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Coulomb problem for a Dirac particle in flat Minkowski space and the Heun functions, extension to curved models

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Abstract

It is shown that there exist several ways to treat the quantum-mechanical Coulomb problem for a Dirac particle in flat Minkowski space with the help of the Heun differential equation, Fuchs's equation with four singular points. When extending the problem to curved spaces of constant curvature, Lobachevsky H_3 and Rimann S_3, there arise 2-nd order differential equations of the Fuchs type with 6 singular points, a method to get relevant equations with five singular points has been elaborated.

Let us start with a Dirac equation written in spherical tetrad of flat Minkowski space (more detail see in [1])

$$\left[i\,\gamma^0\,\partial_t \;+\; i\,(\gamma^3\,\partial_r \;+\; \frac{\gamma^1\,\sigma^{31} \;+\; \gamma^2\,\sigma^{32}}{r}) \;+\; \frac{1}{r}\,\Sigma_{\theta\phi} \;-\; m \right]\;\Psi(x) = 0\;,$$

$$\Sigma_{\theta,\phi} \;=\; i\,\gamma^1\partial_\theta \;+\; \gamma^2\;\frac{i\partial_\phi \;+\; i\;\sigma^{12}}{\sin\theta}\;.$$

In that basis, spherical solutions are constructed on the base of a substitution (Wigner's D functions are noted as $D^j_{-m,\sigma}(\phi,\theta,0) \equiv D_\sigma$):

$$\Psi_{\epsilon jm}(x) \;=\; \frac{e^{-i\epsilon t}}{r}\;\left| \begin{array}{c} f_1(r)\;D_{-1/2} \\ f_2(r)\;D_{+1/2} \\ f_3(r)\;D_{-1/2} \\ f_4(r)\;D_{+1/2} \end{array} \right| \;. \qquad (1)$$

With the use of recurrent relations [2]

$$\partial_\theta\;D_{+1/2} \;=\; a\;D_{-1/2} - b\;D_{+3/2}\;,$$

$$\frac{-m - 1/2\;\cos\theta}{\sin\theta}\;D_{+1/2} \;=\; -a\;D_{-1/2} - b\;D_{+3/2}\;,$$

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$$\partial_\theta \, D_{-1/2} \;=\; b \, D_{-3/2} - a \, D_{+1/2} \; ,$$

$$\frac{-m + 1/2 \;\cos\theta}{\sin\theta} \; D_{-1/2} \;=\; -b \, D_{-3/2} - a \, D_{+1/2} \; ,$$

where

$$a = \frac{j+1/2}{2}, \qquad b = \frac{1}{2}\,\sqrt{(j-1/2)(j+3/2)}\;,$$

we derive (below $\nu = j + 1/2$)

$$\Sigma_{\theta,\phi}\; \Psi_{\epsilon jm}(x) \;=\; i\;\nu\; \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{l} -\; f_4(r)\; D_{-1/2} \\ +\; f_3(r)\; D_{+1/2} \\ +\; f_2(r)\; D_{-1/2} \\ -\; f_1(r)\; D_{+1/2} \end{array} \right| .$$

and after simple calculations obtain radial equations

$$\epsilon f_3 - i\frac{d}{dr} f_3 - i\frac{\nu}{r} f_4 - m f_1 = 0 \; , \qquad \epsilon f_4 + i\frac{d}{dr} f_4 + i\frac{\nu}{r} f_3 - m f_2 = 0 \; ,$$

$$\epsilon f_1 + i\frac{d}{dr} f_1 + i\frac{\nu}{r} f_2 - m f_3 = 0 \; , \qquad \epsilon f_2 - i\frac{d}{dr} f_2 - i\frac{\nu}{r} f_1 - m f_4 = 0 \; . \qquad (1b)$$

