On a Theorem of Bers, with Applications to the Study of Automorphism Groups of Domains[1,2]

Steven G. Krantz

Abstract: We study and generalize a classical theorem of L. Bers that classifies domains up to biholomorphic equivalence in terms of the algebras of holomorphic functions on those domains. Then we develop applications of these results to the study of domains with noncompact automorphism groupg.

1 Introduction

For us a *domain* in complex space is a connected open set. If Ω is a domain then let $\mathcal{O}(\Omega)$ denote the algebra of holomorphic functions on Ω.

In 1948, Lipman Bers [BERS] proved the following elegant result:

Theorem 1.1 *Let Ω, Ω be domains in $\mathbb{C}$. If $\mathcal{O}(\Omega)$ is isomorphic to $\mathcal{O}(\widehat{\Omega})$ as an algebra, then the domain Ω is conformally equivalent to the domain $\widehat{\Omega}$.*

Since that time, this result has been generalized to domains in $\mathbb{C}^n$, and even to domains in Stein manifolds—see for instance [ZAM1], [ZAM2].

In the present paper we offer some other variants of Bers's theorem, and then develop applications of these results to the study of the automorphism groups of domains in complex space.

[1]**Subject Classification Numbers:** 32A38, 30H50, 32A10, 32M99.

[2]**Key Words:** Bers's theorem, algebras of holomorphic functions, noncompact automorphism group, biholomorphic equivalence.

2 Variants of Bers's Theorem

In this section we formulate several variants of Bers's theorem. They all have the same proof. For completeness, we provide here the proof of Bers's original theorem stated in the last section.

Proof of Theorem 1.1: In fact we shall prove the result in $\mathbb{C}^n$. Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $\mathcal{O}(\Omega)$ denote the algebra of holomorphic functions from Ω to $\mathbb{C}$. Bers's theorem says, in effect, that the algebraic structure of $\mathcal{O}(\Omega)$ characterizes Ω. We begin our study by introducing a little terminology.

Definition 2.1 Let $\Omega \subseteq \mathbb{C}$ be a domain. A $\mathbb{C}$-algebra homomorphism $\varphi : \mathcal{O}(\Omega) \to \mathbb{C}$ is called a *character* of $\mathcal{O}(\Omega)$. If $c \in \mathbb{C}$, then the mapping

$$\begin{aligned} e_c : \mathcal{O}(\Omega) &\to \mathbb{C}\,, \\ f &\mapsto f(c)\,, \end{aligned}$$

is called a *point evaluation*. Every point evaluation is a character.

It should be noted that, if $\varphi : \mathcal{O}(\Omega) \to \mathcal{O}(\widehat{\Omega})$ is not the trivial zero homomorphism, then $\varphi(1) = 1$. This follows because $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1) \cdot \varphi(1)$. On any open set where the holomorphic function $\varphi(1)$ does not vanish, we find that $\varphi(1) \equiv 1$. The result follows by analytic continuation.

It turns out that every character of $\mathcal{O}(\Omega)$ is a point evaluation. That is the content of the next lemma.

Lemma 2.2 *Let φ be a character on $\mathcal{O}(\Omega)$. Then $\varphi = e_c$ for some $c \in \Omega$. Indeed, $c = \varphi(id) \in \Omega$. Here id is defined by $id(z) = z$.*

Proof: Let c be defined as in the statement of the lemma. Let $f(z) = z - c$. Then

$$\varphi(f) = \varphi(\mathrm{id}) - \varphi(c) = c - c = 0\,.$$

If it were not the case that $c \in \Omega$ then the function f would be a unit in $\mathcal{O}(\Omega)$. But then

$$1 = \varphi(f \cdot f^{-1}) = \varphi(f) \cdot \varphi(f^{-1}) = 0\,.$$

That is a contradiction. So $c \in \Omega$.

Now let $g \in \mathcal{O}(\Omega)$ be arbitrary. Then we may write

$$g(z) = g(c) + f(z) \cdot \widetilde{g}(z)\,,$$

where $\widetilde{g} \in \mathcal{O}(\Omega)$. Thus

$$\varphi(g) = \varphi(g(c)) + \varphi(f) \cdot \varphi(\widetilde{g}) = g(c) + 0 = g(c) = e_c(g)\,.$$

We conclude that $\varphi = e_c$, as was claimed. □

Now we may prove Bers's theorem. We formulate the result in slightly greater generality than stated heretofore.

