamma_{\nu+1}$, whereas $\nu = 0$ if $u = 1$. Set $\eta := \gamma_\nu$ and $\gamma := \gamma_{\nu+1}$. Furthermore, take $R = (r_1, \dots, r_n) \in \mathbb{N}^n$ and write*

$$\xi := \sum_{j \in J} r_j \rho_{j,\eta} \text{ and } \zeta := \sum_{j \notin J} r_j x_{j,\eta}$$

where we set $J := \{1, \dots, \eta - 1\} \cup \{j \mid \eta \le j < u \text{ and } j \in U\}$. Write also

$$\delta := \tilde{R} \cdot X_\eta + \Sigma X_\eta + 1 - (\zeta + 1) X_\eta^2.$$

Suppose that $\eta \le k \le \gamma$. Then

i) $\Sigma X_k + 1 = x_{k,\eta}(\Sigma X_\eta + 1) + \rho_{k,\eta}$;

ii) $\tilde{R} \cdot X_k^{>\eta} \le \rho_{k,\eta}\zeta + x_{k,\eta}\xi$;

iii) $\tilde{R} \cdot X_k + \Sigma X_k + 1 \le (\delta + \xi) x_{k,\eta} + (\zeta + 1)(x_{k,\eta} X_\eta^2 + \rho_{k,\eta})$;

iv) If $u = 1$, then $\delta = 1$;

v) If $u > 1$, then $R[X_\eta] \sim R[X_u] \Leftrightarrow \delta \sim x_{u,\eta} X_\eta^2 \xi : \rho_{u,\eta}$,

where $\sim$ denotes any of the relations $=, <$ or $>$. Moreover, the equality holds in ii) and iii) if $k = u$.

Proof. (i) Clearly,

$$\Sigma X_k + 1 = \Sigma X_k^{\le \eta} + x_{k,\eta} + \Sigma X_k^{>\eta} + 1 - x_{k,\eta}.$$

By using Proposition 3.12 we get $\Sigma X_k^{\le \eta} + x_{k,\eta} = x_{k,\eta}(\Sigma X_\eta + 1)$, and further, by Proposition 5.8 we obtain $\Sigma X_k^{>\eta} + 1 - x_{k,\eta} = \rho_{k,\eta}$.

(ii) We first observe that

$$\tilde{R} \cdot X_k^{>\eta} = \sum_{j=1}^{n} r_j X_j \cdot X_k^{>\eta}.$$

Because $k \le \gamma$, we have $X_j \cdot X_k^{>\eta} = [X_k \cdot X_j]_\nu$. Proposition 5.2 then yields that

$$\begin{aligned}
\tilde{R} \cdot X_k^{>\eta} &= \sum_{j=1}^{n} r_j [X_k \cdot X_j]_\nu \\
&= \sum_{j=1}^{n} r_j \min\{x_{k,\eta}\rho_{j,\eta}, x_{j,\eta}\rho_{k,\eta}\} \\
&= \sum_{j\in J} r_j \min\{x_{k,\eta}\rho_{j,\eta}, x_{j,\eta}\rho_{k,\eta}\} + \sum_{j\notin J} r_j \min\{x_{k,\eta}\rho_{j,\eta}, x_{j,\eta}\rho_{k,\eta}\} \\
&\leq \sum_{j\in J} r_j x_{j,\eta}\rho_{k,\eta} + \sum_{j\notin J} r_j \rho_{j,\eta} x_{k,\eta} \\
&= \rho_{k,\eta}\zeta + x_{k,\eta}\xi.
\end{aligned}$$

Let us then show that the equality holds here if $k = u$. By the above it is enough to prove that

$$[X_u \cdot X_j]_\nu = \begin{cases} x_{u,\eta}\rho_{j,\eta}, & \text{if } j \in J; \\ x_{j,\eta}\rho_{u,\eta}, & \text{if } j \notin J. \end{cases}$$

Indeed, suppose first that $j \in J$. Then either $j < \eta$ or $j \in U$ with $\eta \leq j < u$. In the first case we have $[X_u \cdot X_j]_\nu = 0 = x_{u,\eta}\rho_{j,\eta}$, while in the second case $[X_u \cdot X_j]_\nu = x_{u,\eta}\rho_{j,\eta}$ by Proposition 5.2 as wanted. Suppose then that $j \notin J$. If $\eta \leq j < u$, then $j \notin U$, and Proposition 5.2 gives $[X_u \cdot X_j]_\nu = x_{j,\eta}\rho_{u,\eta}$. If $j \geq u$, then the same holds, as $u \in U$.

(iii) Using Proposition 3.12 we have

$$\tilde{R} \cdot X_k = \tilde{R} \cdot X_k^{\leq\eta} + \tilde{R} \cdot X_k^{>\eta} = x_{k,\eta}\tilde{R} \cdot X_\eta + \tilde{R} \cdot X_k^{>\eta}.$$

By i) $\Sigma X_k + 1 = x_{k,\eta}(\Sigma X_\eta + 1) + \rho_{k,\eta}$. Putting these together yields

$$\begin{aligned}
\tilde{R} \cdot X_k + \Sigma X_k + 1 &= x_{k,\eta}(\tilde{R} \cdot X_\eta + \Sigma X_\eta + 1) + \rho_{k,\eta} + \tilde{R} \cdot X_k^{>\eta} \\
&= x_{k,\eta}\delta + (\zeta + 1)x_{k,\eta}X_\eta^2 + \rho_{k,\eta} + \tilde{R} \cdot X_k^{>\eta}.
\end{aligned}$$

By ii) $\tilde{R} \cdot X_k^{>\eta} \leq \rho_{k,\eta}\zeta + x_{k,\eta}\xi$, where the equality holds if $k = u$. Thus the claim is clear.

(iv) If $u = 1$, then $\eta = 1$, which implies that $X_\eta^2 = 1 = \Sigma X_\eta$. Furthermore, we have $J = \emptyset$, which yields

$$\zeta = \sum_{j\notin J} r_j x_{j,\eta} = \sum_{j=1}^{n} r_j X_j \cdot X_\eta = \tilde{R} \cdot X_\eta.$$

Thus we see that $\delta = \tilde{R} \cdot X_\eta + \Sigma X_\eta + 1 - \zeta - 1 = 1$.

(v) Observe that $X_u \cdot I = a_\eta x_{u,\eta} X_\eta^2 + [X_u \cdot I]_\nu$ by Corollary 5.3, where $I = (a_1, \ldots, a_n) = X_n$. Because $u \in U$, we have $[X_u \cdot I]_\nu = a_\eta \rho_{u,\eta}$ by Proposition 5.2. Therefore we obtain by using iii)

$$\begin{aligned} R[X_u] &= \frac{\tilde{R} \cdot X_u + \Sigma X_u + 1}{X_u \cdot I} \\ &= \frac{x_{u,\eta}(\delta + \xi) + (\zeta + 1)(x_{u,\eta} X_\eta^2 + \rho_{u,\eta})}{X_u \cdot I} \\ &= \frac{x_{u,\eta}(\tilde{R} \cdot X_\eta + \Sigma X_\eta + 1) + x_{u,\eta}\xi + (\zeta + 1)\rho_{u,\eta}}{a_\eta x_{u,\eta} X_\eta^2 + a_\eta \rho_{u,\eta}}. \end{aligned}$$

Noting that $X_\eta \cdot I = X_\eta \cdot I^{\leq \eta} = a_\eta X_\eta^2$ by Proposition 3.12 we then see that $R[X_\eta] \sim R[X_u]$ is equivalent to

$$\frac{\tilde{R} \cdot X_\eta + \Sigma X_\eta + 1}{X_\eta^2} \sim \frac{x_{u,\eta}(\tilde{R} \cdot X_\eta + \Sigma X_\eta + 1) + x_{u,\eta}\xi + (\zeta + 1)\rho_{u,\eta}}{x_{u,\eta} X_\eta^2 + \rho_{u,\eta}}. \tag{29}$$

If $u > 1$, in which case $u > \eta$, then $\rho_{u,\eta} > 0$. An elementary fact similar to Equation (26) says that if $a, b, c, d \in \mathbb{N}$ such that $bd \neq 0$, then

$$\frac{a + c}{b + d} \sim \frac{a}{b} \Leftrightarrow \frac{c}{d} \sim \frac{a}{b} \tag{30}$$

where $\sim$ is one of the relations $=$, $<$ or $>$. By using this we see that Equation (29) is further equivalent to

$$\frac{\tilde{R} \cdot X_\eta + \Sigma X_\eta + 1}{X_\eta^2} \sim \frac{x_{u,\eta}\xi + (\zeta + 1)\rho_{u,\eta}}{\rho_{u,\eta}}.$$

which is the same as

$$\delta = \tilde{R} \cdot X_\eta + \Sigma X_\eta + 1 - (\zeta + 1) X_\eta^2 \sim \frac{x_{u,\eta} X_\eta^2 \xi}{\rho_{u,\eta}}.$$

The claim has thus been proven. □

Next we will show that the relevant indices in searching the minimum of the $R[X_i]$:s are in the set $\bar{\Gamma} = \Gamma^* \cup \{n\}$, where Γ^* denotes the set of the indices corresponding to the star vertices of the dual graph as before.

Lemma 7.4. *Let $X_1, \ldots, X_n$ be the rows of $P^{-1} := (x_{i,j})_{n \times n}$. Let $i \in \{1, \ldots, n\}$. For $1 < i \leq n$, let $\nu \in \{0, \ldots, g\}$ satisfy $\gamma_\nu < i \leq \gamma_{\nu+1}$, whereas $\nu = 0$ if $i = 1$. Write $\eta := \gamma_\nu$ and $\gamma := \gamma_{\nu+1}$. Then*

$$R[X_\eta] \geq R[X_i] \Rightarrow R[X_i] \geq R[X_\gamma].$$

Moreover, in the case $n > 1$ we have $R[X_1] > R[X_{\gamma_1}]$.

Proof. It follows from Proposition 3.12 that

$$\frac{X_i \cdot X_n}{X_\gamma \cdot X_n} = \frac{X_i^{\leq\gamma} \cdot X_n}{X_\gamma^{\leq\gamma} \cdot X_n} = \frac{X_i \cdot X_n^{\leq\gamma}}{X_\gamma \cdot X_n^{\leq\gamma}} = \frac{x_{n,\gamma} X_i \cdot X_\gamma}{x_{n,\gamma} X_\gamma^2} = \frac{X_i \cdot X_\gamma}{X_\gamma^2},$$

which shows that

$$R[X_i] = \frac{\tilde{R} \cdot X_i + \Sigma X_i + 1}{X_i \cdot X_n} \geq \frac{\tilde{R} \cdot X_\gamma + \Sigma X_\gamma + 1}{X_\gamma \cdot X_n} = R[X_\gamma]$$

is equivalent to

$$\frac{\tilde{R} \cdot X_i + \Sigma X_i + 1}{\tilde{R} \cdot X_\gamma + \Sigma X_\gamma + 1} \geq \frac{X_i \cdot X_\gamma}{X_\gamma^2}. \tag{31}$$

Clearly, $R[X_i] > R[X_\gamma]$ if and only if the inequality here is strict.

Suppose first that $i \notin U$. According to Corollary 5.4 (iv) we have

$$\frac{x_{i,\eta} X_\eta^2 + \rho_{i,\eta}}{x_{\gamma,\eta} X_\eta^2 + \rho_{\gamma,\eta}} = \sigma_2(i,\gamma) \geq \sigma_1(i,\gamma) = \frac{x_{i,\eta}}{x_{\gamma,\eta}} = \frac{X_i \cdot X_\gamma}{X_\gamma^2}.$$

As $1 \in U$ we must have $1 \leq \eta < i \leq \gamma$. Then $\rho_{\gamma,\eta} > 0$, and Equation (30) implies

$$\frac{\rho_{i,\eta}}{\rho_{\gamma,\eta}} \geq \frac{x_{i,\eta}}{x_{\gamma,\eta}}.$$

It follows from Lemma 7.3 (i) that

$$\frac{\Sigma X_i + 1}{\Sigma X_\gamma + 1} = \frac{x_{i,\eta}(\Sigma X_\eta + 1) + \rho_{i,\eta}}{x_{\gamma,\eta}(\Sigma X_\eta + 1) + \rho_{\gamma,\eta}}.$$

An application of Equation (30) then gives

$$\frac{\Sigma X_i + 1}{\Sigma X_\gamma + 1} \geq \frac{x_{i,\eta}}{x_{\gamma,\eta}} = \frac{X_i \cdot X_\gamma}{X_\gamma^2}.$$

In the case $R = 0$ this is the same as Inequality (31). If $R \neq 0$, then Proposition 5.5 yields

$$\frac{\tilde{R} \cdot X_i}{\tilde{R} \cdot X_\gamma} \geq \frac{X_i \cdot X_\gamma}{X_\gamma^2}.$$

Applying Equation (26) to these two inequalities implies Inequality (31).

Suppose then that $i \in U$. According to Corollary 5.4 (iv) we have

$$\frac{x_{i,\eta}}{x_{\gamma,\eta}} = \sigma_1(i,\gamma) \geq \sigma_2(i,\gamma) = \frac{x_{i,\eta} X_\eta^2 + \rho_{i,\eta}}{x_{\gamma,\eta} X_\eta^2 + \rho_{\gamma,\eta}} = \frac{X_i \cdot X_\gamma}{X_\gamma^2}. \tag{32}$$

Choosing $u = i$ and $k = i, \gamma$ in Lemma 7.3 (iii) gives

$$\frac{\tilde{R} \cdot X_i + \Sigma X_i + 1}{\tilde{R} \cdot X_\gamma + \Sigma X_\gamma + 1} \geq \frac{x_{i,\eta}(\delta + \xi) + (\zeta + 1)(x_{i,\eta} X_\eta^2 + \rho_{i,\eta})}{x_{\gamma,\eta}(\delta + \xi) + (\zeta + 1)(x_{\gamma,\eta} X_\eta^2 + \rho_{\gamma,\eta})}. \tag{33}$$

By Lemma 7.3 (iv) and (v) $R[X_\eta] \geq R[X_i]$ implies $\delta \geq 0$. Then $\delta + \xi \geq 0$. If $\delta + \xi = 0$, this already proves Inequality (31) by Inequality (32). Thus it remains to consider the case $\delta + \xi > 0$. Then we may apply Equation (30) to (32) to get

$$\frac{x_{i,\eta}(\delta + \xi) + (\zeta + 1)(x_{i,\eta} X_\eta^2 + \rho_{i,\eta})}{x_{\gamma,\eta}(\delta + \xi) + (\zeta + 1)(x_{\gamma,\eta} X_\eta^2 + \rho_{\gamma,\eta})} \geq \frac{x_{i,\eta} X_\eta^2 + \rho_{i,\eta}}{x_{\gamma,\eta} X_\eta^2 + \rho_{\gamma,\eta}}. \tag{34}$$

Combining this to Inequalities (32) and (33) shows that Inequality (31) holds. The first claim has thus been proven.

The second claim follows from putting $i = 1$ in the inequalities above. Note that Inequality (32) is now strict, which implies that so are Inequalities (34) and (31), too. Indeed, $1 \in U$, and $n > 1$ yields that $\gamma_1 = \gamma > \eta = 1$ so that $\rho_{\gamma,\eta} > 0$ while $\rho_{i,\eta} = 0$. □

Proposition 7.5. *Let $X_1, \ldots, X_n$ be the rows of $P^{-1} := (x_{i,j})_{n \times n}$. Then*

$$c_R = \min\left\{R[X_\gamma] \mid \gamma \in \bar{\Gamma}\right\}$$

for any $R = (r_1, \ldots, r_n) \in \mathbb{N}^n$. Moreover, $c_R < R[X_1]$ if $n > 1$.