Usual P-reflection operator [3] in Cartesian basis $\hat{\Pi}_C = i\gamma^0 \otimes \hat{P}$

$$\hat{\Pi}_{C.} = \left| \begin{array}{cccc} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{array} \right| \;\otimes\; \hat{P}\;, \qquad \hat{P}(\theta,\phi) = (\pi - \theta,\; \phi + \pi)$$

after translation to spherical basis

$$\hat{\Pi}_{sph} = S(\theta,\phi)\; \hat{\Pi}_C \; S^{-1}(\theta,\phi)\;,$$

assumes the form

$$\hat{\Pi}_{sph} \;=\; \left| \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right| \;\otimes\; \hat{P}\;.$$

From eigenvalue equation

$$\hat{\Pi}_{sph}\Psi_{jm} = \Pi\; \Psi_{jm}, \qquad \hat{P}\; D^{j}_{-m,\sigma}(\phi,\theta,0) = (-1)^j\; D^{j}_{-m,-\sigma}(\phi,\theta,0)$$

it follows

$$\Pi = \;\delta\;(-1)^{j+1}, \;\; \delta = \pm 1\;, \qquad f_4 = \;\delta\; f_1, \qquad f_3 = \;\delta\; f_2\;;$$

that is

$$
\Psi(x)_{\epsilon jm\delta} \;=\; \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{c} f_1(r)\; D_{-1/2} \\ f_2(r)\; D_{+1/2} \\ \delta\; f_2(r)\; D_{-1/2} \\ \delta\; f_1(r)\; D_{+1/2} \end{array} \right| . \qquad (2a)
$$

Allowing for (2a), we simplify eqs. (1b):

$$
(\frac{d}{dr} \;+\; \frac{\nu}{r}\;)\; f \;+\; (\epsilon \;+\; \delta\; m)\; g \;=\; 0\;,
$$

$$
(\frac{d}{dr} \;-\; \frac{\nu}{r}\;)\; g \;-\; (\epsilon \;-\; \delta\; m)\; f \;=\; 0\;, \qquad (2b)
$$

where instead of $f_1 \;\; f_2$ new functions are used

$$
f \;=\; \frac{f_1 + f_2}{\sqrt{2}}\;, \qquad g \;=\; \frac{f_1 - f_2}{i\sqrt{2}}\;.
$$

To simplify the wave functions $\Psi_{\epsilon jm} \to \Psi_{\epsilon jm\delta}$ and one can use so-called Dirac (or Johnson-Lippmann) operator [3] $\hat{K}$ (see also [4]),

$$
\hat{K} \;=\; -\gamma^0\gamma^3\; \Sigma_{\theta,\phi} = \gamma^0\gamma^3\; \left[\; \gamma^1\; (\partial_\theta + 1/2) + \frac{\gamma^2}{\sin\theta}\; \partial_\phi \;\right]\;.
$$

Indeed, from the eigenvalue equation $\hat{K}\; \Psi_{\epsilon jm}(x) = K\; \Psi_{\epsilon jm}$ we get

$$
K = -\delta\; (j + 1/2)\;\;,\;\; \delta\; = \pm 1\;,
$$

$$
f_4 = \;\delta\; f_1\;, \qquad f_3 = \;\delta\; f_2\;,
$$

which coincides with the above restrictions.

Transition to a Coulomb problem for a Dirac particle is reached by formal chane the radial system (2b)

$$
(\frac{d}{dr} + \frac{\nu}{r}\;)\; f \;+\; (E + \frac{e}{r} \;+\; m)\; g \;=\; 0\;,
$$

$$
(\frac{d}{dr} \;-\; \frac{\nu}{r}\;)\; g \;-\; (E + \frac{e}{r} \;-\; m)\; f \;=\; 0\;. \qquad (3)
$$

Let us perform a linear transformation over functions $f(r)$ and $g(r)$ (its coefficients may depend on the radial variable, let its determinant obey an identity $a(r)b(r) - c(r)d(r) = 1$)

$$
f(r) = a\; F(r) + c\; G(r)\;, \qquad g(r) = d\; F(r) + b\; G(r)\;,
$$

$$
F(r) = b\; f(r) - c\; g(r)\;, \qquad G(r) = -d\; f(r) + a\; g(r)\;. \qquad (4a)
$$