Theorem 2.3 *Let Ω, $\widehat{\Omega}$ be domains. Suppose that*

$$\varphi : \mathcal{O}(\Omega) \to \mathcal{O}(\widehat{\Omega})$$

is a $\mathbb{C}$-algebra homomorphism. Then there exists one and only one holomorphic mapping $h : \widehat{\Omega} \to \Omega$ such that

$$\varphi(f) = f \circ h \quad \textit{for all } f \in \mathcal{O}(\Omega)\,.$$

In fact, the mapping h is given by $h = \varphi(id)$.

The homomorphism φ is bijective if and only if h is conformal, that is, a one-to-one and onto holomorphic mapping from $\widehat{\Omega}$ to Ω.

Proof: Since we want the mapping h to satisfy $\varphi(f) = f \circ h$ for all $f \in \mathcal{O}(\Omega)$, it must in particular satisfy $\varphi(\mathrm{id}_\Omega) = \mathrm{id}_\Omega \circ h = h$. We take this as our definition of the mapping h.

If $a \in \widehat{\Omega}$, then $e_a \circ \varphi$ is a character of $\mathcal{O}(\Omega)$. Thus our lemma tells us that $e_a \circ \varphi$ must in fact be a point evaluation on Ω. As a result,

$$e_a \circ \varphi = e_c\,, \quad \text{with } c = (e_a \circ \varphi)(\mathrm{id}_\Omega) = e_a(h) = h(a)\,.$$

Thus, if $f \in \mathcal{O}(\Omega)$, then

$$\varphi(f)(a) = e_a(\varphi \circ f) = (e_a \circ \varphi)(f) = e_{h(a)}(f) = f(h(a)) = (f \circ h)(a)$$

for all $a \in \widehat{\Omega}$. We conclude that $\varphi(f) = f \circ h$ for all $f \in \mathcal{O}(\Omega)$.

For the last statement of the theorem, suppose that h is a one-to-one, onto conformal mapping of $\widehat{\Omega}$ to Ω. If $g \in \mathcal{O}(\Omega)$, then set $f = g \circ h^{-1}$. It follows that $\varphi(f) = f \circ h = g$. Hence φ is onto. Likewise, if $\varphi(f_1) = \varphi(f_2)$, then $f_1 \circ h = f_2 \circ h$ hence, composing with h^{-1}, $f_1 \equiv f_2$. So φ is one-to-one. Conversely, suppose that φ is an isomorphism. Let $a \in \Omega$ be arbitrary. Then e_a is a character on $\mathcal{O}(\Omega)$; hence $e_a \circ \varphi^{-1}$ is a character on $\mathcal{O}(\widehat{\Omega})$. By the lemma, there is a point $c \in \widehat{\Omega}$ such that $e_a \circ \varphi^{-1} = e_c$. It follows that

$$e_a = e_c \circ \varphi \,.$$

Applying both sides to id_Ω yields

$$e_a(\mathrm{id}_\Omega) = (e_c \circ \varphi)(\mathrm{id}_\Omega) \,.$$

Unraveling the definitions gives

$$a = e_c(\mathrm{id}_\Omega \circ h) = h(c) \,.$$

Thus $h(c) = a$ and h is surjective. The argument in fact shows that the pre-image c is uniquely determined. So h is also one-to-one. □

Now we formulate some variants of Bers's theorem. Again we stress that each has the same proof (the proof that we just presented).

In what follows, we shall be dealing with the space $L(\Omega)$ of Lipschitz functions on Ω. These are functions that satisfy a condition of the form

$$\sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|} \leq C \,. \tag{2.4}$$

As usual, we use the expression (2.4) to define a norm $\| \quad \|_{L(\Omega)}$ on $L(\Omega)$.

Proposition 2.5 *If Ω is a domain in $\mathbb{C}^n$, then let $L(\Omega)$ denote the algebra of Lipschitz holomorphic functions on Ω. The domains Ω and $\widehat{\Omega}$ in $\mathbb{C}^n$ are biholomorphically equivalent if and only if the algebras $L(\Omega)$ and $L(\widehat{\Omega})$ are isomorphic as algebras.*

Proposition 2.6 *The bounded domains Ω and $\widehat{\Omega}$ in $\mathbb{C}^n$ are biholomorphically equivalent, with a biholomorphism that is bi-Lipschitz, if and only if the algebras $L(\Omega)$ and $L(\widehat{\Omega})$ are isomorphic as algebras.*

We remark that it is possible to formulate versions of these results for Sobolev spaces of holomorphic functions, for Besov spaces of holomorphic functions, and in other contexts as well. We leave the details for the interested reader.