Proof. According Proposition 7.2 we have $c_R = \min\{R[X_j] \mid j = 1, \ldots, n\}$. Hence the claim is obvious in the case $n = 1$.

Suppose that $n > 1$. By Lemma 7.4 we have $c_R \leq R[X_{\gamma_1}] < R[X_1]$. Thus the last claim is clear, and furthermore, if i is such that $c_R = R[X_i]$ for some i, then $i > 1$. It follows that $\gamma_\nu < i \leq \gamma_{\nu+1}$ for some $\nu \in \{0, \ldots, g^*\}$. Because $R[X_{\gamma_\nu}] \geq R[X_i]$, Lemma 7.4 yields $R[X_i] \geq R[X_{\gamma_{\nu+1}}]$, which further implies that $c_R = R[X_{\gamma_{\nu+1}}]$. □

Lemma 7.6. *Suppose that $\gamma \in \Gamma^*$, and take any $R \in \mathbb{N}^n$ such that $c_R = R[X_\gamma]$. Then $c_{R^{<\gamma}} = R^{<\gamma}[X_\gamma] \leq a_\gamma^{-1}$, and for some $m \in \mathbb{N}$,*

$$c_R = c_{R^{<\gamma}} + \frac{m}{a_\gamma}.$$

Proof. Write $R = (r_1, \ldots, r_n)$ and set $R' := R^{<\gamma}$. In order to show that $c_{R'} = R'[X_\gamma]$, it is by Proposition 7.5 enough to verify, that $R'[X_\gamma] \leq R'[X_\eta]$ for any $\eta \in \bar{\Gamma}$.

Write $R = R' + R''$, where $R'' := R^{\geq\gamma}$. Furthermore, set $\tilde{R}' := R'P^{-1}$ and $\tilde{R}'' := R''P^{-1}$. Then

$$\tilde{R} = RP^{-1} = R'P^{-1} + R''P^{-1} = \tilde{R}' + \tilde{R}'',$$

and for any row X of P^{-1} we may write

$$R[X] = \frac{(\tilde{R}' + \tilde{R}'') \cdot X + \Sigma X + 1}{I \cdot X} = R'[X] + \frac{\tilde{R}'' \cdot X}{I \cdot X}. \tag{35}$$

Assume that $\eta \in \bar{\Gamma}$. Proposition 7.2 implies that $c_R = R[X_\gamma] \leq R[X_\eta]$. So

$$R'[X_\gamma] - R'[X_\eta] \leq \frac{\tilde{R}'' \cdot X_\eta}{I \cdot X_\eta} - \frac{\tilde{R}'' \cdot X_\gamma}{I \cdot X_\gamma} =: \Delta.$$

We will show that $\Delta \leq 0$. Note that the claim is trivial, if $R = R'$. Thus we may presume that $R'' \neq 0$.

Suppose first that $\eta \leq \gamma$. Then $X_\eta = X_\eta^{\leq\gamma}$. Since $\tilde{R}'' = \sum_{i=\gamma}^n r_i X_i$, we see by using Proposition 3.12 that for any $\eta \leq \gamma$

$$\frac{\tilde{R}'' \cdot X_\eta}{I \cdot X_\eta} = \frac{\tilde{R}'' \cdot X_\eta^{\leq\gamma}}{I \cdot X_\eta^{\leq\gamma}} = \frac{\sum_{i=\gamma}^n r_i X_i^{\leq\gamma} \cdot X_\eta}{I^{\leq\gamma} \cdot X_\eta} = \frac{\sum_{i=\gamma}^n r_i x_{i,\gamma} X_\gamma \cdot X_\eta}{a_\gamma X_\gamma \cdot X_\eta} = \frac{m}{a_\gamma}, \tag{36}$$

where $m := \sum_{i=\gamma}^n r_i x_{i,\gamma}$. Hence $\Delta = (m - m)/a_\gamma = 0$, if $\eta \leq \gamma$.

Assume next that $\eta > \gamma$. Then Proposition 3.12 implies that

$$\frac{I \cdot X_\gamma}{I \cdot X_\eta} = \frac{I \cdot X_\gamma^{\leq\eta}}{I \cdot X_\eta^{\leq\eta}} = \frac{I^{\leq\eta} \cdot X_\gamma}{I^{\leq\eta} \cdot X_\eta} = \frac{a_\eta X_\eta \cdot X_\gamma}{a_\eta X_\eta^2} = \frac{X_\eta \cdot X_\gamma}{X_\eta^2}.$$

By Proposition 5.5 we have for every $i = 1, \dots, n$

$$\frac{X_i \cdot X_\gamma}{X_i \cdot X_\eta} \geq \frac{X_\eta \cdot X_\gamma}{X_\eta^2}.$$

By applying Equation (26) we then get

$$\frac{\tilde{R}'' \cdot X_\gamma}{\tilde{R}'' \cdot X_\eta} = \frac{\sum_{i=\gamma}^n r_i X_i \cdot X_\gamma}{\sum_{i=\gamma}^n r_i X_i \cdot X_\eta} \geq \frac{X_\eta \cdot X_\gamma}{X_\eta^2} = \frac{I \cdot X_\gamma}{I \cdot X_\eta}.$$

Thus

$$\Delta = \frac{\tilde{R}'' \cdot X_\eta}{I \cdot X_\eta} - \frac{\tilde{R}'' \cdot X_\gamma}{I \cdot X_\gamma} \leq 0,$$

and therefore $c_{R'} = R'[X_\gamma]$. Subsequently, by Equations (35) and (36)

$$c_R = R[X_\gamma] = R'[X_\gamma] + \frac{\tilde{R}'' \cdot X_\gamma}{I \cdot X_\gamma} = c_{R'} + \frac{m}{a_\gamma}.$$

It remains to show $c_{R'} \le 1/a_\gamma$. Set $i := \min\{\eta \in \bar{\Gamma} \mid \gamma < \eta\}$. Observe that such an index exists, because by assumption $\gamma < n$. Recall that $i \in U$ (see Remark 5.1). Since $c_{R'} = R'[X_\gamma]$, we have $R'[X_\gamma] \le R'[X_i]$. By Lemma 7.3 (v) this implies

$$\tilde{R}' \cdot X_\gamma + \Sigma X_\gamma + 1 - (\zeta + 1) X_\gamma^2 \le \frac{x_{i,\gamma} X_\gamma^2 \xi}{\rho_{i,\gamma}}. \tag{37}$$

We have $\rho_{i,\gamma}\zeta + x_{i,\gamma}\xi = \tilde{R}' \cdot X_i^{>\gamma}$ by Lemma 7.3 (ii). On the other hand,

$$\tilde{R}' \cdot X_i^{>\gamma} = \sum_{j=1}^{\gamma-1} r_j X_j \cdot X_i^{>\gamma} = 0,$$

Therefore Equation (37) gives

$$\tilde{R}' \cdot X_\gamma + \Sigma X_\gamma + 1 \le \left(\frac{\rho_{i,\gamma}\zeta + x_{i,\gamma}\xi}{\rho_{i,\gamma}} + 1 \right) X_\gamma^2 = X_\gamma^2.$$

Since $X_\gamma \cdot I = X_\gamma \cdot I^{\le\gamma} = a_\gamma X_\gamma^2$ by Proposition 3.12, we obtain from this

$$R'[X_\gamma] = \frac{\tilde{R}' \cdot X_\gamma + \Sigma X_\gamma + 1}{X_\gamma \cdot I} \le \frac{1}{a_\gamma}.$$

Because $c_{R'} = R'[X_\gamma]$, we get the claim. □

8 Jumping numbers of a simple ideal

In this section we will give a formula for the jumping numbers of a simple complete ideal $\mathfrak{a}$ in a two-dimensional regular local ring α in terms of the multiplicities of the point basis $I = (a_1, \dots, a_n)$ of $\mathfrak{a}$. Let $\gamma_0, \dots, \gamma_{g+1}$ and Γ^* be as in Notation 3.3. Recall from Proposition 4.3 that Γ^* the set of indices corresponding to the stars of the associated dual graph. Before we proceed to the main theorem, we prove the following lemma:

Lemma 8.1. *Let α be a two-dimensional regular local ring, and let $\mathfrak{a}$ be a simple complete $\mathfrak{m}_\alpha$-primary ideal in α having the point basis $I = (a_1, \dots, a_n)$. For $\nu = 0, \dots, g$, write $\gamma_\nu = \eta$ and $\gamma_{\nu+1} = \gamma$, and set*

$$b_\nu := \frac{I \cdot I^{\le\gamma}}{a_\eta}.$$

Then

$$b_\nu = \frac{a_\gamma I \cdot X_\gamma}{a_\eta} = a_\gamma (x_{\gamma,\eta} X_\eta^2 + \rho_\nu).$$

Especially, b_ν is an integer, for which

$$\gcd\{a_\eta, b_\nu\} = a_\gamma.$$

Moreover, $a_\eta \le b_\nu$, where the equality holds if and only if $n = 1$.

Proof. Proposition 3.12 gives $I^{\le\gamma} = a_\gamma X_\gamma$. Therefore

$$b_\nu = \frac{I \cdot I^{\le\gamma}}{a_\eta} = \frac{a_\gamma I \cdot X_\gamma}{a_\eta}.$$

Using Corollary 5.3 we obtain $X_\gamma \cdot I = a_\eta(x_{\gamma,\eta}X_\eta^2 + \rho_\nu)$, and so we get the first claim. Clearly, b_ν is an integer, and since $a_\eta^2 \le I \cdot I^{\le\gamma} = a_\eta b_\nu$, we see that $a_\eta \le b_\nu$. Here the equality holds if only if $n = 1$. By Corollary 3.15 $a_\gamma x_{\gamma,\eta} = a_\eta$ and $a_\gamma \rho_\nu = \rho_{n,\eta}$. Then

$$\begin{aligned} \gcd\{a_\eta, b_\nu\} &= \gcd\{a_\eta, a_\gamma(x_{\gamma,\eta}X_\eta^2 + \rho_\nu)\} \\ &= \gcd\{a_\eta, a_\eta X_\eta^2 + \rho_{n,\eta}\} \\ &= \gcd\{a_\eta, \rho_{n,\eta}\}. \end{aligned}$$

By definition $\rho_{n,\eta} = a_{\eta+1} + \cdots + a_{\tau_{\nu+1}}$, and $a_\eta = \cdots = a_{\tau_{\nu+1}-1}$ by Proposition 3.5. Thus $\gcd\{a_\eta, \rho_{n,\eta}\} = \gcd\{a_\eta, a_{\tau_{\nu+1}}\}$, and the claim follows from Proposition 3.7. □

Remark 8.2. The integers $a_1, b_0, \ldots, b_g$ in fact coincide with the so called *Zariski exponents* $\bar\beta_0, \ldots, \bar\beta_{g+1}$. The Zariski exponents can be defined recursively as follows. Let $\beta'_1, \ldots, \beta'_{g+1}$ be the Puiseux exponents (see Notation 3.6 and Remark 3.8). Proposition 3.7 gives that $\gcd\{a_{\gamma_{\nu-1}} + \cdots + a_{\tau_\nu}, a_{\gamma_{\nu-1}}\} = a_{\gamma_\nu}$ for every $\nu = 1, \ldots, g+1$, and then by using Corollary 3.15 we see that

$$\beta'_\nu = \frac{x_{\gamma_\nu,\gamma_{\nu-1}} + \cdots + x_{\gamma_\nu,\tau_\nu}}{x_{\gamma_\nu,\gamma_{\nu-1}}},$$

where $\gcd\{x_{\gamma_\nu,\gamma_{\nu-1}} + \cdots + x_{\gamma_\nu,\tau_\nu}, x_{\gamma_\nu,\gamma_{\nu-1}}\} = 1$. Recall that by Corollary 3.15 $a_{\gamma_\nu} = x_{\gamma_{\nu+1},\gamma_\nu} \cdots x_{\gamma_{g+1},\gamma_g}$ for every $\nu = 0, \ldots, g$. Note that the integers $a_{\gamma_0}, \ldots, a_{\gamma_{g+1}}$ and $x_{\gamma_1,\gamma_0}, \ldots, x_{\gamma_{g+1},\gamma_g}$ are usually denoted by $e_0, \ldots, e_{g+1}$ and $n_1, \ldots, n_{g+1}$, respectively (cf. [20, p. 130]). Write $n_0 := 1$. Following [20, Equation 6.1] (clearly, e_{i+1} is a misprint in the cited equation), set

$$\bar\beta_0 := e_0, \text{ and } \bar\beta_\nu := (\beta'_\nu - 1)e_{\nu-1} + \bar\beta_{\nu-1}n_{\nu-1} \text{ for } 1 \le \nu \le g+1.$$

Let us prove that these are the integers $a_{\gamma_0}, b_0 \ldots, b_g$. From the definition above we see that $\bar\beta_0 = a_{\gamma_0} = I \cdot X_{\gamma_0}$. In order to verify that $\bar\beta_\nu = b_{\nu-1}$ for $1 \le \nu \le g+1$, let us first show that $\bar\beta_\nu n_\nu = I \cdot X_{\gamma_\nu}$ also for $\nu > 0$. Suppose that $\bar{\beta}_{\nu-1}n_{\nu-1} = I \cdot X_{\gamma_{\nu-1}}$ holds for some $1 \leq \nu \leq g+1$. Remark 3.14 shows that $(\beta'_\nu - 1)e_{\nu-1} = \rho_{n,\gamma_{\nu-1}}$, and $I \cdot X_{\gamma_{\nu-1}} = a_{\gamma_{\nu-1}} X^2_{\gamma_{\nu-1}}$ by Proposition 3.12. Subsequently, we obtain by using Corollary 5.3

$$\bar{\beta}_\nu n_\nu = n_\nu(I \cdot X_{\gamma_{\nu-1}} + \rho_{n,\gamma_{\nu-1}}) = x_{\gamma_\nu,\gamma_{\nu-1}}(a_{\gamma_{\nu-1}} X^2_{\gamma_{\nu-1}} + \rho_{n,\gamma_{\nu-1}}) = I \cdot X_{\gamma_\nu}. \quad (38)$$

Thus an induction on ν shows that

$$\bar{\beta}_\nu = \frac{I \cdot X_{\gamma_\nu}}{x_{\gamma_\nu,\gamma_{\nu-1}}} = \frac{a_{\gamma_\nu} I \cdot X_{\gamma_\nu}}{a_{\gamma_{\nu-1}}}$$

for $1 \leq \nu \leq g+1$, where the last equality follows from Corollary 3.15. Together with Lemma 8.1 this completes the proof.

We have also an alternative characterization for the Zariski exponents:

$$\bar{\beta}_\nu = v_{\tau_\nu}(\mathfrak{a}) = v(\mathfrak{p}_{\tau_\nu}) \qquad (\nu = 0, \ldots, g+1).$$

Clearly, this holds for $\nu = 0$. Let us then verify this in the case $\nu > 0$. Equation (38) yields $\bar{\beta}_\nu = I \cdot X_{\gamma_{\nu-1}} + \rho_{n,\gamma_{\nu-1}}$ for $1 \leq \nu \leq g+1$. On the other hand, since $x_{\tau_\nu,i} = 1$ for every $\gamma_{\nu-1} \leq i \leq \tau_\nu$ by Proposition 3.10, we get $I \cdot X^{>\gamma_{\nu-1}}_{\tau_\nu} = \rho_{n,\gamma_{\nu-1}}$. An application of Proposition 3.12 then gives $I \cdot X^{\leq\gamma_{\nu-1}}_{\tau_\nu} = I \cdot X_{\gamma_{\nu-1}}$. Subsequently, for $1 \leq \nu \leq g+1$,

$$\bar{\beta}_\nu = I \cdot X_{\gamma_{\nu-1}} + \rho_{n,\gamma_{\nu-1}} = I \cdot X^{\leq\gamma_{\nu-1}}_{\tau_\nu} + I \cdot X^{>\gamma_{\nu-1}}_{\tau_\nu} = I \cdot X_{\tau_\nu}.$$

Because $I \cdot X_{\tau_\nu} = v_{\tau_\nu}(\mathfrak{a}) = v(\mathfrak{p}_{\tau_\nu})$, as observed in Remark 5.6, we thus obtain the desired characterization.