Let us combine eqs. (3) as follows: the first equation is multiplied by $+b$, the second by $-c$, and add the results; analogously, the firs equation multiply by $-d$, the second by $+a$, and sum the results. Thus we arrive at

$$
\left[\; \frac{d}{dr} - b'a + c'd + \frac{\nu}{r}\; (ba + cd) + (E + \frac{e}{r} + m)\; bd + (E + \frac{e}{r} - m)\; ca \;\right] F
$$

$$
= \left[\; b'c - bc' - \frac{\nu}{r} 2bc - (E + \frac{e}{r} + m)b^2 - (E + \frac{e}{r} - m)c^2 \;\right] G\;,
$$

$$\Big[\ \frac{d}{dr} + d'c - a'b - \frac{\nu}{r}\ (dc+ab) - (E+\frac{e}{r}+m)\ bd - (E+\frac{e}{r}-m)\ ca\ \Big]\ G$$
$$= \Big[\ -d'a + da' + \frac{\nu}{r} 2ad + (E+\frac{e}{r}+m)d^2 + (E+\frac{e}{r}-m)a^2\ \Big]\ F\ . \tag{4b}$$

For simplicity, let the transformation (2) does not depend on r, and let it be orthogonal:

$$S = \left| \begin{array}{cc} a & c \\ d & b \end{array} \right| = \left| \begin{array}{cc} \cos A/2 & \sin A/2 \\ -\sin A/2 & \cos A/2 \end{array} \right| ,$$

which simplifies eqs. (3)

$$\Big(\frac{d}{dr} + \frac{\nu}{r}\ \cos A - m\ \sin A\Big)\ F = \Big(-\frac{\nu}{r}\sin A - \frac{e}{r} - E - m\cos A\Big)\ G\ ,$$
$$\Big(\frac{d}{dr} - \frac{\nu}{r}\ \cos A + m\ \sin A\Big)\ G = \Big(-\frac{\nu}{r}\sin A + \frac{e}{r} + E - m\cos A\Big)\ F\ . \tag{5}$$

There exist four possibilities (in fact, only two of them are different)

1)

$$-\frac{\nu}{r}\sin A + \frac{e}{r} = 0\ , \qquad \sin A = \frac{e}{\nu}\ , \qquad \cos A = \sqrt{1-e^2/\nu^2}\ ,$$
$$\cos\frac{A}{2} = \sqrt{\frac{\nu+\sqrt{\nu^2-e^2}}{2\nu}}\ , \qquad \sin\frac{A}{2} = \sqrt{\frac{\nu-\sqrt{\nu^2-e^2}}{2\nu}}\ ;$$

1′)

$$-\frac{\nu}{r}\sin A - \frac{e}{r} = 0\ , \qquad \sin A = -\frac{e}{\nu}\ , \qquad \cos A = \sqrt{1-e^2/\nu^2}\ ,$$
$$\cos\frac{A}{2} = \sqrt{\frac{\nu-\sqrt{\nu^2-e^2}}{2\nu}}\ , \qquad \sin\frac{A}{2} = \sqrt{\frac{\nu+\sqrt{\nu^2-e^2}}{2\nu}}\ ; \tag{6a}$$

2)

$$E - m\cos A = 0\ , \qquad \cos A = +\frac{E}{m}\ , \qquad \sin A = \sqrt{1-E^2/m^2}\ ,$$
$$\cos\frac{A}{2} = \sqrt{\frac{m+E}{2m}}\ , \qquad \sin\frac{A}{2} = \sqrt{\frac{m-E}{2m}}\ ;$$

2′)

$$-E - m\cos A = 0\ , \qquad \cos A = -\frac{E}{m}\ , \qquad \sin A = \sqrt{1-E^2/m^2}\ ,$$
$$\cos\frac{A}{2} = \sqrt{\frac{m-E}{2m}}\ , \qquad \sin\frac{A}{2} = \sqrt{\frac{m+E}{2m}}\ . \tag{6b}$$