3 Applications

Our intention here is to study the automorphism groups of domains in $\mathbb{C}^n$. Here, if $\Omega \subseteq \mathbb{C}^n$ is a domain, then the automorphism group of Ω (denoted $\mathrm{Aut}\,(\Omega)$) is the collection of biholomorphic mappings of Ω to itself. The usual topology on $\mathrm{Aut}\,(\Omega)$ is that of uniform convergence on compact sets (equivalently, the compact-open topology). For a bounded domain Ω, this topology turns $\mathrm{Aut}\,(\Omega)$ into a real Lie group. Note, however, that the automorphism group of $\Omega = \mathbb{C}^n$ with $n > 1$ is infinite dimensional hence certainly *not* a Lie group.

If Ω is a fixed domain in $\mathbb{C}^n$ and if $f \in L(\Omega)$, then let us say that f is *noncompact* if there is a sequence $\varphi_j \in \mathrm{Aut}\,(\Omega)$ such that $\{f \circ \varphi_j\}$ is a noncompact set in $L(\Omega)$. Notice that, obversely, f is compact if $\{f \circ \varphi_j\}$ is a compact set in $L(\Omega)$ for every choice of φ_j.

Proposition 3.1 *Let Ω be a smoothly bounded, pseudoconvex domain in $\mathbb{C}^n$. Then Ω has noncompact automorphism group if and only if there exists an $f \in L(\Omega)$ such that f is noncompact.*

Proof: If the automorphism group is noncompact, then (by a classical result of H. Cartan), there exist $\varphi_j \in \mathrm{Aut}\,(\Omega)$, $P \in \Omega$, and $X \in \partial\Omega$ such that $\varphi_j(P) \to X$. By a result of Ohsawa (see [OHS]), the Bergman metric is complete. Fix a nonconstant $f \in L(\Omega)$. Choose $p, q \in \Omega$, $p \neq q$, so that

$$|p - q| \approx (1/|f\|_{L(\Omega)}) \cdot |f(p) - f(q)|\,.$$

We may suppose without loss of generality that $|p - q| = 1$.

Now certainly $|\varphi_j(p) - \varphi_j(q)| \to 0$ (since, by the completeness of the metric, both $\varphi_j(p)$ and $\varphi_j(q)$ must both tend to X). We may now calculate that

$$\begin{aligned} C &= C|p-q| \\ &\approx (1/\|f\|_{L(\Omega)}) \cdot |f(p) - f(q)| \\ &= (1/\|f\|_{L(\Omega)}) \cdot |f(\varphi_j^{-1}(\varphi_j(p))) - f(\varphi_j^{-1}(\varphi_j(q)))|\,. \end{aligned}$$

Since $|\varphi_j(p) - \varphi_j(q)| \to 0$, we see that $\{f \circ \varphi_j^{-1}\}$ has Lipschitz norm which is blowing up. So f is noncompact.

Conversely, if $\mathrm{Aut}\,(\Omega)$ is compact, then let $f \in L(\Omega)$ and consider $\{f \circ \varphi_j\}$ for $\varphi_j \in \mathrm{Aut}\,(\Omega)$. Examine

$$|f \circ \varphi_j(p) - f \circ \varphi_j(q)| \,. \tag{3.1.1}$$

Clearly, by compactness, $|\nabla \varphi_j|$ is bounded above and below, uniformly in j, on any compact set $K \subset\subset \Omega$. By the Ascoli-Arzela theorem applied on compact sets, we see from (3.1.1) that $f \circ \varphi_j$ has a convergent subsequence. $\square$

The next well-known result, due to Bun Wong [WON], is a cornerstone of the modern theory of automorphism groups of smoothly bounded domains. Now we present some new proofs of this result.