Theorem 8.3. *Let α be a two-dimensional regular local ring, and let $\mathfrak{a}$ be a simple complete $\mathfrak{m}_\alpha$-primary ideal in α having the point basis $I = (a_1, \ldots, a_n)$. Let $\Gamma^* = \{\gamma_1, \ldots, \gamma_{g^*}\}$ the set of indices corresponding to the stars of the dual graph associated to $\mathfrak{a}$, and write $\gamma_0 := 1$ and $\gamma_{g^*+1} := n$. For $\nu = 0, \ldots, g^*$, set*

$$b_\nu := \frac{I \cdot I^{\leq\gamma_{\nu+1}}}{a_{\gamma_\nu}}$$

and then define for $s, t, m \in \mathbb{N}$

$$H_\nu := \left\{ \frac{s+1}{a_{\gamma_\nu}} + \frac{t+1}{b_\nu} + \frac{m}{a_{\gamma_{\nu+1}}} \;\middle|\; s,t,m \in \mathbb{N}, \frac{s+1}{a_{\gamma_\nu}} + \frac{t+1}{b_\nu} \leq \frac{1}{a_{\gamma_{\nu+1}}} \right\}$$

for $\nu = 0, \ldots, g^* - 1$ and

$$H_{g^*} := \left\{ \frac{s+1}{a_{\gamma_{g^*}}} + \frac{t+1}{b_{g^*}} \;\middle|\; s,t \in \mathbb{N} \right\}.$$

The set of the jumping numbers of the ideal $\mathfrak{a}$ is then

$$\mathcal{H}_\mathfrak{a} = H_0 \cup \cdots \cup H_{g^*}.$$

Proof. If $n = 1$, then $\mathfrak{a} = \mathfrak{m}_\alpha$. Subsequently, $\mathcal{H}_\mathfrak{a} = \{m \in \mathbb{N} \mid m > 1\}$, and the case is clear. Thus we may suppose throughout the proof that $n > 1$.

To see that every jumping number is in $H_0 \cup \cdots \cup H_{g^*}$, recall first that by Proposition 6.7 any element in $\mathcal{H}_\mathfrak{a}$ is of the form c_R for some $R \in \mathbb{N}^n$. Take an arbitrary $R = (r_1, \ldots, r_n) \in \mathbb{N}^n$. By Proposition 7.5 there is $\nu \in \{0, \ldots, g^*\}$ (recall that $\bar{\Gamma} = \{\gamma_1, \ldots, \gamma_{g^*+1}\}$) such that

$$c_R = R[X_\gamma] < R[X_\eta], \tag{39}$$

where $\gamma := \gamma_{\nu+1}$ and $\eta := \gamma_\nu$. As observed in Remark 5.1, $\gamma \in U$. Thus we may apply Lemma 7.3 (iii) to get

$$R[X_\gamma] = \frac{\tilde{R} \cdot X_\gamma + \Sigma X_\gamma + 1}{I \cdot X_\gamma} = \frac{(x_{\gamma,\eta} X_\eta^2 + \rho_\nu)(\zeta + 1) + x_{\gamma,\eta}(\xi + \delta)}{I \cdot X_\gamma}.$$

By Corollary 3.15 we have $a_\eta / a_\gamma = x_{\gamma,\eta}$. It then follows from Lemma 8.1 that

$$I \cdot X_\gamma = a_\eta (x_{\gamma,\eta} X_\eta^2 + \rho_\nu) = x_{\gamma,\eta} b_\nu. \tag{40}$$

Putting all together we get

$$c_R = \frac{\zeta + 1}{a_\eta} + \frac{\xi + \delta}{b_\nu}. \tag{41}$$

Clearly, $\zeta+1$ is a positive integer, and by Lemma 7.3 (v) we see from Equation (39) that δ and thereby also $\xi + \delta$ are positive integers.

If $\nu = g^*$, then Equation (41) proves that $c_R \in H_{g^*}$. If $\nu < g^*$, then it follows from Lemma 7.6 that c_R is in H_ν exactly, when $c_{R^{<\gamma}}$ is. Thus we may assume $R = R^{<\gamma}$. This case is now clear by Equation (41), as Lemma 7.6 guarantees that $c_{R^{<\gamma}} \leq a_\gamma^{-1}$. Subsequently,

$$\mathcal{H}_\mathfrak{a} \subset H_0 \cup \cdots \cup H_{g^*}$$

as wanted.

In order to prove the opposite inclusion, we first need two more lemmas.

Lemma 8.4. *Let $\nu \in \{0, \ldots, g\}$, where g is the number of the terminal satellites of $\mathfrak{a}$, and let $t_{\nu+1}, m_{\nu+1} \in \mathbb{N}$. Then there exists two sequences of pairs $(s_1, t_1), \ldots, (s_\nu, t_\nu)$ and $(s_{\nu+2}, m_{\nu+2}), \ldots, (s_g, m_g)$ of non negative integers satisfying the following conditions:*

i) for every $1 \leq i \leq \nu$

$$t_{i+1} + 1 + X_{\gamma_i}^2 = (s_i + 1)(x_{\gamma_i,\gamma_{i-1}} X_{\gamma_{i-1}}^2 + \rho_{i-1}) + (t_i + 1) x_{\gamma_i,\gamma_{i-1}},$$

ii) for every $\nu < i < g$ *we have* $m_i = m_{i+1}x_{\gamma_{i+1},\gamma_i} + s_{i+1}$ *and* $s_{i+1} < x_{\gamma_{i+1},\gamma_i}$. *Moreover, writing* $s_{g+1} := m_g$ *we get for every* $\nu < \mu \leq g$

$$m_\mu = \sum_{i=\mu}^{g} s_{i+1}x_{\gamma_i,\gamma_\mu}.$$

iii) For any $k = \nu+2, \dots, g+1$, *we have*

$$\Phi_k := \frac{(m_{\nu+1}+1)\rho_{\gamma_k,\gamma_{\nu+1}} + \cdots + (m_{k-1}+1)\rho_{\gamma_k,\gamma_{k-1}}}{a_{\gamma_{\nu+1}}\rho_{\gamma_k,\gamma_{\nu+1}} + \cdots + a_{\gamma_{k-1}}\rho_{\gamma_k,\gamma_{k-1}}} \geq \frac{m_{\nu+1}+1}{a_{\gamma_{\nu+1}}}.$$

Proof. (i) Using descending induction, suppose that $t_{i+1} \in \mathbb{N}$ is given for some $1 < i \leq \nu$. By Corollary 5.3

$$x_{\gamma_i,\gamma_{i-1}}(x_{\gamma_i,\gamma_{i-1}}X_{\gamma_{i-1}}^2 + \rho_{i-1}) = X_{\gamma_i}^2.$$

Note that $x_{\gamma_i,\gamma_{i-1}} = a_{\gamma_{i-1}}/a_{\gamma_i}$ by Corollary 3.15. Moreover, we also have $x_{\gamma_i,\gamma_{i-1}}X_{\gamma_{i-1}}^2 + \rho_{i-1} = b_{i-1}/a_{\gamma_i}$ by Lemma 8.1, which then further yields

$$\gcd\{x_{\gamma_i,\gamma_{i-1}}, x_{\gamma_i,\gamma_{i-1}}X_{\gamma_{i-1}}^2 + \rho_{i-1}\} = \frac{\gcd\{a_{\gamma_{i-1}}, b_{i-1}\}}{a_{\gamma_i}} = 1.$$

The existence of a pair (s_i, t_i) now follows from Lemma 8.5 below, and thereby we obtain a sequence $(s_1, t_1), \dots, (s_\nu, t_\nu)$.

(ii) Given a non negative integer m_i for $\nu < i < g$, we have $m_{i+1}, s_{i+1} \in \mathbb{N}$ with $m_i = m_{i+1}x_{\gamma_{i+1},\gamma_i} + s_{i+1}$ and $s_{i+1} < x_{\gamma_{i+1},\gamma_i}$. Arguing inductively, the existence of the pairs $(s_{\nu+2}, m_{\nu+2}), \dots, (s_g, m_g)$ is then clear. Observe that $m_g = s_{g+1}x_{\gamma_g,\gamma_g}$. Moreover, assuming that

$$m_{\mu+1} = \sum_{i=\mu+1}^{g} s_{i+1}x_{\gamma_i,\gamma_{\mu+1}}$$

holds for any $\nu < \mu < g$ we see by using Corollary 3.15 that

$$m_\mu = \sum_{i=\mu+1}^{g} s_{i+1}x_{\gamma_i,\gamma_{\mu+1}}x_{\gamma_{\mu+1},\gamma_\mu} + s_{\mu+1} = \sum_{i=\mu}^{g} s_{i+1}x_{\gamma_i,\gamma_\mu}.$$

Hence this holds for every $\nu < \mu \leq g$.

(iii) To prove the last claim we first note that for every $\nu < i < g$

$$\frac{m_{i+1}+1}{a_{\gamma_{i+1}}} = \frac{m_{i+1}x_{\gamma_{i+1},\gamma_i} + x_{\gamma_{i+1},\gamma_i}}{a_{\gamma_{i+1}}x_{\gamma_{i+1},\gamma_i}}.$$

As $x_{\gamma_{i+1},\gamma_i} \geq s_{i+1} + 1$ and $a_{\gamma_{i+1}} x_{\gamma_{i+1},\gamma_i} = a_{\gamma_i}$ by Corollary 3.15, we see that

$$\frac{m_{i+1}+1}{a_{\gamma_{i+1}}} \geq \frac{m_{i+1}x_{\gamma_{i+1},\gamma_i} + s_{i+1} + 1}{a_{\gamma_{i+1}} x_{\gamma_{i+1},\gamma_i}} = \frac{m_i+1}{a_{\gamma_i}}.$$

Because $\rho_{\gamma_k,\gamma_i} > 0$ for every $\nu < i < k$, the above implies

$$\frac{(m_i+1)\rho_{\gamma_k,\gamma_i}}{a_{\gamma_i}\rho_{\gamma_k,\gamma_i}} \geq \frac{m_{\nu+1}+1}{a_{\gamma_{\nu+1}}}.$$

The claim follows now from Equation (26). □

Lemma 8.5. *If $a \leq b$ are positive integers and $\gcd\{a, b\} = 1$, then for any positive integer t there exist positive integers u and v such that $ua+vb = ab+t$.*

Proof. We can assume $a < b$, as the case $a = b = 1$ is trivial, and clearly, we may reduce to the case $t < a$. Since $\gcd\{a, b\} = 1$, we can find a positive integer p such that $pa = 1 \mod b$. Then $tpa = t + qb$ for some integer q. We see that $b \nmid tp$, as otherwise $b \mid tpa - qb = t$, which is impossibile as $0 < t < a < b$. Thus we may find integers r and u such that $tp = rb + u$ and $0 < u < b$. We get $tpa = rba + ua = t + qb$, i.e., $ua = t + (q - ra)b$. As $0 < ua$ and $t < b$, we see that $0 \leq q - ra$. Since $u < b$ we get $ua = t + (q - ra)b < ab$. Especially, this gives $q - ra < a$. Set $v := (r+1)a - q$. Then we observe that $v > 0$ and

$$\begin{aligned} ua + vb &= ua + ((r+1)a - q)b \\ &= t + (q - ra)b + (r+1)ab - qb \\ &= t + ab. \end{aligned}$$

□

Choose any $\nu \in \{0, \ldots, g^*\}$, and take an arbitrary element $c \in H_\nu$. Then there exist $s, t, m \in \mathbb{N}$ such that

$$c = \frac{s+1}{a_\eta} + \frac{t+1}{b_\nu} + \frac{m}{a_\gamma}, \text{ moreover, } c \leq \frac{m+1}{a_\gamma} \text{ if } \nu < g^*.$$

As above, write $\eta = \gamma_\nu$ and $\gamma = \gamma_{\nu+1}$. In order to prove that $c = c_R$ for some $R \in \mathbb{N}^n$, we shall proceed in three steps.

A) To begin with, we shall construct a suitable $R \in \mathbb{N}^n$. Let g be the number of the terminal satellites of $\mathfrak{a}$. Set $t_{\nu+1} := t$ and $m_{\nu+1} := m$, and let $(s_1, t_1), \ldots, (s_\nu, t_\nu)$ and $(s_{\nu+2}, m_{\nu+2}), \ldots, (s_g, m_g)$ be sequences constructed as in Lemma 8.4. From these we obtain a sequence $s_0, \ldots, s_{g+1} \in \mathbb{N}$ by setting $s_{\nu+1} := s$, $s_0 := t_1$ and $s_{g+1} := m_g$. Define $R = (r_1, \ldots, r_n)$ so that

$$r_i = \begin{cases} s_\nu & \text{if } i = \tau_\nu \text{ for some } \nu = 0, \ldots, g+1; \\ 0 & \text{otherwise.} \end{cases}$$

We observe that

$$\tilde{R} = s_0 X_{\tau_0} + \cdots + s_{g+1} X_{\tau_{g+1}} = \tilde{R}_k + \tilde{S}_k$$

for any $k = 0, \ldots, g+1$, where

$$\tilde{R}_k := \sum_{i=0}^{k} s_i X_{\tau_i} \text{ and } \tilde{S}_k := \sum_{i=k+1}^{g+1} s_i X_{\tau_i}.$$

B) We shall show $c = R[X_\gamma]$. For $k = 0, \ldots, g+1$, set

$$\psi(k) := \tilde{R}_k \cdot X_{\gamma_k} + \Sigma X_{\gamma_k} + 1.$$

Note that

$$R[X_\gamma] = \frac{\tilde{R} \cdot X_\gamma + \Sigma X_\gamma + 1}{I \cdot X_\gamma} = \frac{\psi(\nu+1)}{I \cdot X_\gamma} + \frac{\tilde{S}_{\nu+1} \cdot X_\gamma}{I \cdot X_\gamma}. \tag{42}$$

According to Proposition 3.12 we have $X_{\tau_{i+1}}^{\leq \gamma_i} = x_{\tau_{i+1},\gamma_i} X_{\gamma_i}$, where $x_{\tau_{i+1},\gamma_i} = 1$ by Proposition 3.10. If $i \geq k$, then $\gamma_k \leq \gamma_i$ so that

$$X_{\tau_{i+1}} \cdot X_{\gamma_k} = X_{\tau_{i+1}} \cdot X_{\gamma_k}^{\leq \gamma_i} = X_{\tau_{i+1}}^{\leq \gamma_i} \cdot X_{\gamma_k} = X_{\gamma_i} \cdot X_{\gamma_k}.$$

Moreover, an application of Proposition 3.12 gives

$$I \cdot X_{\gamma_k} = I^{\leq \gamma_i} \cdot X_{\gamma_k} = a_{\gamma_i} X_{\gamma_i} \cdot X_{\gamma_k}.$$

Therefore

$$\frac{\tilde{S}_k \cdot X_{\gamma_k}}{I \cdot X_{\gamma_k}} = \sum_{i=k}^{g} \frac{s_{i+1} X_{\tau_{i+1}} \cdot X_{\gamma_k}}{I \cdot X_{\gamma_k}} = \sum_{i=k}^{g} \frac{s_{i+1} X_{\gamma_i} \cdot X_{\gamma_k}}{a_{\gamma_i} X_{\gamma_i} \cdot X_{\gamma_k}} = \sum_{i=k}^{g} \frac{s_{i+1}}{a_{\gamma_i}}.$$

As $a_\gamma / a_{\gamma_i} = x_{\gamma_i,\gamma}$ by Corollary 3.15, it follows from Lemma 8.4 (ii) that

$$\frac{\tilde{S}_{\nu+1} \cdot X_\gamma}{I \cdot X_\gamma} = \sum_{i=\nu+1}^{g} \frac{s_{i+1}}{a_{\gamma_i}} = \sum_{i=\nu+1}^{g} \frac{s_{i+1} x_{\gamma_i,\gamma}}{a_\gamma} = \frac{m}{a_\gamma}. \tag{43}$$

We aim to show that

$$\psi(\nu+1) = (t+1) x_{\gamma,\eta} + (s+1)(x_{\gamma,\eta} X_\eta^2 + \rho_\nu). \tag{44}$$

Suppose for a moment that this holds. As we already saw in Equation (40), $I \cdot X_\gamma = a_\eta (x_{\gamma,\eta} X_\eta^2 + \rho_\nu) = x_{\gamma,\eta} b_\nu$. By Equations (42) and (43) we then have

$$\begin{aligned} R[X_\gamma] &= \frac{(t+1) x_{\gamma,\eta}}{I \cdot X_\gamma} + \frac{(s+1)(x_{\gamma,\eta} X_\eta^2 + \rho_\nu)}{I \cdot X_\gamma} + \frac{m}{a_\gamma} \\ &= \frac{(t+1)}{b_\nu} + \frac{(s+1)}{a_\eta} + \frac{m}{a_\gamma} = c \end{aligned}$$

as desired.