First, consider the case 1). Eqs. (5) takes the form

$$\left(\frac{d}{dr} + \frac{\nu}{r}\cos A - m\sin A\right) F = \left(-\frac{2e}{r} - E - m\cos A\right) G \;,$$

$$\left(\frac{d}{dr} - \frac{\nu}{r}\cos A + m\sin A\right) G = (E - m\cos A)\; F \;. \qquad (7)$$

After excluding the function F, we get a second order equation for G

$$\left(\frac{d}{dr} + \frac{\nu}{r}\cos A - m\sin A\right)\left(\frac{d}{dr} - \frac{\nu}{r}\cos A + m\sin A\right) G$$

$$= (E - m\cos A)\left(-\frac{2e}{r} - E - m\cos A\right) G \;, \qquad (8a)$$

or in more detailed form

$$\left(\frac{d^2}{dr^2} + E^2 - m^2 + \frac{\nu\cos A - \nu^2\cos^2 A}{r^2}\right.$$

$$\left. + \frac{2eE - 2em\cos A + 2m\nu\sin A\cos A}{r}\right) G = 0 \;.$$

Taking in mind identity $\sin A = e/\nu$, thus equation reduces to

$$\left(\frac{d^2}{dr^2} + E^2 - m^2 + \frac{\nu\cos A - \nu^2\cos^2 A}{r^2} + \frac{2eE}{r}\right) G = 0 \;. \qquad (8b)$$

After changing the variable, $x = 2\sqrt{m^2 - E^2}\, r$, it reads

$$\frac{d^2 G}{dx^2} + \left(-\frac{1}{4} - \frac{\nu\cos A\,(\nu\cos A - 1)}{x^2} + \frac{eE}{\sqrt{m^2 - E^2}\, x}\right) G = 0 \,.$$

With the use of a substitution $G(x) = x^a e^{bx}\varphi(x)$, for φ we get

$$x\,\frac{d^2\varphi}{dx^2} + (2\,a + 2\,b\,x)\,\frac{d\varphi}{dx}$$

$$+ \left[(b^2 - \frac{1}{4})\,x + \frac{a^2 - a - \nu\cos A\;(\nu\cos A - 1)}{x} + 2ab + \frac{eE}{\sqrt{m^2 - E^2}}\right]\varphi = 0 \,.$$

When

$$a = +\nu\;\cos A = \sqrt{\nu^2 - e^2}\,, \qquad b = -\frac{1}{2}\;,$$

this equation for φ becomes simpler

$$x\,\frac{d^2\varphi}{dx^2} + (2\,a - \;x)\,\frac{d\varphi}{dx} - \left[a - \frac{eE}{\sqrt{m^2 - E^2}}\right]\varphi = 0\,,$$

which is a confluent hypergeometric equation

$$x\,Y'' + (\gamma - x)Y' - \alpha Y = 0\;, \qquad \alpha = a - \frac{eE}{\sqrt{m^2 - E^2}}\,, \qquad \gamma = 2a\,.$$

To polynomials there correspond the know restriction $\alpha = -n, \ n = 0, 1, 2, ...,$ which gives the known energy quantization rule

$$a - \frac{eE}{\sqrt{m^2 - E^2}} = -n \qquad \Longrightarrow \qquad E = \frac{m}{\sqrt{1 + e^2/(n + \sqrt{\nu^2 - e^2})^2}} \ . \tag{8c}$$