Theorem 3.2 *Let Ω be a smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^n$. Suppose that there a point $P \in \Omega$ and a strongly pseudoconvex boundary point $X \in \partial\Omega$ and that there exist $\varphi_j \in Aut\,(\Omega)$ such that $\varphi_j(P) \to X$. Then Ω is biholomorphic to the unit ball $B \subseteq \mathbb{C}^n$.*

Proof: As advertised, we shall sketch three proofs. We first note that, according to H. Cartan's theorem and our previous result, the hypotheses imply that there is an $f \in L(\Omega)$ which is noncompact.

First Proof of the Theorem: If Ω is *not* biholomorphic to the ball then, by a celebrated result of Lu Qi-Keng [LQK] (see [GKK] for thorough discussion), there is a point Q in Ω where the holomorphic sectional curvature of the Bergman metric is not the constant holomorphic sectional curvature of the ball.

As noted in the proof of the preceding result, the Bergman metric is complete on Ω. So in fact any compact set $K \subset\subset \Omega$ has the property that $\{\varphi_j\}$ converges uniformly on K to X. In particular, $\varphi_j(Q) \to X$. But it can be calculated (see [KLE], [GK1], [GKK]) that the holomorphic sectional curvature of the Bergman metric tends to the constant curvature of the ball at points that approach a strongly pseudoconvex boundary point X. That contradicts the last sentence of the previous paragraph.

We conclude that Ω *is* biholomorphic to the ball, as claimed. $\square$

Second Proof of the Theorem: It is convenient for this argument to equip $\mathcal{O}(\Omega)$ with the topology of uniform convergence on compact sets (i.e., the compact-open topology). For convenience, and without any loss of generality, we restrict attention now to ambient dimension 2.

Let U be a small neighborhood of X. Since X is a peak point (see [KRA1]), it is standard to argue that, for any compact set $K \subseteq \Omega$, there is a J so large that $j > J$ implies that $\varphi_j(K) \subseteq U \cap \Omega$. Let X' be a point of $U \cap \Omega$ that is very near to X. Let $\delta = \delta_j = \text{dist}\,(X', \partial\Omega)$. After a normalization of coordinates, we may suppose that the complex normal direction at X is z_1 and the complex tangential direction at X is z_2.

Define $\psi(z_1, z_2) = (X_1' + (z_1 - X_1')/\delta, X_2' + (z_2 - X_2')/\sqrt{\delta})$. Then $\psi \circ \varphi_j$, with j as above, will have Lipschitz norm that is bounded, independent of j. As a result, using a sequence of compact sets K_j that exhausts Ω, and neighborhoods U that shrink to X, we may derive a subsequence, convergent on compact sets. And it will converge to a mapping of Ω to the Siegel upper half space. [This is just the standard method of scaling, which is described in detail in [GKK]). So Ω is biholomorphic to the Siegel upper half space, which is in turn biholomorphic to the unit ball. $\square$

Third Proof of the Theorem: For this proof we examine the Fefferman asymptotic expansion for the Bergman kernel near a strongly pseudoconvex boundary point (see [FEF] and also [GKK]). This says that, in suitable local coordinates,

$$K(z, \zeta) = \frac{\psi(z, \zeta)}{[-X(z, \zeta)]^{n+1}} + \widetilde{\psi}(z, \zeta) \cdot \log[-X](z, \zeta)\,. \tag{3.2.1}$$

Here ψ, $\widetilde{\psi}$ are smooth functions on $\overline{\Omega} \times \overline{\Omega}$ and X is the Levi polynomial (see [KRA1, Ch. 3]) on Ω.

An interesting feature of Fefferman's work, and subsequent work of Burns and Graham [GRA], is that the logarithmic term is always present near a boundary point that is not spherical.

Arguing as usual, if P and X exist then any other point $Q \in \Omega$ has the property that $\varphi_j(Q) \to X$ as $j \to \infty$. We begin with a point Q near the boundary at which the Fefferman expansion (3.2.1) is valid. If Ω is not the ball then we can take Q to be very near to a boundary point that is not spherical.