In order to verify Equation (44), recall first that by Proposition 5.8

$$\Sigma X_{\gamma_k}^{>\gamma_0} + 1 - x_{\gamma_k,\gamma_0} = \rho_{\gamma_k,\gamma_0} + \cdots + \rho_{\gamma_k,\gamma_{k-1}}.$$

By Corollary 3.15 we get $\rho_{\gamma_k,\gamma_i} = x_{\gamma_k,\gamma_{i+1}}\rho_i$ for $0 \le i < k$, and so

$$\Sigma X_{\gamma_k} + 1 = 2x_{\gamma_k,\gamma_0} + x_{\gamma_k,\gamma_1}\rho_0 + \cdots + x_{\gamma_k,\gamma_k}\rho_{k-1}.$$

Setting $\rho_{(-1)} := 2$ allows us to write

$$\psi(k) = \tilde{R}_k \cdot X_{\gamma_k} + \Sigma X_{\gamma_k} + 1 = \sum_{i=0}^{k} \big(s_i X_{\tau_i} \cdot X_{\gamma_k} + x_{\gamma_k,\gamma_i}\rho_{i-1}\big). \tag{45}$$

If $i < k$, then $\gamma_{k-1} \ge \gamma_i \ge \tau_i$, and Proposition 3.12 implies that

$$X_{\tau_i} \cdot X_{\gamma_k} = X_{\tau_i} \cdot X_{\gamma_k}^{\le\gamma_{k-1}} = x_{\gamma_k,\gamma_{k-1}} X_{\tau_i} \cdot X_{\gamma_{k-1}}.$$

Furthermore, by Corollary 3.15 $x_{\gamma_k,\gamma_i} = x_{\gamma_k,\gamma_{k-1}} x_{\gamma_{k-1},\gamma_i}$. Hence for $i < k$

$$s_i X_{\tau_i} \cdot X_{\gamma_k} + x_{\gamma_k,\gamma_i}\rho_{i-1} = x_{\gamma_k,\gamma_{k-1}}\big(s_i X_{\tau_i} \cdot X_{\gamma_{k-1}} + x_{\gamma_{k-1},\gamma_i}\rho_{i-1}\big).$$

Therefore we get by Equation (45)

$$\begin{aligned}
\psi(k) &= \sum_{i=0}^{k-1} \big(s_i X_{\tau_i} \cdot X_{\gamma_k} + x_{\gamma_k,\gamma_i}\rho_{i-1}\big) + s_k X_{\tau_k} \cdot X_{\gamma_k} + x_{\gamma_k,\gamma_k}\rho_{k-1} \\
&= x_{\gamma_k,\gamma_{k-1}} \sum_{i=0}^{k-1} \big(s_i X_{\tau_i} \cdot X_{\gamma_{k-1}} + x_{\gamma_{k-1},\gamma_i}\rho_{i-1}\big) + s_k X_{\tau_k} \cdot X_{\gamma_k} + \rho_{k-1}.
\end{aligned}$$

But a look at Equation (45) again shows that this is the same as

$$\psi(k) = x_{\gamma_k,\gamma_{k-1}}\psi(k-1) + s_k X_{\tau_k} \cdot X_{\gamma_k} + \rho_{k-1}.$$

Proposition 3.10 says that $x_{\tau_k,\gamma_{k-1}} = \cdots = x_{\tau_k,\tau_k} = 1$, which implies that

$$\begin{aligned}
X_{\tau_k} \cdot X_{\gamma_k} &= X_{\tau_k}^{\le\gamma_{k-1}} \cdot X_{\gamma_k}^{\le\gamma_{k-1}} + X_{\tau_k} \cdot X_{\gamma_k}^{>\gamma_{k-1}} \\
&= x_{\tau_k,\gamma_{k-1}} x_{\gamma_k,\gamma_{k-1}} X_{\gamma_{k-1}}^2 + \rho_{k-1} \\
&= x_{\gamma_k,\gamma_{k-1}} X_{\gamma_{k-1}}^2 + \rho_{k-1},
\end{aligned}$$

as $X_{\gamma_k}^{\le\gamma_{k-1}} = x_{\gamma_k,\gamma_{k-1}} X_{\gamma_{k-1}}$ and $X_{\tau_k}^{\le\gamma_{k-1}} = x_{\tau_k,\gamma_{k-1}} X_{\gamma_{k-1}}$ by Proposition 3.12. This yields a recursion formula

$$\psi(k) = x_{\gamma_k,\gamma_{k-1}}\psi(k-1) + s_k(x_{\gamma_k,\gamma_{k-1}} X_{\gamma_{k-1}}^2 + \rho_{k-1}) + \rho_{k-1}. \tag{46}$$

We claim that for $k = 0, \dots, \nu$

$$\psi(k) = t_{k+1} + 1 + X_{\gamma_k}^2.$$

Taking $k = \nu$, Equation (44) results from the recursion formula above.

We use induction on k. By definition $s_0 = t_1$ and $\gamma_0 = \tau_0 = 1$, so that Equation (45) gives

$$\psi(0) = s_0 + 2 = t_1 + 1 + X_{\gamma_0}^2.$$

Assume next that $\psi(k-1) = t_k + 1 + X_{\gamma_{k-1}}^2$ for some $1 \le k \le \nu$. Then the recursion in Equation (46) yields

$$\begin{aligned} \psi(k) &= (t_k + 1 + X_{\gamma_{k-1}}^2) x_{\gamma_k,\gamma_{k-1}} + s_k(x_{\gamma_k,\gamma_{k-1}} X_{\gamma_{k-1}}^2 + \rho_{k-1}) + \rho_{k-1} \\ &= (t_k + 1) x_{\gamma_k,\gamma_{k-1}} + (s_k + 1)(x_{\gamma_k,\gamma_{k-1}} X_{\gamma_{k-1}}^2 + \rho_{k-1}). \end{aligned}$$

Subsequently, by Lemma 8.4 (i) we get $\psi(k) = t_{k+1} + 1 + X_{\gamma_k}^2$ as needed. This completes the step B).

C) It remains to show that $R[X_\gamma] = c_R$. If this is not the case, then by Proposition 7.5 there is an integer $k \neq \nu + 1$ satisfying

$$c_R = R[X_{\gamma_k}] < R[X_\gamma] \tag{47}$$

($\gamma = \gamma_{\nu+1}$). Assume first that $k < \nu + 1$. Then $\tau_k < \gamma_k < \tau_{k+1}$ by Equation (11), which yields $\tilde{R}_k = (R^{<\gamma_k})^{\sim}$, and subsequently,

$$R^{<\gamma_k}[X_{\gamma_k}] = \frac{\tilde{R}_k \cdot X_{\gamma_k} + \Sigma X_{\gamma_k} + 1}{I \cdot X_{\gamma_k}} = \frac{\psi(k)}{I \cdot X_{\gamma_k}}.$$

By Lemma 7.6 this gives

$$\frac{\psi(k)}{I \cdot X_{\gamma_k}} \le \frac{1}{a_{\gamma_k}}.$$

On the other hand, using Proposition 3.12 we get

$$I \cdot X_{\gamma_k} = I \cdot X_{\gamma_k}^{\le \gamma_k} = I^{\le \gamma_k} \cdot X_{\gamma_k} = a_{\gamma_k} X_{\gamma_k}^2,$$

and then by Equation (8)

$$\frac{\psi(k)}{I \cdot X_{\gamma_k}} = \frac{t_{k+1} + 1 + X_{\gamma_k}^2}{a_{\gamma_k} X_{\gamma_k}^2} > \frac{1}{a_{\gamma_k}},$$

which is a contradiction. Therefore we must have $k > \nu + 1$. Then $\nu < g^*$, and by assumption and the step B)

$$R[X_\gamma] = c \le \frac{m+1}{a_\gamma}. \tag{48}$$

Clearly, we may rewrite inequality (47)

$$\begin{aligned}
& \frac{\tilde{R} \cdot X_{\gamma_k} + \Sigma X_{\gamma_k} + 1}{I \cdot X_{\gamma_k}} \\
= \quad & \frac{\tilde{R} \cdot X_{\gamma_k}^{\leq\gamma} + \Sigma X_{\gamma_k}^{\leq\gamma} + x_{\gamma_k,\gamma} + \tilde{R} \cdot X_{\gamma_k}^{>\gamma} + \Sigma X_{\gamma_k}^{>\gamma} + 1 - x_{\gamma_k,\gamma}}{I \cdot X_{\gamma_k}^{\leq\gamma} + I \cdot X_{\gamma_k}^{>\gamma}} \\
= \quad & \frac{x_{\gamma_k,\gamma}(\tilde{R} \cdot X_\gamma + \Sigma X_\gamma + 1) + \tilde{R} \cdot X_{\gamma_k}^{>\gamma} + \Sigma X_{\gamma_k}^{>\gamma} + 1 - x_{\gamma_k,\gamma}}{x_{\gamma_k,\gamma} I \cdot X_\gamma + I \cdot X_{\gamma_k}^{>\gamma}} \\
< \quad & \frac{\tilde{R} \cdot X_\gamma + \Sigma X_\gamma + 1}{I \cdot X_\gamma},
\end{aligned}$$

where the second equality follows from Proposition 3.12. Applying Equation (30) to this, we get

$$\frac{\tilde{R} \cdot X_{\gamma_k}^{>\gamma} + \Sigma X_{\gamma_k}^{>\gamma} + 1 - x_{\gamma_k,\gamma}}{I \cdot X_{\gamma_k}^{>\gamma}} < R[X_\gamma]. \tag{49}$$

We aim to prove that

$$\frac{\tilde{R} \cdot X_{\gamma_k}^{>\gamma} + \Sigma X_{\gamma_k}^{>\gamma} + 1 - x_{\gamma_k,\gamma}}{I \cdot X_{\gamma_k}^{>\gamma}} = \Phi_k,$$

where Φ_k is as given in Lemma 8.4. This will lead to a contradiction proving the claim, because $\Phi_k \geq (m+1)/a_\gamma$ by Lemma 8.4 (iii), and then

$$R[X_\gamma] \leq \frac{m+1}{a_\gamma} \leq \Phi_k < R[X_\gamma]. \tag{50}$$

Because

$$\tilde{R} = \sum_{i=0}^{g+1} s_i X_{\tau_i} \text{ and } X_{\gamma_k}^{>\gamma} = \sum_{j=\nu+1}^{k-1} X_{\gamma_k}^{(\gamma_j,\gamma_{j+1}]},$$

we get

$$\tilde{R} \cdot X_{\gamma_k}^{>\gamma} = \sum_{i=0}^{g+1} \sum_{j=\nu+1}^{k-1} s_i X_{\tau_i} \cdot X_{\gamma_k}^{(\gamma_j,\gamma_{j+1}]} = \sum_{i=0}^{g+1} \sum_{j=\nu+1}^{k-1} s_i [X_{\tau_i} \cdot X_{\gamma_k}]_j.$$

Here $[X_{\tau_i} \cdot X_{\gamma_k}]_j = 0$ whenever $i \leq j < k$, because then $\tau_i < \gamma_j$. Therefore

$$\tilde{R} \cdot X_{\gamma_k}^{>\gamma} = \sum_{j=\nu+1}^{k-1} \sum_{i=j+1}^{g+1} s_i [X_{\tau_i} \cdot X_{\gamma_k}]_j.$$

As observed in Remark 5.1, $\gamma_k \in U$ and $\tau_i \notin U$ when $0 < i < g+1$. Note also that $\gamma_k \leq \tau_{g+1} = n$. Therefore $[X_{\tau_i} \cdot X_{\gamma_k}]_j = x_{\tau_i,\gamma_j}\rho_{\gamma_k,\gamma_j}$ for $0 < i \leq g+1$ by Proposition 5.2. If $j < i$, then $\gamma_j \leq \gamma_{i-1} \leq \tau_i$, and by Corollary 3.15 we get $x_{\tau_i,\gamma_j} = x_{\tau_i,\gamma_{i-1}} x_{\gamma_{i-1},\gamma_j}$, where $x_{\tau_i,\gamma_{i-1}} = 1$ according to Proposition 3.10. Hence

$$[X_{\tau_i} \cdot X_{\gamma_k}]_j = x_{\tau_i,\gamma_j}\rho_{\gamma_k,\gamma_j} = x_{\gamma_{i-1},\gamma_j}\rho_{\gamma_k,\gamma_j} \tag{51}$$

for $\nu < j < k$ and $j < i < g$. Subsequently, by Lemma 8.4 (ii)

$$\tilde{R} \cdot X_{\gamma_k}^{>\gamma} = \sum_{j=\nu+1}^{k-1} \left(\sum_{i=j+1}^{g+1} s_i x_{\gamma_{i-1},\gamma_j} \right) \rho_{\gamma_k,\gamma_j} = \sum_{j=\nu+1}^{k-1} m_j \rho_{\gamma_k,\gamma_j}.$$

By Proposition 5.8 we know that

$$\Sigma X_{\gamma_k}^{>\gamma} + 1 - x_{\gamma_k,\gamma} = \sum_{j=\nu+1}^{k-1} \rho_{\gamma_k,\gamma_j}.$$

Thus we get

$$\tilde{R} \cdot X_{\gamma_k}^{>\gamma} + \Sigma X_{\gamma_k}^{>\gamma} + 1 - x_{\gamma_k,\gamma} = \sum_{j=\nu+1}^{k-1} (m_j + 1)\rho_{\gamma_k,\gamma_j}.$$

Moreover, $[I \cdot X_{\gamma_k}]_j = a_{\gamma_j}\rho_{\gamma_k,\gamma_j}$ for every $j < k$ by Proposition 5.2, and so

$$I \cdot X_{\gamma_k}^{>\gamma} = \sum_{j=\nu+1}^{k-1} I \cdot X_{\gamma_k}^{(\gamma_j,\gamma_{j+1}]} = \sum_{j=\nu+1}^{k-1} [I \cdot X_{\gamma_k}]_j = \sum_{j=\nu+1}^{k-1} a_{\gamma_j}\rho_{\gamma_k,\gamma_j}.$$

Hence we finally obtain

$$\frac{\tilde{R} \cdot X_{\gamma_k}^{>\gamma} + \Sigma X_{\gamma_k}^{>\gamma} + 1 - x_{\gamma_k,\gamma}}{I \cdot X_{\gamma_k}^{>\gamma}} = \frac{\sum_{j=\nu+1}^{k-1}(m_j+1)\rho_{\gamma_k,\gamma_j}}{\sum_{j=\nu+1}^{k-1} a_{\gamma_j}\rho_{\gamma_k,\gamma_j}} = \Phi_k,$$

which leads to the contradiction (50). The proof is thus complete. □

As a corollary, we give the result for the monomial case.