In turn, from (7) it follows a second order equation for F

$$\left(\frac{d}{dr} - \frac{\nu}{r} \ \cos A + m \ \sin A \right) \frac{r}{2e + (E + m \cos A) r} \left(\frac{d}{dr} + \frac{\nu}{r} \ \cos A - m \ \sin A \right) F$$
$$= (-E + m \cos A) F \ , \tag{9a}$$

or

$$\Big[(\frac{d}{dr} - \frac{\nu}{r} \ \cos A + m \ \sin A)(\frac{d}{dr} + \frac{\nu}{r} \ \cos A - m \ \sin A)$$
$$+ \frac{d}{dr} \ln \left(\frac{r}{2e + (E + m \cos A) r} \right) (\frac{d}{dr} + \frac{\nu}{r} \ \cos A - m \ \sin A) \Big] F$$
$$= (\frac{2e}{r} + E + m \cos A) \ (-E + m \cos A) \ F \ ,$$

and further

$$\Big[\frac{d^2}{dr^2} + \left(\frac{1}{r} - \frac{E + m \cos A}{2e + (E + m \cos A) r} \right) \frac{d}{dr}$$
$$+ \frac{\nu}{r^2} \ \cos A - \frac{m \ \sin A}{r} - \left(\frac{1}{r} - \frac{2e}{r[2e + (E + m \cos A) r]} \right) (\frac{\nu}{r} \ \cos A - m \ \sin A)$$
$$+ E^2 - m^2 - \frac{\nu \cos A}{r^2} + \frac{e^2 - \nu^2}{r^2} + \frac{2eE}{r} \Big] F = 0 \ .$$

Finally, we arrive at

$$\Big[\frac{d^2}{dr^2} + (\frac{1}{r} - \frac{E + m \cos A}{2e + (E + m \cos A) r}) \frac{d}{dr}$$
$$+ E^2 - m^2 + \frac{2eE}{r} + \frac{e^2 - \nu^2}{r^2} - \frac{2em \sin A + \nu \cos A (E + m \cos A)}{r \ [2e + (E + m \cos A) r]} \Big] F = 0 \ .$$

Let us introduce special designation for the additional singular point

$$- \frac{2e}{E + m \cos A} = R \ ;$$

then we obtain

$$\Big[\frac{d^2}{dr^2} + (\frac{1}{r} - \frac{1}{r - R}) \frac{d}{dr}$$
$$+ E^2 - m^2 + \frac{2eE}{r} + \frac{e^2 - \nu^2}{r^2} + \frac{mR \ \sin A - \nu \cos A}{r \ (r - R)} \Big] F = 0 \ . \tag{9b}$$

After changing the variable $x = r/R$, it reads

$$\frac{d^2F}{dx^2} + \left[\frac{1}{x} - \frac{1}{x-1}\right] \frac{dF}{dx} + \left[(E^2 - m^2) R^2 - \frac{\nu^2 - e^2}{x^2}\right.$$
$$\left. + \frac{-\nu \cos A + mR \sin A}{x-1} + \frac{2eRE - mR \sin A + \nu \cos A}{x}\right] F = 0 . \tag{9c}$$

Let us search solutions in the form $F = x^a \, e^{bx} \, \varphi(x)$, the function φ obeys

$$\frac{d^2\varphi}{dx^2} + \left[\frac{2a+1}{x} + 2b - \frac{1}{x-1}\right] \frac{d\varphi}{dx}$$
$$+ \left[b^2 + (E^2 - m^2) R^2 + \frac{a^2 - \nu^2 + e^2}{x^2} - \frac{a + b + \nu \cos A - mR \sin A}{x-1}\right.$$
$$\left. + \frac{a + b + 2ab + R(2eE - m \sin A) + \nu \cos A}{x}\right] \varphi = 0 .$$

When a, b taken according to (below we will use underlines values)

$$a = \underline{+\sqrt{\nu^2 - e^2}} , \quad -\sqrt{\nu^2 - e^2} ,$$
$$b = +\sqrt{m^2 - E^2} \, R , \underline{-\sqrt{m^2 - E^2} \, R} \tag{10a}$$

the above equation becomes simpler

$$\frac{d^2\varphi}{dx^2} + \left[2b + \frac{2a+1}{x} - \frac{1}{x-1}\right] \frac{d\varphi}{dx}$$
$$+ \left[\frac{a + b + 2ab + 2eRE - mR \sin A + \nu \cos A}{x} - \frac{a + b + \nu \cos A - mR \sin A}{x-1}\right] \varphi = 0 . \tag{10b}$$