Of course the Bergman kernel transforms under a biholomorphic mapping F of Ω by the standard formula ([KRA1, Ch. 1])

$$\mathrm{Jac}_{\mathbb{C}} F(z) K(F(z), F(\zeta)) \overline{\mathrm{Jac}_{\mathbb{C}} F(\zeta)} = K(z, \zeta) \,. \qquad (3.2.2)$$

So, when we think of $\varphi_j(Q) \to X$, then we may understand how the Bergman kernel transforms by applying the transformation formula (3.2.2) to the Fefferman expansion (3.2.1). On the one hand, this should give rise to another Fefferman-type formula based at the point $\varphi_j(Q)$. But the problem is that the logarithmic expression does not scale. The result, as $j \to \infty$, will not be a valid Fefferman formula. That is a contradiction. So Ω must be biholomorphic to the ball. □

A consequence of the first two results is this:

Corollary 3.3 *A strongly pseudoconvex domain $\Omega \subseteq \mathbb{C}^N$ is biholomorphic to the ball if and only if the algebra $L(\Omega)$ of Lipschitz functions is noncompact.*

4 An Analysis of Algebra Isomorphisms

First suppose that A is an annulus in the complex plane. Suppose that

$$\Phi : \mathcal{O}(A) \longrightarrow \mathcal{O}(A)$$

is an algebra isomorphism. We claim that $\Phi(z) = z$. That is to say, Φ maps the holomorphic identity function to itself.

First of all, it cannot be that $\Phi(z) = z^2$ or any other higher-order polynomial (or power series or Laurent series) because then it is clear that Φ would not be onto. A similar argument shows that $\Phi(z)$ cannot be a Laurent series with initial term having negative index.

So $\Phi(z)$ is either a power series beginning with a degree-zero term or a power series beginning with a degree-one term. But obviously $\Phi(1) = 1$ and $\Phi(0) = 0$. So $\Phi(z)$ is a power series beginning with a first-degree term. But in fact if that power series contains any term beyond the first-degree term, then there is no holomorphic function that will map to z^2 under Φ. So the power series is simply of the form αz. So any Laurent series of the form

$$\sum_{j=-\infty}^{\infty} a_j z^j \tag{4.1}$$

is mapped under Φ to

$$\sum_{j=-\infty}^{\infty} \alpha^j a_j z^j \,. \tag{4.2}$$

If the Laurent series in (4.1) is chosen so that the function it defines has A as its natural domain of definition, and if the modulus of α is not 1, then it follows that the image function given by (4.2) will have a *different* natural domain of definition. And that is impossible.

We conclude that α has modulus 1. We may as well take $\alpha = 1$.

This example illustrates Theorem 3.2. For the automorphism group of an annulus is just two copies of the circle group. So it is compact. As a result, any $f \in L(A)$ will be compact.

5 Further Results

The next result is classical. See [KRA2, Ch. 12] for a more traditional proof.

Proposition 5.1 *Fix a bounded domain $\Omega \subseteq \mathbb{C}^n$. Let $\{\varphi_j\}$ be automorphisms of Ω. Assume that the φ_j converge normally (i.e., uniformly on compact sets) to a limit f. Then either*

(1) *The mapping f is an automorphism of Ω;*

or

(2) *The mapping f is a constant.*

Proof: We adopt the point of view of Bers's theorem.

With $\varphi_j \in \mathrm{Aut}\,(\Omega)$ as in the statement of the theorem, and $g \in L(\Omega)$, examine $\{g \circ \varphi_j\}$.

Now either $g \circ \varphi_j$ is compact or it is not. If $g \circ \varphi_j$ is compact, then there exists a subsequence φ_{j_k} and a τ such that $g \circ \varphi_{j_k} \to \tau$ with $\tau \in L(\Omega)$. So

$g \circ f = \tau$, with $f \in \mathrm{Aut}\,(\Omega)$ (because it is a nondegenerate mapping, and a limit of automorphisms). Specifically, the mapping f is univalent because it is the limit of univalent mappings. Also f is onto because we can apply our reasoning to φ_j^{-1}. That is part **(1)** of our conclusion (formulated in the language of the present paper).

If instead $g \circ \varphi_j$ is noncompact, then $\{g \circ \varphi_j\}$ has no convergent subsequence. So $g \circ \varphi_j$ blows up in norm. Hence there are a point $P \in \Omega$ and a point $X \in \partial\Omega$ such that $\varphi_{j_k}(P) \to X$ (for some subsequence φ_{j_k}). Hence $g \circ \varphi_{j_k} \to g(X)$. That completes the proof of **(2)**. □

Now we have

Proposition 5.2 *Suppose that $f : \Omega \to \Omega$ is a holomorphic mapping. Assume that, for some sequence $\{\varphi_j\}$ of automorphisms of Ω, $f \circ \varphi_{j_k}$ converges normally to a function $g \in \mathcal{O}(\Omega)$. Then*

- **(a)** *If $g \in \mathrm{Aut}\,(\Omega)$, then $f \in \mathrm{Aut}\,(\Omega)$;*
- **(b)** *If g is not constant then every convergent subsequence of $h_k \equiv f \circ \varphi_{j_{k+1}} \circ \varphi_{j_k}^{-1}$ has limit id_Ω.*

Proof: This result is like a converse to compactness.