Corollary 8.6. *Let $\mathfrak{a}$ be a simple complete $\mathfrak{m}_\alpha$-primary ideal in a two-dimensional regular local ring α. Let I be a point basis of $\mathfrak{a}$. Suppose that $\Gamma^* = \emptyset$. Write $a := \mathrm{ord}(\mathfrak{a})$ and $b := I^2/a$. Then the set $\mathcal{H}_\mathfrak{a}$ of the jumping numbers of the ideal $\mathfrak{a}$ is*

$$H := \left\{ \frac{s+1}{a} + \frac{t+1}{b} \,\middle|\, s,t \in \mathbb{N} \right\}.$$

Lemma 8.7. *In the setting of Theorem 8.3, we have for every $\nu \in \{0, \dots, g^*\}$*

$$\frac{1}{a_{\gamma_\nu}} + \frac{1}{b_\nu} \in H_\nu.$$

In particular, the subsets $H_0, \dots, H_{g^*}$ are non empty.

Proof. Obviously, the claim holds for $\nu = g^*$. Suppose that $0 \leq \nu < g^*$. Note that then $\nu < g$, which implies that $n > 1$, and further, $b_\nu > a_{\gamma_\nu}$ by Lemma 8.1. Recall also that $a_{\gamma_\nu} > a_{\gamma_{\nu+1}}$ by Proposition 3.5. Because $a_{\gamma_{\nu+1}} \mid a_{\gamma_\nu}$ by Proposition 3.7, it follows that $b_\nu > a_{\gamma_\nu} \geq 2a_{\gamma_{\nu+1}}$. Therefore

$$\frac{1}{a_{\gamma_\nu}} + \frac{1}{b_\nu} \leq \frac{1}{a_{\gamma_{\nu+1}}},$$

and then the claim follows from Theorem 8.3. Hence H_ν is a non empty, in fact an infinite set for every $\nu \in \{0, \dots, g^*\}$. □

Notation 8.8. *For every $\nu \in \{0, \dots, g^*\}$, write*

$$\xi'_\nu := \frac{1}{a_{\gamma_\nu}} + \frac{1}{b_\nu} \quad \big(= \min H_\nu \big).$$

Proposition 8.9. *Let $\mathfrak{a}$ be a simple complete ideal in a two-dimensional regular local ring α with the point basis $(a_1, \dots, a_n)$. Then $a_{\gamma_i}^{-1} \notin \mathcal{H}_\mathfrak{a}$ for any $i \in \{0, \dots, g^* + 1\}$. In particular, $1 \notin \mathcal{H}_\mathfrak{a}$.*

Proof. Let $\nu \in \{0, \dots, g^*\}$ and let us write $\gamma := \gamma_{\nu+1}$ and $\eta := \gamma_\nu$. According to Corollary 3.15 we have $a_\eta = a_\gamma x_{\gamma,\eta}$. Furthermore, by Lemma 8.1 we have $b_\nu = a_\gamma(x_{\gamma,\eta} X_\eta^2 + \rho_\nu)$ and $\gcd\{a_\eta, b_\nu\} = a_\gamma$. Let us write $a := a_\eta / a_\gamma$ and $b := b_\nu / a_\gamma$. Then a and b are positive integers with $\gcd\{a, b\} = 1$.

Assume that $1/a_\gamma \in H_\nu$. Theorem 8.3 then shows that there are positive integers u and v with

$$\frac{v}{a_\eta} + \frac{u}{b_\nu} = \frac{1}{a_\gamma}.$$

So $vb = (b - u)a$. This is impossible, because $\gcd\{a, b\} = 1$. Therefore $1/a_\gamma \notin H_\nu$, which further implies that $m/a_\gamma \notin H_\nu$ for any $m \in \mathbb{N}$ and $\nu < g^*$. Especially, if $i > \nu$, then we may choose $m = x_{\gamma_i,\gamma}$, and we see by Corollary 3.15 that $1/a_{\gamma_i} \notin H_\nu$. If $i \leq \nu$, then $a_{\gamma_i} \geq a_\eta$, which shows that

$$\frac{1}{a_{\gamma_i}} < \frac{1}{a_\eta} + \frac{1}{b_\nu} = \min H_\nu.$$

Therefore $a_{\gamma_i}^{-1} \notin H_\nu$ for any i and for any ν, i.e., $a_{\gamma_i}^{-1} \notin \mathcal{H}_\mathfrak{a}$. □

Remark 8.10. By Proposition 8.9 we observe that we could define the sets H_ν for $\nu = 0, \dots, g^* - 1$ in Theorem 8.3 as follows:

$$H_\nu := \left\{ \frac{s+1}{a_{\gamma_\nu}} + \frac{t+1}{b_\nu} + \frac{m}{a_{\gamma_{\nu+1}}} \,\middle|\, s,t,m \in \mathbb{N}, \frac{s+1}{a_{\gamma_\nu}} + \frac{t+1}{b_\nu} < \frac{1}{a_{\gamma_{\nu+1}}} \right\}.$$

Proposition 8.11. *Let $\mathfrak{a}$ be a simple complete ideal in a two-dimensional regular local ring α with the point basis $I = (a_1, \dots, a_n)$. Then*

$$\{c \in H_{g^*} \mid c > 1\} = \left\{ 1 + \frac{k+1}{I^2} \,\middle|\, k \in \mathbb{N} \right\}.$$

Especially, every integer greater than one is a jumping number. Moreover, $1 + 1/I^2$ is the smallest jumping number at least one, while $1 - 1/(I^{\leq \gamma_g})^2$ is the greatest jumping number at most one, whenever $g > 0$.

Proof. Let the sets $H_0, \dots, H_{g^*}$ be as in Theorem 8.3. For $0 \leq \nu \leq g^*$, let us write $\eta := \gamma_\nu$ and $\gamma := \gamma_{\nu+1}$. Set $a := a_\eta / a_\gamma$ and $b := b_\nu / a_\gamma$. Proposition 3.12 yields $I^{\leq \gamma} = a_\gamma X_\gamma$, and thus $ab = X_\gamma^2$. By Lemma 8.1 $\gcd\{a_\eta, b_\nu\} = a_\gamma$, i.e., $\gcd\{a, b\} = 1$. It follows now from Lemma 8.5 that we can find $s, t \in \mathbb{N}$ for any $k \in \mathbb{N}$ satisfying

$$ab + k + 1 = (t+1)a + (s+1)b. \tag{52}$$

Recall also that by Theorem 8.3 every element in H_ν is of the form

$$\frac{(t+1)a + (s+1)b + mab}{a_\gamma ab} \tag{53}$$

for some $s, t, m \in \mathbb{N}$. Thereby we observe that

$$H_\nu \subset \left\{ \frac{k+1}{(I^{\leq \gamma})^2} \,\middle|\, k \in \mathbb{N} \right\} \tag{54}$$

for every $\nu \in \{0, \dots, g^*\}$.

Choosing $\nu = g^*$ gives $X_\gamma = X_{\gamma_{g^*+1}} = I$ and $a_\gamma = a_{\gamma_{g^*+1}} = 1$. Moreover, then $a = a_{\gamma_{g^*}}$ and $b = b_{\gamma_{g^*}}$. Equations (52) and (54) now yield that

$$\left\{ 1 + \frac{k+1}{I^2} \,\middle|\, k \in \mathbb{N} \right\} = \{c \in H_{g^*} \mid c > 1\}.$$

Subsequently, $1+1/I^2$ is the least element in H_{g^*} greater than one, and every integer greater than one is in H_{g^*}. Furthermore, Equation (54) implies that $1 + 1/I^2$ must be the smallest jumping number at least one, as $1 \notin \mathcal{H}_\mathfrak{a}$ by Proposition 8.9, For the last claim, note first that if $g = 0$, then $g^* = 0$. In that case Theorem 8.3 implies that $\mathcal{H}_{\mathfrak{a}} = H_{g^*}$, and $a_{\gamma_{g^*+1}} = a_{\gamma_{g^*}} = 1$ by Proposition 3.5. Especially, Equation (54) then shows that every jumping number of $\mathfrak{a}$ is greater than one.

Suppose next that $g > 0$. Observe that if $c \in H_\nu$ and $c \leq 1$, then it follows from Equation (53) that $\nu < g$, because $a_{\gamma_g} = a_{\gamma_{g+1}} = 1$ by Proposition 3.5. Set $\nu = g-1$.Then we get $a > a_\gamma = 1$ by Proposition 3.5, and By Lemma 8.1 $a \leq b$. Let $k = a-2$ in Equation (52), and take $s, t \in \mathbb{N}$ accordingly so that $ab + a - 1 = (s+1)b + (t+1)a$. Note that t must be positive as otherwise $b \mid ab-1$ implying $b = 1$. But then $1 < a \leq b = 1$, which is impossible. Thus

$$(I^{\leq \gamma_g})^2 - 1 = ab - 1 = (s+1)b + (t'+1)a,$$

where $t' = t - 1 \in \mathbb{N}$. In particular, Theorem 8.3 now shows that

$$1 - \frac{1}{(I^{\leq \gamma_g})^2} = \frac{s+1}{a} + \frac{t'+1}{b} \in H_{g-1}.$$

Moreover, it follows now from Equation (54) that this must be the maximal jumping number at most one. □

Corollary 8.12. *Let $\mathfrak{a}$ be a simple complete ideal in a two-dimensional regular local ring α and let $\mathcal{H}_{\mathfrak{a}}$ denote the set of the jumping numbers of $\mathfrak{a}$. The Hilbert-Samuel multiplicity of $\mathfrak{a}$ is*

$$e(\mathfrak{a}) = (\xi' - 1)^{-1},$$

where $\xi' := \min\{\xi \in \mathcal{H}_{\mathfrak{a}} \mid \xi > 1\}$.

Proof. It follows from the Hoskin-Deligne formula that the Hilbert-Samuel multiplicity of $\mathfrak{a}$ is I^2, where I denotes the point basis vector of $\mathfrak{a}$ (see [13, Corollary 3.8]). By Proposition 8.11 we now know that $\xi' = (1 + I^2)^{-1}$. □

Lemma 8.13. *For $\mu, \nu \in \{0, \ldots, g^*\}$ set*

$$\theta_{\mu,\nu} := \min\left\{\xi \in H_\mu \mid \xi \geq \frac{1}{a_{\gamma_\nu}}\right\}.$$

Then

$$\theta_{\mu,\nu} = \begin{cases} \xi'_\mu + \dfrac{1}{a_{\gamma_\nu}} & \text{if } \mu < \nu; \\ \xi'_\mu & \text{if } \mu \geq \nu. \end{cases}$$

In particular, $\theta_{\mu,\nu} \geq \xi'_\nu$, where the equality holds if and only if $\mu = \nu$.

Proof. Since $1/a_{\gamma_\nu} < \xi'_\nu = \min H_\nu$, we observe that $\theta_{\nu,\nu} = \xi'_\nu$.

Suppose that $\mu > \nu$. Then $a_{\gamma_{\nu+1}} \geq a_{\gamma_\mu}$ by Proposition 3.5. As $\nu < g^*$ and $\xi'_\nu \in H_\nu$ by Lemma 8.7, Theorem 8.3 shows that $\xi'_\nu \leq 1/a_{\gamma_{\nu+1}}$. Hence

$$\xi'_\nu = \frac{1}{a_{\gamma_\nu}} + \frac{1}{b_\nu} \leq \frac{1}{a_{\gamma_{\nu+1}}} \leq \frac{1}{a_{\gamma_\mu}} < \frac{1}{a_{\gamma_\mu}} + \frac{1}{b_\mu} = \xi'_\mu,$$

which yields $\xi'_\nu < \xi'_\mu = \theta_{\mu,\nu}$.

Suppose next that $\mu < \nu$. In particular, this gives $\mu < g^*$. As $\theta_{\mu,\nu} \in H_\mu$, we then know by Theorem 8.3 that $\theta_{\mu,\nu} = c + m/a_{\gamma_{\mu+1}}$ for some $m \in \mathbb{N}$ and $c \in H_\mu$ with $c \leq 1/a_{\gamma_{\mu+1}}$. By Proposition 8.9 we know that $\theta_{\mu,\nu} > 1/a_{\gamma_\nu}$, and since $1/a_{\gamma_\nu} = x_{\gamma_\nu,\gamma_{\mu+1}}/a_{\gamma_{\mu+1}}$ by Corollary 3.15, we observe that $m = x_{\gamma_\nu,\gamma_{\mu+1}}$ and $c = \xi'_\mu$. Thereby $\theta_{\mu,\nu} = \xi'_\mu + 1/a_{\gamma_\nu}$.