In can be recognized as a confluent Heun equation for $G(\alpha, \beta, \gamma, \delta, \eta, z)$

$$G'' + \left(\alpha + \frac{1+\beta}{z} + \frac{1+\gamma}{z-1}\right) G'$$
$$+ \left(\frac{1}{2} \, \frac{\alpha + \alpha\beta - \beta - \beta\gamma - \gamma - 2\eta}{z} + \frac{1}{2} \, \frac{\alpha + \alpha\gamma + \beta + \beta\gamma + \gamma + 2\delta + 2\eta}{z-1}\right) G = 0 \tag{10c}$$

with parameters

$$\alpha = 2b , \qquad \beta = 2a , \qquad \gamma = -2 ,$$
$$\delta = 2 \, eER , \qquad \eta = 1 + mR \sin A - 2 \, eER - \nu \cos A . \tag{10d}$$

Let us use the known condition for polynomial solutions

$$\delta = -\left(n + \frac{\beta + \gamma + 2}{2}\right) \alpha , \qquad n = 0, 1, 2, ..., \tag{11a}$$

it results the energy quantization rule

$$a = +\sqrt{\nu^2 - e^2}\,, \qquad b = -\sqrt{m^2 - E^2}\,R\,,$$
$$eER = (n + \sqrt{\nu^2 - e^2})\,\sqrt{m^2 - E^2}\;R\,,$$

so we arrive at

$$E = \frac{m}{\sqrt{1 + e^2/(n + \sqrt{\nu^2 - e^2})^2}}\,, \tag{11b}$$

which coincides with the known formula for energy levels.

Let us consider the case 2). Eq. (5) takes the form

$$\left(\frac{d}{dr} + \frac{\nu}{r}\;\cos A - m\;\sin A\right) F = \left(-\frac{\nu \sin A + e}{r} - 2m\cos A\right) G\,,$$
$$\left(\frac{d}{dr} - \frac{\nu}{r}\;\cos A + m\;\sin A\right) G = \frac{e - \nu\sin A}{r}\;F\,. \tag{12}$$

One can obtain a second order equation for $G(r)$:

$$\Big[(\frac{d}{dr} + \frac{\nu}{r}\;\cos A - m\;\sin A)\; r (\frac{d}{dr} - \frac{\nu}{r}\;\cos A + m\;\sin A)$$
$$+ (e - \nu\sin A)(\frac{e + \nu\sin A}{r} + 2m\cos A) \Big]\, G = 0\,, \tag{13a}$$

from whence it follows

$$\left[\frac{d}{dr} - \frac{\nu}{r}\;\cos A + m\;\sin A + r\left(\frac{d^2}{dr^2} + \frac{\nu\cos A}{r^2}\;-(\frac{\nu}{r}\;\cos A - m\;\sin A)^2\right)\right.$$
$$\left. + (e - \nu\sin A)\left(\frac{e + \nu\sin A}{r} + 2m\cos A\right)\right] G = 0\,,$$

or

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - m^2\;\sin^2 A + \frac{e^2 - \nu^2}{r^2} + \frac{2me\cos A}{r} + \frac{m\sin A}{r}\right) G = 0\,.$$

And further, taking into account identity $\cos A = E/m$, we arrive at

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + E^2 - m^2 + \frac{e^2 - \nu^2}{r^2} + \frac{2eE}{r} + \frac{\sqrt{m^2 - E^2}}{r}\right) G = 0\,. \tag{13b}$$

Making the change of variables $x = 2\sqrt{m^2 - E^2}\,r$, we get

$$\frac{d^2 G}{dx^2} + \frac{1}{x}\,\frac{dG}{dx} + \left(-\frac{1}{4} - \frac{\nu^2 - e^2}{x^2} + \frac{1}{2}\,\frac{m^2 - E^2 + 2Ee\sqrt{m^2 - E^2}}{(m^2 - E^2)\,x}\right) G = 0\,.$$