If $f(a) = f(b)$ for some distinct points $a, b \in \Omega$ then

$$f(\varphi_{j_k} \circ \varphi_{j_k}^{-1}(a)) = f(\varphi_{j_k} \circ \varphi_{j_k}^{-1}(b))\,.$$

Now, if the φ_{j_k} converge to some ψ, then we see that

$$g(\psi(a)) = g(\psi(b))\,.$$

If ψ is an automorphism then this is certainly a contradiction.

Of course $f \circ \varphi_{j_k}(\Omega) \subseteq f(\Omega)$ for all k. So $g(\Omega) \subseteq f(\Omega) \subseteq \Omega$. But $g(\Omega) = \Omega$. So $f(\Omega) = \Omega$. Thus f is onto. It is also one-to-one. This proves **(a)**.

For part **(b)**, we take g to be holomorphic and nonconstant. Let h be a subsequential limit of $f \circ \varphi_{j_{k+1}} \circ \varphi_{j_k}^{-1} \equiv h_k$. As a result, $f \circ \varphi_{j_{k+1}} = h_k \circ \varphi_{j_k}$ so $g = h \circ \psi$. But then $h = g \circ \psi^{-1}$. So h differs from g by an automorphism. Certianly then h is nonconstant. We note further that $g = h \circ \psi$ so that g is an automorphism. □

References

[BERS] L. Bers, On rings of analytic functions, *Bull. Amer. Math. Soc.* 54(1948), 311–315.

[FEF] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Invent. Math.* 26(1974), 1–65.

[GRA] C. R. Graham, Scalar boundary invariants and the Bergman kernel, *Complex analysis*, II (College Park, Md., 1985–86), 108–135, *Lecture Notes in Math.*, 1276, Springer, Berlin, 1987.

[GKK] R. E. Greene, K.-T. Kim, and S. G. Krantz, *The Geometry of Complex Domains*, Birkhäuser Publishing, Boston, 2011.

[GK1] R. E. Greene and S. G. Krantz, Deformations of complex structure, estimates for the $\overline{\partial}$-equation, and stability of the Bergman kernel, *Advances in Math.* 43(1982), 1–86.

[KLE] P. Klembeck, Kähler metrics of negative curvature, the Bergman metric near the boundary and the Kobayashi metric on smooth bounded strictly pseudoconvex sets, *Indiana Univ. Math. J.* 27(1978), 275–282.

[KRA1] S. G. Krantz, *Function Theory of Several Complex Variables*, 2nd. ed., American Mathematical Society, Providence, RI, 2001.

[KRA2] S. G. Krantz, *Cornerstones of Geometric Function Theory: Explorations in Complex Analysis*, Birkhäuser Publishing, Boston, 2006.

[LQK] Lu Qi-Keng, On Kähler manifolds with constant curvature, *Acta. Math. Sinica* 16(1966), 269–281 (Chinese) (= *Chinese Math.* 9(1966), 283–298).

[OHS] T. Ohsawa, A remark on the completeness of the Bergman metric, *Proc. Japan Acad.* Ser. A Math. Sci. 57(1981), 238–240.

[WON] B. Wong, Characterizations of the ball in $\mathbb{C}^n$ by its automorphism group, *Invent. Math.* 41(1977), 253–257.

[ZAM1] W. R. Zame, Homomorphisms of rings of germs of analytic functions, *Proc. Amer. Math. Soc.* 33(1972), 410–414.

[ZAM2] W. R. Zame, Induced homomorphisms of algebras of analytic germs, *Complex Analysis*, 1972 (Proc. Conf., Rice Univ., Houston, Tex., 1972), Vol. II: Analysis on singularities. Rice Univ. Studies 59 (1973), no. 2, 157–163.

Steven G. Krantz
Department of Mathematics
Washington University in St. Louis
St. Louis, Missouri 63130
sk@math.wustl.edu