Because $a_{\gamma_\mu} b_\mu = (I^{\leq\gamma_{\mu+1}})^2 \leq (I^{\leq\gamma_{\nu+1}})^2 = a_{\gamma_\nu} b_\nu$, we have $b_\mu \leq b_\nu$. Thus

$$\xi'_\mu = \frac{1}{a_{\gamma_\mu}} + \frac{1}{b_\mu} > \frac{1}{b_\nu} = \xi'_\nu - \frac{1}{a_{\gamma_\nu}},$$

which shows that $\theta_{\mu,\nu} > \xi'_\nu$, as wanted. □

Proposition 8.14. *In the setting of Theorem 8.3 we get for $\nu \in \{0, \dots, g^*\}$*

$$\xi'_\nu = \min\left\{\xi \in \mathcal{H}_\mathfrak{a} \,\middle|\, \xi \geq \frac{1}{a_{\gamma_\nu}}\right\}.$$

Moreover, $\xi'_\nu \in H_\mu$ if and only if $\mu = \nu$, and

$$\frac{1}{a_{\gamma_0}} < \xi'_0 < \cdots < \frac{1}{a_{\gamma_{g^*}}} < \xi'_{g^*}.$$

Proof. Note first that $1/a_{\gamma_\nu} < \xi'_\nu$ for every $\nu = 0, \dots, g^*$. Furthermore, $\xi'_\nu \in H_\nu$ by Lemma 8.7. By Remark 8.10 we then see that $\xi'_\nu < 1/a_{\gamma_{\nu+1}}$ for $\nu < g^*$. If $\xi'_\nu \in H_\mu$ for some $\mu, \nu \in \{0, \dots, g^*\}$, then $\theta_{\mu,\nu} \leq \xi'_\nu$ by definition as $\xi'_\nu > 1/a_{\gamma_\nu}$. It follows from Lemma 8.13 that $\theta_{\mu,\nu} = \xi'_\nu$, and further, $\mu = \nu$. Moreover, $\xi'_\nu = \min\{\theta_{\mu,\nu} \mid \mu = 0, \dots, g^*\}$ for any $\nu = 0, \dots, g^*$. Hence

$$\xi'_\nu = \min\left\{\xi \in \mathcal{H}_\mathfrak{a} \,\middle|\, \xi \geq \frac{1}{a_{\gamma_\nu}}\right\}.$$

□

Corollary 8.15. *Let $\mathfrak{a}$ be a simple complete ideal of finite colength in a two-dimensional regular local ring α. The sequence of pairs $(a_{\gamma_0}, b_0), \dots, (a_{\gamma_{g^*}} b_{g^*})$ and thereby the set of the jumping numbers of $\mathfrak{a}$, is totally determined by the numbers $\xi'_0, \dots, \xi'_{g^*}$.*

Proof. For $\nu \in \{0,\dots,g^*\}$, write $\gamma := \gamma_{\nu+1}$, $\eta := \gamma_\nu$, $u_\nu := a_\eta/a_\gamma$ and $v_\nu := b_\nu/a_\gamma$. Corollary 3.15 gives $u_\nu = x_{\gamma,\eta}$ while $v_\nu = x_{\gamma,\eta}X_\eta^2 + \rho_\nu$ by Lemma 8.1. Then

$$\xi'_\nu = \frac{1}{a_\eta} + \frac{1}{b_\nu} = \frac{a_\eta + b_\nu}{a_\eta b_\nu} = \frac{1}{a_\gamma} \cdot \frac{u_\nu + v_\nu}{u_\nu v_\nu}.$$

Moreover, by Lemma 8.1 we have

$$\gcd\{u_\nu + v_\nu, u_\nu v_\nu\} = \gcd\{u_\nu, v_\nu\} = \frac{\gcd\{a_\eta, b_\nu\}}{a_\gamma} = 1.$$

Thus a_γ and ξ'_ν determine $u_\nu + v_\nu$ and $u_\nu v_\nu$. Note that u_ν and v_ν are the roots of the quadratic equation $\omega^2 - (u_\nu + v_\nu)\omega + u_\nu v_\nu = 0$. So u_ν and v_ν are uniquely determined by a_γ and ξ'_ν.

Given all the numbers $\xi'_0,\dots,\xi'_{g^*}$, suppose that we would know the integer a_γ as well as the pair (u_ν, v_ν) for some $\nu > 0$. Then we obtain $a_\eta = a_\gamma u_\nu$, and from the product $a_\eta \xi'_{\nu-1}$ we get the pair $(u_{\nu-1}, v_{\nu-1})$ as described above. Because $a_{\gamma_{g^*+1}} = 1$, we see that ξ_{g^*} yields $(u_{g^*}, v_{g^*}) = (a_{\gamma_{g^*}}, b_{g^*})$ so that we eventually get all the pairs $(u_0, v_0), \dots, (u_{g^*}, v_{g^*})$ as well as the integers $a_{\gamma_0},\dots,a_{\gamma_{g^*}}$. Subsequently, we get the sequence $(a_{\gamma_0}, b_0),\dots,(a_{\gamma_{g^*}}, b_{g^*})$, and the claim now follows from Theorem 8.3. □

Lemma 8.16. *In the setting of Theorem 8.3, consider the set*

$$H' := \left\{\xi \in \mathcal{H}_{\mathfrak{a}} \mid \xi \le \frac{1}{a_{\gamma_1}}\right\}.$$

Write $a := a_{\gamma_0}/a_{\gamma_1}$ *and* $b := b_0/a_{\gamma_1}$. *Then* $H' = \{\xi \in H_0 \mid \xi < 1/a_{\gamma_1}\}$, *and*

(i) H' *is empty, if and only if* $g = 0$;

(ii) H' *has exactly one element, if and only if* $(a, b) = (2, 3)$;

(ii) H' *has exactly two elements, if and only if* $(a, b) = (2, 5)$.

Proof. Observe that Lemma 8.1 yields $\gcd\{a_{\gamma_0}, b_0\} = a_{\gamma_1}$ and $a_{\gamma_0} \le b_0$. Thus $a \le b$ are positive integers. Note also that $\xi'_\nu > 1/a_{\gamma_1}$ whenever $\nu \ge 1$ by Proposition 8.14. Thereby $H' = \{\xi \in H_0 \mid \xi < 1/a_{\gamma_1}\}$ where the inequality is strict by Remark 8.10.

(i) $H' = \emptyset$ exactly when $a_{\gamma_1}\xi'_0 = 1/a + 1/b > 1$. This is the case if and only if $a = 1$, i.e., $a_{\gamma_0} = a_{\gamma_1}$. By Proposition 3.5 this is equivalent to $g = 0$.

(ii) Theorem 8.3 implies that H' has exactly one element, if and only if $\xi'_0 \in H'$, while $1/a_{\gamma_0} + 2/b_0 \notin H'$. This happens exactly when

$$\frac{1}{a} + \frac{1}{b} < 1 \text{ and } \frac{1}{a} + \frac{2}{b} > 1.$$

Clearly, this takes place if and only if $(a,b)=(2,3)$.

(iii) It follows from Theorem 8.3 that H' has exactly two elements, if and only if both ξ'_0 and $1/a_{\gamma_0}+2/b_0$ are in H' while $1/a_{\gamma_0}+3/b_0$ and $2/a_{\gamma_0}+1/b_0$ are not. This is equivalent to the condition

$$\frac{1}{a}+\frac{2}{b}<1<\min\left\{\frac{1}{a}+\frac{3}{b},\frac{2}{a}+\frac{1}{b}\right\}.$$

Obviously $a>1$. If $a\geq 3$ then $2/a+1/b\leq 1$ for any $b\geq a$. Therefore $a=2$, and the inequality on the left implies that $b>4$. If $b\geq 6$ then $1/2+2/b<1$. Thus the only possible solution to this is $(a,b)=(2,5)$. □

Theorem 8.17. *The point basis $I=(a_1,\ldots,a_n)$ of a simple complete ideal $\mathfrak{a}$ of finite colength in a two-dimensional regular local ring α can be read off from the set $\mathcal{H}_\mathfrak{a}$ of its jumping numbers. In particular, the jumping numbers belonging to the subset $\mathcal{H}':=\{c\in\mathcal{H}_\mathfrak{a}\mid 0<c<1\}$ determine the multiplicities $a_1,\ldots,a_{\gamma_g}$.*

Proof. As $a_{\gamma_1}\geq 1$, Lemma 8.16 (i) implies that $\mathcal{H}'=\emptyset$, if and only if $g=0$. Then $a_1=\cdots=a_n=1$, i.e., $I^2=n$. Subsequently, Proposition 8.11 yields $\xi'_0=1+1/n$ so that $n=1/(\xi'_0-1)$, and we are done in this case.

Consider then the case $\mathcal{H}'\neq\emptyset$, i.e., $g\geq 1$. Suppose first that besides the jumping numbers of $\mathfrak{a}$, we would already know the multiplicities $a_1,\ldots,a_{\gamma_g}$. According to Proposition 3.5 $a_i=1$ for every $i=\gamma_g,\ldots,n$. Thus it remains to determine the number n. Let ξ' be as above, and let $\xi'':=\max\{c\in\mathcal{H}_\mathfrak{a}\mid c<1\}$. By Proposition 8.11 we know that $\xi'=1+1/I^2$, while $\xi''=1-1/(I^{\leq\gamma_g})^2$. Therefore we see that

$$n-\gamma_g=I^2-(I^{\leq\gamma_g})^2=\frac{1}{\xi'-1}+\frac{1}{\xi''-1}.$$

Let us then prove that the set $\mathcal{H}'$ determines the multiplicities $a_1,\ldots,a_{\gamma_g}$. According to Proposition 3.7 it is sufficient to find the rational numbers $\beta'_1,\ldots,\beta'_g$. Recall that $\beta'_{\nu+1}:=1+\rho_{n,\gamma_\nu}/a_{\gamma_\nu}$ by Remark 3.14. Since

$$a_{\gamma_\nu}b_\nu=(I^{\leq\gamma_{\nu+1}})^2=a_{\gamma_0}^2+[I^2]_0+\cdots+[I^2]_\nu$$

for every $\nu=0,\ldots,g-1$, we get by Proposition 5.2

$$a_{\gamma_\nu}b_\nu=a_{\gamma_0}^2+a_{\gamma_0}\rho_{n,\gamma_0}+\cdots+a_{\gamma_\nu}\rho_{n,\gamma_\nu}.$$

As $a_{\gamma_\nu}b_\nu-a_{\gamma_{\nu-1}}b_{\nu-1}=a_{\gamma_\nu}\rho_{n,\gamma_\nu}$, it is enough to find out all the pairs

$$(a_{\gamma_0},b_0),\ldots,(a_{\gamma_{g-1}},b_{g-1}).$$

Arguing inductively, suppose that for some $1 \leq k \leq g-1$ we know the pairs $(a_{\gamma_0}, b_0), \ldots, (a_{\gamma_{k-1}}, b_{k-1})$. Then we obtain by Lemma 8.1

$$a_{\gamma_k} = \gcd\{a_{\gamma_{k-1}}, b_{k-1}\},$$

and by Proposition 3.5 we know that $a_{\gamma_k} > a_{\gamma_g} = 1$. The least element in $\mathcal{H}_{\mathfrak{a}}$ greater than $1/a_{\gamma_k}$ is by Proposition 8.14

$$\xi'_k = \frac{1}{a_{\gamma_k}} + \frac{1}{b_k},$$

and because $1 < a_{\gamma_k} \leq b_k$ by Lemma 8.1 we see that ξ'_k lies in the set $\mathcal{H}'$. Now $b_k = (\xi'_k - 1/a_{\gamma_k})^{-1}$. Thereby we get also the pair (a_{γ_k}, b_k).

The problem is to find the first pair of integers (a_{γ_0}, b_0). By Theorem 8.18 below the three smallest jumping numbers determine the order of the ideal $\mathfrak{a}$, which is precisely the integer a_{γ_0}. Then $b_0 = (\xi'_0 - 1/a_{\gamma_0})^{-1}$. Thus everything is clear, if $\mathcal{H}'$ has at least three elements.

It remains to consider the two special cases where $\mathcal{H}'$ has either two or only one element. Let us show that then $g = 1$. We already saw that $\mathcal{H}' \neq \emptyset$ implies $g \geq 1$. Suppose that we would have $g > 1$. Then it follows from Proposition 3.5 that $a_{\gamma_1} > a_{\gamma_g} = 1$, i.e., $a_{\gamma_1} \geq 2$. Proposition 8.14 implies that $\xi'_0 < 1/a_{\gamma_1}$, and further, $\xi'_0 < \xi'_1 < \xi'_0 + 1/a_{\gamma_1} < 2/a_{\gamma_1} \leq 1$. This means that ξ'_0, ξ'_1 and $\xi'_0 + 1/a_{\gamma_1}$ are all in $\mathcal{H}'$, which is impossible. Therefore we have $g = 1$ in both cases so that $a_{\gamma_1} = 1$ by Proposition 3.5. Write $a := a_{\gamma_0}$ and $b := b_0$.

Suppose first that ξ'_0 is the only element in $\mathcal{H}'$. By Lemma 8.16 (ii) we must then have $(a, b) = (2, 3)$. In this case we get

$$(a_1, \ldots, a_{\gamma_g}) = (2, 1, 1).$$

Suppose next that $\mathcal{H}'$ has just two elements. Then by Lemma 8.16 (iii) we have $(a, b) = (2, 5)$, in which case

$$(a_1, \ldots, a_{\gamma_g}) = (2, 2, 1, 1).$$

Thereby we are done as soon as we have proven the next theorem. □

Theorem 8.18. *Let $\mathfrak{a}$ be a simple complete ideal in a two-dimensional regular local ring α. Let $\xi < \psi < \zeta$ be the three smallest jumping numbers of $\mathfrak{a}$, and let H_0 be as in Theorem 8.3. Then the order of the ideal $\mathfrak{a}$ is*

$$\operatorname{ord}(\mathfrak{a}) = \begin{cases} \dfrac{5}{3\xi} & \text{if } 6\xi = 10\psi - 5\zeta; \\[2ex] \dfrac{1}{2\xi - \psi} & \text{if } 6\xi \neq 10\psi - 5\zeta. \end{cases}$$

Moreover, $\operatorname{ord}(\mathfrak{a}) = 1$ *if and only if* $\xi > 1$*, and if* $\xi < 1 < \zeta$*, then* $\operatorname{ord}(\mathfrak{a}) = 2$.

Proof. Let $a_{\gamma_0}, \ldots, a_{\gamma_g}$ and $b_0, \ldots, b_{g^*}$ be as in Theorem 8.3. Note that $\mathrm{ord}(\mathfrak{a}) = a_{\gamma_0}$. By Proposition 8.14 we know that

$$\xi = \xi'_0 = \frac{1}{a_{\gamma_0}} + \frac{1}{b_0} \in H_0.$$

We shall first show that $\psi \in H_0$ implies $\mathrm{ord}(\mathfrak{a}) = 1/(2\xi - \psi)$, while $\psi \notin H_0$ gives $\mathrm{ord}(\mathfrak{a}) = 5/3\xi$. We shall then verify that $6\xi = 10\psi - 5\zeta$ is equivalent to $\psi \notin H_0$. This will prove the first claim.

Suppose first that $\psi \in H_0$. Because $a_{\gamma_0} \leq b_0$ by Lemma 8.1, it follows from Theorem 8.3 that necessarily

$$\psi = \frac{1}{a_{\gamma_0}} + \frac{2}{b_0},$$

provided that in the case $g^* \geq 1$ we can prove that $\psi \leq 1/a_{\gamma_1}$. Indeed, suppose that we would have $\psi > 1/a_{\gamma_1}$. As $g^* \geq 1$, we know by Proposition 8.14 that $\xi'_1 \notin H_0$. This means that $\psi < \xi'_1$. Lemma 8.13 yields $\xi'_1 < \theta_{0,1}$ so that $1/a_{\gamma_1} < \psi < \theta_{0,1}$ contradicting the definition of $\theta_{0,1}$. Thus we observe that if $\psi \in H_0$, then

$$a_{\gamma_0} = \frac{1}{2\xi - \psi}.$$

Suppose next that $\psi \notin H_0$. Then $g^* \geq 1$. By Proposition 8.14 we get $\psi = \xi'_1 > 1/a_{\gamma_1}$. So ξ is the only jumping number at most $1/a_{\gamma_1}$, in which case Lemma 8.16 (ii) gives $(a_{\gamma_0}, b_0) = (2a_{\gamma_1}, 3a_{\gamma_1})$. Hence

$$\xi = \frac{5}{6a_{\gamma_1}}, \text{ or equivalently } a_{\gamma_0} = \frac{5}{3\xi},$$

as wanted.

Let us now verify that in the case $\psi \notin H_0$ we have $6\xi = 10\psi - 5\zeta$. It is enough to prove that

$$\zeta = \frac{1}{a_{\gamma_1}} + \frac{2}{b_1}, \tag{55}$$

because then $2\psi - \zeta = 1/a_{\gamma_1} = 6\xi/5$, i.e., $6\xi = 10\psi - 5\zeta$. Observe that if $g^* = 1$ or if $g^* > 1$ and $1/a_{\gamma_1} + 2/b_1 < 1/a_{\gamma_2}$, then $1/a_{\gamma_1} + 2/b_1 \in H_1$ by Theorem 8.3. Subsequently it is the smallest jumpling number in H_1 greater than ψ, as $a_{\gamma_1} < b_1$.