Let $G(x) = x^a e^{bx}\varphi(x)$, the function φ satisfies

$$x\,\frac{d^2\varphi}{dx^2} + (2\,a + 1 + 2\,b\,x)\,\frac{d\varphi}{dx}$$

$$+\left[(b^2-\frac{1}{4})\,x+\frac{a^2-\nu^2+e^2}{x}+2ab+b+\frac{1}{2}\,\frac{m^2-E^2+2Ee\sqrt{m^2-E^2}}{m^2-E^2}\right]\varphi=0\,.$$

When

$$a=\sqrt{\nu^2-e^2}\,,\qquad b=-\frac{1}{2}\,,$$

we get

$$x\,\frac{d^2\varphi}{dx^2}+(2\,a+1-x)\,\frac{d\varphi}{dx}-\left(a-\frac{Ee}{\sqrt{m^2-E^2}}\right)\varphi=0\,,$$

which is a confluent hypergeometric equation

$$x\,Y''+(\gamma-x)Y'-\alpha Y=0\,,\qquad \alpha=a-\frac{Ee}{\sqrt{m^2-E^2}}\,,\qquad \gamma=2a+1\,.$$

Solutions become polynomials if $\alpha=-n$, $n=0,1,2,...$, this provides us with the energy spectrum

$$E=\frac{m}{\sqrt{1+e^2/(n+\sqrt{\nu^2-e^2})^2}}\,. \qquad (13c)$$

In turn, from (12) it follows a second order equation for $F(r)$

$$\left(\frac{d}{dr}-\frac{\nu}{r}\,\cos A+m\,\sin A\right)\frac{r}{\nu\sin A+e+2m\cos A\,r}\left(\frac{d}{dr}+\frac{\nu}{r}\,\cos A-m\,\sin A\right)F$$
$$=\frac{\nu\sin A-e}{r}\,F\,, \qquad (14a)$$

that is

$$\left[(\frac{d}{dr}-\frac{\nu}{r}\,\cos A+m\,\sin A)\,(\frac{d}{dr}+\frac{\nu}{r}\,\cos A-m\,\sin A)\right.$$
$$+\left(\frac{d}{dr}\ln\frac{r}{\nu\sin A+e+2m\cos A\,r}\right)(\frac{d}{dr}+\frac{\nu}{r}\,\cos A-m\,\sin A)$$
$$\left.+\frac{e+\nu\sin A+2m\cos A\,r}{r}\,\frac{e-\nu\sin A}{r}\right]F=0\,.$$

After simple transformation, we arrive at

$$\left[\frac{d^2}{dr^2}+\left(\frac{1}{r}-\frac{2m\cos A}{\nu\sin A+e+2m\cos A\,r}\right)\frac{d}{dr}\right.$$
$$-\frac{2m\cos A}{\nu\sin A+e+2m\cos A\,r}\left(\frac{\nu}{r}\,\cos A-m\,\sin A\right)$$
$$\left.+E^2-m^2+\frac{e^2-\nu^2}{r^2}+\frac{2eE-m\sin A}{r}\right]F=0\,.$$

With special notation for the additional singular point

$$R=-\frac{e+\nu\sin A}{2m\cos A}\ ,$$

it reads shorter

$$\left[\frac{d^2}{dr^2}+\left(\frac{1}{r}-\frac{1}{r-R}\right)\frac{d}{dr}+\frac{m\sin A}{r-R}-\frac{\nu\cos A}{R}(\frac{1}{r-R}-\frac{1}{r}) \right.$$
$$\left. +E^2-m^2+\frac{e^2-\nu^2}{r^2}+\frac{2eE-m\sin A}{r}\right]F=0\ . \tag{14b}$$