We aim to show that $1/a_{\gamma_1} + 2/b_1 < \theta_{0,1}$ and that $\theta_{0,1} < 1/a_{\gamma_2}$ whenever $g^* > 1$. These will then yield that $1/a_{\gamma_1} + 2/b_1 \in H_1$ and that $\zeta \notin H_0$. The condition $\theta_{0,1} < 1/a_{\gamma_2}$ will also guarantee by Proposition 8.14 that $\zeta \notin H_\nu$ for any $\nu > 1$ in the case $g^* > 1$. But then the only possibility is $\zeta \in H_1$, which will then prove Equation (55).

Since $[I^2]_1 = a_{\gamma_1}\rho_{n,\gamma_1}$ by Proposition 5.2, we see that

$$b_1 = \frac{(I^{\leq\gamma_2})^2}{a_{\gamma_1}} = \frac{(I^{\leq\gamma_1})^2 + [I^2]_1}{a_{\gamma_1}} = \frac{a_{\gamma_0}b_0 + a_{\gamma_1}\rho_{n,\gamma_1}}{a_{\gamma_1}} = 6a_{\gamma_1} + \rho_{n,\gamma_1}.$$

This, together with Lemma 8.13, implies that

$$\frac{1}{a_{\gamma_1}} + \frac{2}{b_1} < \frac{1}{a_{\gamma_1}} + \frac{2}{6a_{\gamma_1}} < \xi + \frac{1}{a_{\gamma_1}} = \theta_{0,1}.$$

If $g^* > 1$, then it follows from Proposition 3.5 that $a_{\gamma_1} > a_{\gamma_2}$ so that

$$\theta_{0,1} = \xi + \frac{1}{a_{\gamma_1}} < \xi + \frac{1}{a_{\gamma_2}} = \theta_{0,2}$$

by Lemma 8.13. This means that $\theta_{0,1} < 1/a_{\gamma_2}$. The proof of Equation (55) is thus complete.

Assume then that $6\xi = 10\psi - 5\zeta$, and that we would have $\psi \in H_0$. As we saw above $2\xi - \psi = 1/a_{\gamma_0}$, which further yields $\psi - \xi = 1/b_0$.

Suppose first that $\zeta < 1/a_{\gamma_1}$. It follows from Proposition 8.14 that $\zeta \in H_0$. By applying Theorem 8.3 we observe that

$$\zeta = \min\left\{ \frac{s+1}{a_{\gamma_0}} + \frac{t+1}{b_0} \,\middle|\, (s,t) \in \mathbb{N}^2 \smallsetminus \{(0,0);(0,1)\} \right\}.$$

so that

$$\zeta = \frac{1}{a_{\gamma_0}} + \frac{3}{b_0} \quad \text{or} \quad \zeta = \frac{2}{a_{\gamma_0}} + \frac{1}{b_0}.$$

If $\zeta = 1/a_{\gamma_0} + 3/b_0$, then we get

$$10\psi - 6\xi = 5\zeta = 5(2\xi - \psi) + 15(\psi - \xi) = 9\xi + 10\psi,$$

which is impossible. If $\zeta = 2/a_{\gamma_0} + 1/b_0$, then

$$10\psi - 6\xi = 5\zeta = 10(2\xi - \psi) + 5(\psi - \xi) = 15\xi - 5\psi,$$

which yields $9/b_0 = 9\psi - 9\xi = 12\xi - 6\psi = 6/a_{\gamma_0}$, i.e., $a_{\gamma_0}/b_0 = 2/3$. It follows from Lemma 8.16 (ii) that $\psi \notin H_0$, which is a contradiction.

We must thus have $\zeta > 1/a_{\gamma_1}$. Note that $1/a_{\gamma_1}$ is not a jumping number by Proposition 8.9. Now ξ and ψ are the only jumping numbers at most $1/a_{\gamma_1}$. By Lemma 8.16 (iii) we then have $a_{\gamma_0} = 2a_{\gamma_1}$ and $b_0 = 5a_{\gamma_1}$. This gives

$$\frac{1}{a_{\gamma_1}} \leq \zeta = \frac{10\psi - 6\xi}{5} = \frac{4}{5a_{\gamma_0}} + \frac{14}{5b_0} = \frac{24}{25a_{\gamma_1}} < \frac{1}{a_{\gamma_1}},$$

which is a contradiction. The first claim has thus been proven.

It remains to prove the last two claims. Proposition 3.5 implies that $a_{\gamma_0} = 1$ if and only if $g = 0$. Moreover, it follows from Lemma 8.16 (i) that this happens exactly when $\xi > 1/a_{\gamma_1} = 1$.

Assume then that $\xi < 1 < \zeta$, in which case $g > 0$. This implies that the set H' in Lemma 8.16 has either one or two elements. Indeed, otherwise we would have $\zeta \in H'$ in which case $\zeta \leq 1/a_{\gamma_1} \leq 1$. So $a_{\gamma_0} = 2a_{\gamma_1}$ by Lemma 8.16 (ii) and (iii). We need to verify that $a_{\gamma_1} = 1$. If a_{γ_1} were greater than one, then we would have $g^* \geq 1$ because of Proposition 3.5. Lemma 8.13 would then yield $\xi_1' < \theta_{0,1} = \xi + 1/a_{\gamma_1}$. Since $\xi < 1/a_{\gamma_1} < \xi_1'$ by Proposition 8.14, we would then have $\xi < \xi_1' < \theta_{0,1}$ so that

$$1 < \zeta \leq \theta_{0,1} < \frac{2}{a_{\gamma_1}} \leq 1,$$

which is a contradiction. Therefore $a_{\gamma_1} = 1$, i.e., $a_{\gamma_0} = 2$. □

Example 8.19. Suppose that $I = (2, 1, 1)$. It follows from Theorem 8.3 that in Theorem 8.18 $\xi = 5/6$, $\psi = 7/6$ and $\zeta = 8/6$. Then $6\xi = 10\psi - 5\zeta$. On the other hand, in the case $I = (3, 1, 1, 1)$ $\xi = 7/12$, $\psi = 10/12$ and $\zeta = 11/12$ so that $6\xi \neq 10\psi - 5\zeta$.

9 Jumping numbers of an analytically irreducible plane curve

In this section, we aim to utilize Theorem 8.3 in determining the jumping numbers of an analytically irreducible plane curve with an isolated singularity at the origin. As jumping numbers are compatible with localization, it is enough to consider the local situation. Therefore in the following we mean by a plane curve the subscheme $\mathcal{C}_f$ of $\operatorname{Spec} \alpha$ determined by an element f in the maximal ideal $\mathfrak{m}_\alpha$. We will next recall some basic facts about plane curves. For more details, we refer to [2], [7] and [4].

As before, we assume that the residue field $\Bbbk$ of α is algebraically closed. If $\beta \supset \alpha$, then the *strict transform* of $\mathcal{C}_f$ is $\mathcal{C}_f^{(\beta)} := \operatorname{Spec} \beta/(f^{(\beta)})$, where $f^{(\beta)}$ denotes any generator of the transform $(f)^{(\beta)}$. The *multiplicity* of $\mathcal{C}_f$ at β is $m_\beta(\mathcal{C}_f) := \operatorname{ord}_\beta(f^{(\beta)})$. Following the terminology of [7], we call the set of those $\beta \supset \alpha$ for which $f^{(\beta)} \neq \beta$ the *point locus* of f (or $\mathcal{C}_f$).

Suppose that f is analytically irreducible. Then by [7, Corollary 4.8] there is a unique quadratic transform $\beta \supset \alpha$ belonging to the point locus of f. By [7, Corollary 4.8] $f^{(\beta)}$ is analytically irreducible. It follows that the point locus of f consists of a quadratic sequence

$$\alpha = \alpha_1 \subset \alpha_2 \subset \cdots .$$

This corresponds to a sequence of point blow ups

$$\operatorname{Spec} \alpha = \mathcal{X}_1 \xleftarrow{\pi_1} \mathcal{X}_2 \xleftarrow{\pi_2} \cdots . \tag{56}$$

There exists the smallest ν such that the total transform $(\pi_\nu \circ \cdots \circ \pi_1)^* \mathcal{C}_f$ has normal crossing support. The morphism $\bar{\pi} := \pi_\nu \circ \cdots \circ \pi_1 : \mathcal{X}_{\nu+1} \to \mathcal{X}_1$ is called the *standard resolution* of $\mathcal{C}_f$. The *multiplicity sequence* of $\mathcal{C}_f$ is now

$$(m_1, \dots, m_\nu),$$

where $m_i = m_{\alpha_i}(\mathcal{C}_f)$.

Let us recall the notion of a *general element* of an ideal. Fix a minimal system $f_1, \dots, f_\mu$ of generators for an ideal $\mathfrak{a}$ in α. Set $\bar{\lambda} := \lambda + \mathfrak{m}_\alpha \in \Bbbk$ for $\lambda \in \alpha$. One says that a general element of $\mathfrak{a}$ has some property $\mathcal{P}$, if there is a non empty open subset $V \subset \Bbbk^\mu$ such that $f = \lambda_1 f_1 + \cdots + \lambda_\mu f_\mu$ has $\mathcal{P}$ whenever $(\bar{\lambda}_1, \dots, \bar{\lambda}_\mu) \in V$. Note that the ideal can always be generated by general elements.

Suppose that $\mathfrak{a}$ is a simple complete ideal. It follows from [7, Corollary 4.10] that a general element $f \in \mathfrak{a}$ is analytically irreducible. As f is general, it easily follows that if $I = (a_1, \dots, a_n)$ is the point basis of $\mathfrak{a}$, then $(a_1, \dots, a_\nu)$ is the multiplicity sequence of $\mathcal{C}_f$. It is clear that the resolution (2) of $\mathfrak{a}$ contains the standard resolution of $\mathcal{C}_f$, i.e., if $\pi : \mathcal{X}_{n+1} \to \mathcal{X}_1$ is the resolution of $\mathfrak{a}$, then

$$\pi = \pi_n \circ \cdots \circ \pi_{\nu+1} \circ \bar{\pi}.$$

Let $\mathcal{C}_f^{(i)}$ denote the strict transform of $\mathcal{C}_f$ on $\mathcal{X}_i$. Since ν is the least integer such that $\bar{\pi}^* \mathcal{C}_f$ has only normal crossings, we observe that either $\nu = 1, 2$ or $\mathcal{C}_f^{\nu-1}$ intersects transversely the exceptional divisor of $\pi_{\nu-1}$ and the strict transform of some other exceptional divisor going through the center $\varsigma_\nu \in \mathcal{X}_\nu$ of π_ν. Therefore α_ν must be a satellite point or $\nu = 1, 2$. Furthermore, since $\mathcal{C}_f^{(i+1)}$ intersects for every $i \in \{\nu, \dots, n\}$ only one of the exceptional divisors and that transversely, we see that the points $\alpha_{\nu+1}, \dots, \alpha_n$ are free. It follows that $\nu = \gamma_g = \max \Gamma_{\mathfrak{a}}$.

In a lack of a suitable reference we state the following lemma:

Lemma 9.1. *Let $\mathfrak{a}$ be a simple complete ideal of finite colength in a two-dimensional regular local ring α having the resolution (2) and the point basis $(a_1, \dots, a_n)$. Let f be an analytically irreducible element in $\mathfrak{m}_\alpha$. The following conditions are then equivalent:*

(i) $m_{\alpha_i}(\mathcal{C}_f) = a_i$ *for* $i = 1, \ldots, n$;

(ii) $\pi^*\mathcal{C}_f = \mathcal{C}_f^{(n+1)} + \hat{E}_n$;

(iii) $\mathcal{C}_f^{(n+1)} \cdot E_i = \delta_{i,n}$ *for* $i = 1, \ldots, n$;

(iv) *if* $E_i^{(n)}$ *passes through* $\varsigma_n \in \mathcal{X}_n$ *for some* $i < n$*, then* $\mathcal{C}_f^{(n)}$ *intersects* $E_i^{(n)}$ *transversely at* ς_n*.*

Proof. We know that

$$\pi^*\mathcal{C}_f - \mathcal{C}_f^{(n+1)} = \sum_{j=1}^{n} m_{\alpha_j}(\mathcal{C}_f)E_j^*.$$

Since $\hat{E}_n = \sum_{j=1}^{n} a_j E_j^*$, this proves the equivalence of (i) and (ii). Write

$$\pi^*\mathcal{C}_f - \mathcal{C}_f^{(n+1)} = \sum_{j=1}^{n} \hat{d}_j \hat{E}_j.$$

By the projection formula $\pi^*\mathcal{C}_f \cdot E_i = 0$ for all $i = 1, \ldots, n$. As $\hat{E}_j \cdot E_i = \delta_{i,j}$ for all $j = 1, \ldots, n$, we now obtain $\hat{d}_i = \mathcal{C}_f^{(n+1)} \cdot E_i$. It thus follows that (ii) and (iii) are equivalent.

In order to prove the equivalence of (iii) and (iv) we first observe that in any case $\varsigma_n \in \mathcal{C}_f^{(n)}$. Because $\mathcal{C}_f$ is analytically irreducible, we see that if $E_i^{(n)}$ passes through ς_n, then ς_n is the only point of intersection of $\mathcal{C}_f^{(n)}$ and $E_i^{(n)}$. This implies that (iv) is equivalent to $\mathcal{C}_f^{(n)} \cdot E_i^{(n)} = 1$. As $\pi_n^* E_i^{(n)} = E_i + E_n$, we get by the projection formula

$$\mathcal{C}_f^{(n)} \cdot E_i^{(n)} = \mathcal{C}_f^{(n+1)} \cdot \pi_n^* E_i^{(n)} = \mathcal{C}_f^{(n+1)} \cdot E_i + \mathcal{C}_f^{(n+1)} \cdot E_n.$$

This immediately shows that (iii) implies (iv). Conversely, assuming (iv) and observing that the both summands on the right hand side are non negative and $\mathcal{C}_f^{(n+1)} \cdot E_n \neq 0$, we get $\mathcal{C}_f^{(n+1)} \cdot E_i^{(n+1)} = 0$ and $\mathcal{C}_f^{(n+1)} \cdot E_n = 1$. Thereby we see that (iii) holds. □

Remark 9.2. Following [20, Definition 7.1] and [3, Definition 1] we could define an element $f \in \mathfrak{a}$ to be general, if the corresponding curve $\mathcal{C}_f$ is analytically irreducible and $\mathcal{C}_f^{(n)}$ intersects transversely at $\varsigma_n \in \mathcal{X}_n$ every $E_i^{(n)}$ $(i < n)$ passing through ς_n.

We will now describe the correspondence between simple complete ideals and classes of equisingular plane curves following the exposition given in [19, II.5, p. 433]. The class of equisingular curves $\mathcal{L}$ corresponding to the ideal $\mathfrak{a}$ is defined to be the set of the analytically irreducible plane curves whose strict transform on $\mathcal{X}_n$ intersects transversely at the point ς_n the strict transform of any exceptional divisor passing through ς_n. Note that $\mathcal{L}$ is specified by a pair $(\mathcal{C}, t)$, where $\mathcal{C} \in \mathcal{L}$ is a curve and $t := n - \nu$, so that $\mathcal{L}$ is the collection of all the curves equisingular to $\mathcal{C}$ and sharing the $\nu + t$ first points of its point locus with $\mathcal{C}$.

Conversely, let $\mathcal{L}$ be the class of equisingular plane curves specified by the pair $(\mathcal{C}, t)$. Then the corresponding simple ideal $\mathfrak{a}$ is generated by the defining equations of the elements of $\mathcal{L}$. If the standard resolution of $\mathcal{C}$ is $\bar{\pi} : \mathcal{X}_{\nu+1} \to \mathcal{X}_1 = \operatorname{Spec} \alpha$, then $\mathfrak{a}$ is the ideal, whose resolution is $\pi = \pi_n \circ \cdots \circ \pi_{\nu+1} \circ \bar{\pi} : \mathcal{X}_{n+1} \to \mathcal{X}_1$, where $n = \nu + t$ and $\pi_i : \mathcal{X}_{i+1} \to \mathcal{X}_i$ is the blow up emerging in the sequence (56) corresponding to the point locus of $\mathcal{C}$.

For the convenience of the reader we state the following variant of [10, Proposition 9.2.28] adjusted to our case:

Proposition 9.3. *Let α be a two-dimensional regular local ring and let $\mathfrak{a}$ be a simple complete ideal in α having the resolution (2). Suppose that $\mathcal{C} \subset \operatorname{Spec} \alpha$ is an analytically irreducible plane curve, whose strict transform intersects transversely at the point ς_n the strict transform of any exceptional divisor passing through ς_n. Then the multiplier ideals of the curve $\mathcal{C}$ and the multiplier ideals of the ideal $\mathfrak{a}$ coincide in the interval $[0, 1[$. In particular, the jumping numbers of the curve $\mathcal{C}$ and the ideal $\mathfrak{a}$ coincide in the interval $[0, 1[$.*

Proof. Take a non-negative rational number c. The multiplier ideal $\mathcal{J}(c \cdot \mathcal{C}) \subset \alpha$ is defined by

$$\mathcal{J}(c \cdot \mathcal{C}) := \Gamma\big(\mathcal{X}, \mathcal{O}_{\mathcal{X}}\big(K_{\mathcal{X}} - \lfloor c \cdot \pi^* \mathcal{C} \rfloor\big)\big),$$

where $\pi^*\mathcal{C}$ is the total transform of $\mathcal{C}$ on $\mathcal{X}$. Since $\mathcal{C}^{(n)}$ intersects transversely at the point ς_n the strict transform of any exceptional divisor passing through ς_n, Lemma 9.1 (ii) implies $\pi^*\mathcal{C} = \mathcal{C}^{(n)} + \hat{E}_n$. Moreover, $\mathfrak{a}\mathcal{O}_{\mathcal{X}} = O_{\mathcal{X}}(-\hat{E}_n)$, and thus

$$\mathcal{J}(c \cdot \mathcal{C}) = \Gamma\big(\mathcal{X}, \mathcal{O}_{\mathcal{X}}\big(K_{\mathcal{X}} - \lfloor c \cdot \hat{E}_n \rfloor - \lfloor c \cdot \mathcal{C}^{(n)} \rfloor\big)\big),$$

Suppose that $0 \leq c < 1$. Then $\lfloor c \cdot \mathcal{C}^{(n)} \rfloor$ vanish, and we obtain

$$\mathcal{J}(c \cdot \mathcal{C}) = \Gamma\big(\mathcal{X}, \mathcal{O}_{\mathcal{X}}\big(K_{\mathcal{X}} - \lfloor c \cdot \hat{E}_n \rfloor\big)\big) = \mathcal{J}(\mathfrak{a}^c).$$

□

Theorem 9.4. *Let α be a two-dimensional regular local ring, and let $\mathfrak{a}$ be a simple complete ideal in α having the resolution (2). Let $\mathcal{C} \subset \operatorname{Spec} \alpha$ be an analytically irreducible plane curve, whose strict transform intersects transversely at the point ς_n the strict transform of any exceptional divisor passing through ς_n. Then the set of the jumping numbers of $\mathcal{C}$ is*

$$\mathcal{H}_{\mathcal{C}} = \{c + m \mid c \in \mathcal{H}_{\mathfrak{a}} \cup \{1\}, 0 < c \leq 1 \text{ and } m \in \mathbb{N}\},$$

where $\mathcal{H}_{\mathfrak{a}}$ denotes the set of the jumping numbers of $\mathfrak{a}$.

Proof. By the periodicity of the jumping numbers for integral divisors (see e.g. [6, Remark 1.15]) we know that $c > 0$ is a jumping number of $\mathcal{C}$ if and only if $c + 1$ is. Thus it is enough to find out the jumping numbers of $\mathcal{C}$ in the interval $]0, 1]$.

If $c < 1$, then by Proposition 9.3 $\mathcal{J}(c \cdot \mathcal{C}) = \mathcal{J}(\mathfrak{a}^c)$. Because $\lfloor c \cdot \mathcal{C}^{(n)} \rfloor = 0$ for $c < 1$, we must have $\mathcal{J}(c \cdot \mathcal{C}) \supsetneq \mathcal{J}(\mathcal{C})$ for $c < 1$ and thus 1 must be also a jumping number of $\mathcal{C}$. Hence $c \in]0, 1]$ is a jumping number of $\mathcal{C}$, if and only if $c \in \mathcal{H}_{\mathfrak{a}}$ or $c = 1$. □

This result can be utilized in determining the jumping numbers of an arbitrary analytically irreducible plane curve.

Corollary 9.5. *Let α be a two-dimensional regular local ring and let $\mathcal{C} \subset \operatorname{Spec} \alpha$ be an analytically irreducible plane curve. Let $\mathfrak{a}$ be the simple complete ideal in α corresponding to the class of equisingular plane curves specified by the pair $(\mathcal{C}, 0)$. Then the set of the jumping numbers of $\mathcal{C}$ is*

$$\mathcal{H}_{\mathcal{C}} = \{c + m \mid c \in \mathcal{H}_{\mathfrak{a}} \cup \{1\}, 0 < c \leq 1 \text{ and } m \in \mathbb{N}\}.$$

Proof. As we observed above, the standard resolution of $\mathcal{C}$ coincides with the resolution of $\mathfrak{a}$. Then the claim is a direct consequence of Theorem 9.4. □

Remark 9.6. By Corollary 9.5 we observe that the jumping numbers of $\mathcal{C}$ depend only on the equisingularity class of $\mathcal{C}$, because by Theorem 8.3 the jumping numbers of $\mathfrak{a}$ less than one are totally determined by the multiplicities $a_1, \ldots, a_{\gamma_g}$ of the point basis $(a_1, \ldots, a_n)$ of $\mathfrak{a}$.

Remark 9.7. Let $\mathcal{C}$ be an analytically irreducible plane curve, and let $(a_1, \ldots, a_\nu)$ be the multiplicity sequence of $\mathcal{C}$. Let X_i denote the i:th row of the inverse of the corresponding proximity matrix, and let $\{\gamma_1, \ldots, \gamma_g\}$ be the indices corresponding to the terminal satellites. Note that g is the genus of the curve $\mathcal{C}$, (see e.g. [1, Definition 3.2.1]). The *characteristic exponents* of $\mathcal{C}$ are

$$\beta_k := a_1 + \rho_{n,\gamma_0} + \cdots + \rho_{n,\gamma_{k-1}}, \text{ where } k = 0, \ldots, g.$$

It is easy to see by induction on k that $\gcd\{\beta_0, \ldots, \beta_k\} = a_{\gamma_k}$. Indeed, we first observe that $\beta_0 = a_1$ and $\beta_k = \beta_{k-1} + \rho_{n,\gamma_{k-1}}$ for every $0 < k \leq g$. Assume that $\gcd\{\beta_0, \ldots, \beta_{k-1}\} = a_{\gamma_{k-1}}$. Then by Proposition 3.7

$$\gcd\{\beta_0, \ldots, \beta_k\} = \gcd\{a_{\gamma_{k-1}}, \rho_{n,k-1}\} = a_{\gamma_k}.$$

The *characteristic pairs* or *Puiseux pairs* (in the case $\Bbbk = \mathbb{C}$) are the pairs of integers (m_k, n_k) for $k = 1, \ldots, g$, where

$$m_k := \frac{\beta_k}{a_{\gamma_k}} \text{ and } n_k := x_{\gamma_k,\gamma_{k-1}}$$

(see, e.g., [1, Remark 3.1.6]). We can obtain the pairs $(a_{\gamma_0}, b_0), \ldots, (a_{\gamma_g}, b_g)$ from these as follows: Corollary 3.15 yields

$$m_i = x_{\gamma_i,\gamma_0} + \rho_{\gamma_i,\gamma_0} + \cdots + \rho_{\gamma_i,\gamma_{i-1}}$$

and $a_{\gamma_{i-1}} = n_i \cdots n_g$ for $i = 1, \ldots, g$. Moreover, $\rho_{\gamma_i,\gamma_{i-1}} = m_i - n_i m_{i-1}$ for $i = 2, \ldots, g$, which further yields $\rho_{n,\gamma_{i-1}} = (n_{i+1} \cdots n_g)(m_i - n_i m_{i-1})$. Then also $a_{\gamma_{i-1}}/a_{\gamma_k} = n_i \cdots n_k$ for $i = 1, \ldots, k+1$. Proposition 5.2 implies that

$$a_{\gamma_k} b_k = I \cdot I^{\leq \gamma_{k+1}} = a_{\gamma_0}^2 + a_{\gamma_0}\rho_{n,\gamma_0} + \cdots + a_{\gamma_k}\rho_{n,\gamma_k}.$$

Writing $\varphi_1 = m_1$ and $\varphi_i = m_i - n_i m_{i-1}$, we then get

$$a_{\gamma_k} = n_{k+1} \cdots n_g \text{ and } b_k = \sum_{i=1}^{k+1} (n_{i+1} \cdots n_g)(n_i \cdots n_k)\varphi_i$$

for every $k = 1, \ldots, g$.

Theorem 9.8. *Let α be a two-dimensional regular local ring. The jumping numbers of an analytically irreducible plane curve $\mathcal{C} \subset \operatorname{Spec} \alpha$ less than one determine the equisingularity class of $\mathcal{C}$.*

Proof. Take the ideal $\mathfrak{a}$ corresponding to the class of equisingular curves specified by the pair $(\mathcal{C}, 0)$. Then the point basis of $\mathfrak{a}$ is the same as the multiplicity sequence $(a_1, \ldots, a_{\gamma_g})$ of $\mathcal{C}$. By Theorem 9.4 we know that the the jumping numbers of $\mathfrak{a}$, which are less than one, coincide with those of the curve $\mathcal{C}$. By Theorem 8.17 they determine the sequence of multiplicities $(a_1, \ldots, a_{\gamma_g})$. ∎

In Theorem 9.4 we saw how to obtain the jumping numbers of the analytically irreducible plane curve determined by a general element of a simple complete ideal from the jumping numbers of the ideal. In the next theorem we consider the converse situation.

Theorem 9.9. *Let α be a two-dimensional regular local ring, and let $\mathfrak{a}$ be a simple complete ideal in α having the resolution (2). Let $\mathcal{C} \subset \operatorname{Spec} \alpha$ be an analytically irreducible plane curve, whose strict transform intersects transversely at the point ς_n the strict transform of any exceptional divisor passing through ς_n. Then the set $\mathcal{H}_{\mathfrak{a}}$ of the jumping numbers of $\mathfrak{a}$ is*

$$\mathcal{H}_{\mathfrak{a}} = (\mathcal{H}_{\mathcal{C}} \setminus \{1\}) \cup \left\{ 1 + \frac{k+1}{v(\mathfrak{a})} \,\middle|\, k \in \mathbb{N} \right\},$$

where $v = v_n$ denotes the divisorial valuation associated to $\mathfrak{a}$.

Proof. As before, let $I = (a_1, \dots, a_n)$ denote the point basis of $\mathfrak{a}$ and let $\mathcal{H}_{\mathfrak{a}} = H_0 \cup \dots \cup H_{g^*}$ be the set of the jumping numbers of $\mathfrak{a}$ where the subsets H_ν are as in Theorem 8.3. Recall that $v(\mathfrak{a}) = I^2$ (see Remark 5.6). By Proposition 8.11

$$\left\{ 1 + \frac{k+1}{I^2} \,\middle|\, k \in \mathbb{N} \right\} = \{ c \in H_{g^*} \mid c > 1 \}.$$

Proposition 9.3 shows that $0 < c < 1$ is a jumping number of $\mathfrak{a}$, if and only if $c \in \mathcal{H}_{\mathcal{C}}$. Moreover, Proposition 8.9 says that $1 \notin \mathcal{H}_{\mathfrak{a}}$. In order to prove the claim, it is thereby enough to show that every element of H_ν greater than one is in $\mathcal{H}_{\mathcal{C}}$ for any $\nu < g^*$, and that every element of $\mathcal{H}_{\mathcal{C}}$ gerater than one is in $\mathcal{H}_{\mathfrak{a}}$.

Suppose that $\xi > 1$ and $\xi \in H_\nu$ for some $\nu < g^*$. Theorem 8.3 implies that $\xi = c + m/a_{\gamma_{\nu+1}}$ for some $c \in H_\nu$ and $m \in \mathbb{N}$, where $0 < c \leq a_{\gamma_{\nu+1}}^{-1}$. By Proposition 8.9 we even know that $c < a_{\gamma_{\nu+1}}^{-1}$. Now $m > (1-c)a_{\gamma_{\nu+1}}$ yields $m \geq a_{\gamma_{\nu+1}}$. Write $m = m' a_{\gamma_{\nu+1}} + m''$, where $m' \in \mathbb{N}$ and $m'' < a_{\gamma_{\nu+1}}$. Set $c' := c + m''/a_{\gamma_{\nu+1}}$. Then $c' \in H_\nu$ and $0 < c' < 1$. By Proposition 9.3 $c' \in \mathcal{H}_{\mathcal{C}}$. As $\xi = c' + m'$, it follows that $\xi \in \mathcal{H}_{\mathcal{C}}$.

Take $\xi \in \mathcal{H}_{\mathcal{C}}$ so that $\xi > 1$. By Proposition 8.11 the case is clear, if ξ is an integer. Let us then assume that ξ is not an integer. Proposition 9.4 implies that $\xi = c + m$, where m is a positive integer and $c \in H_\nu$ with $0 < c < 1$ for some $\nu \in \{0, \dots, g^*\}$. It follows from Theorem 8.3 that $c = c' + m'/a_{\gamma_{\nu+1}}$ for some $m' \in \mathbb{N}$ and for some $c' \in H_\nu$ satisfying $c' < a_{\gamma_{\nu+1}}^{-1}$. Thus $\xi = c' + (a_{\gamma_{\nu+1}} m + m')/a_{\gamma_{\nu+1}}$, and $\xi \in \mathcal{H}_{\mathfrak{a}}$ according to Theorem 8.3. This completes the proof. ∎

Remark 9.10. In the setting of Theorem 9.9, $\mathfrak{a}$ is the ideal corresponding to the pair $(\mathcal{C}, t)$, where $t = n - \gamma_g$. The jumping numbers of the curve $\mathcal{C}$ together with the integer t, or equivalently the integer n, then determine the jumping numbers of $\mathfrak{a}$. Indeed, according to Theorem 9.8 the integer γ_g as well as the entire multiplicity sequence $(a_1, \dots, a_{\gamma_g})$ of $\mathcal{C}$ can be obtained from the jumping numbers of the curve $\mathcal{C}$. Because $a_{\gamma_g} = \cdots = a_n = 1$ by Proposition 3.5, we then get $v(\mathfrak{a}) = a_1^2 + \cdots + a_{\gamma_g}^2 + n - \gamma_g$.

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