In the variable $x=r/R$, it looks simpler

$$\frac{d^2F}{dx^2}+\left[\frac{1}{x}-\frac{1}{x-1}\right]\frac{dF}{dx}+\left[(E^2-m^2)\,R^2-\frac{\nu^2-e^2}{x^2}\right.$$
$$\left.+\frac{-\nu\cos A+mR\,\sin A}{x-1}+\frac{R(2eE-m\sin A)+\nu\cos A}{x}\right]F=0\,. \tag{14c}$$

Let $F=x^a\,e^{bx}\,\varphi(x)$, the function φ satisfies

$$\frac{d^2\varphi}{dx^2}+\left[\frac{2a+1}{x}+2b-\frac{1}{x-1}\right]\frac{d\varphi}{dx}$$
$$+\left[b^2+(E^2-m^2)\,R^2+\frac{a^2-\nu^2+e^2}{x^2}-\frac{a+b+\nu\cos A-mR\,\sin A}{x-1}\right.$$
$$\left.+\frac{a+b+2ab+R(2eE-m\sin A)+\nu\cos A}{x}\right]\varphi=0\,.$$

When a and b are

$$a=\underline{+\sqrt{\nu^2-e^2}}\,,\quad -\sqrt{\nu^2-e^2}\ ,$$
$$b=+\sqrt{m^2-E^2}\,R\,,\quad \underline{-\sqrt{m^2-E^2}\,R}\ , \tag{15a}$$

it reads

$$\frac{d^2\varphi}{dx^2}+\left(2b+\frac{2a+1}{x}-\frac{1}{x-1}\right)\frac{d\varphi}{dx}$$
$$+\left(\frac{a+b+2ab+R(2eE-m\sin A)+\nu\cos A}{x}\right.$$
$$\left.-\frac{a+b+\nu\cos A-mR\,\sin A}{x-1}\right)\varphi=0\,, \tag{15b}$$

which is a confluent Heun equation for $G(\alpha,\beta,\gamma,\delta,\eta,z)$

$$G''+\left(\alpha+\frac{1+\beta}{z}+\frac{1+\gamma}{z-1}\right)G'$$
$$+\left(\frac{1}{2}\,\frac{\alpha+\alpha\beta-\beta-\beta\gamma-\gamma-2\eta}{z}+\frac{1}{2}\,\frac{\alpha+\alpha\gamma+\beta+\beta\gamma+\gamma+2\delta+2\eta}{z-1}\right)G=0\,,$$

$$(15c)$$

with parameters

$$\alpha = 2b\,, \qquad \beta = 2a\,, \qquad \gamma = -2\,,$$
$$\delta = 2\,eER\,, \qquad \eta = 1 + mR\,\sin A - 2\,eER - \nu\cos A\,. \tag{15d}$$

Imposing the known condition for polynomial solutions

$$\delta = -\left(n + \frac{\beta+\gamma+2}{2}\right)\,\alpha\,, \qquad n = 0, 1, 2, ... \tag{16a}$$

we derive the energy quantization rule

$$a = +\sqrt{\nu^2 - e^2}\,, \qquad b = -\sqrt{m^2 - E^2}\,R\,,$$
$$eER = (n + \sqrt{\nu^2 - e^2})\,\sqrt{m^2 - E^2}\;R \qquad \Rightarrow$$
$$E = \frac{m}{\sqrt{1 + e^2/(n + \sqrt{\nu^2 - e^2})^2}}\,, \tag{16b}$$

which coincides with the known result.

It should be emphasized that confluent Heun equations in cases 1) and 2) formally coincide, however all parameters are different in fact:

$$1) \qquad \alpha = 2b\,, \qquad \beta = 2a\,, \qquad \gamma = -2\,,$$
$$\delta = 2\,eER\,, \qquad \eta = 1 + mR\,\sin A - 2\,eER - \nu\cos A\,, \tag{17a}$$

$$2) \qquad \alpha = 2b\,, \qquad \beta = 2a\,, \qquad \gamma = -2\,,$$
$$\delta = 2\,eER\,, \qquad \eta = 1 + mR\,\sin A - 2\,eER - \nu\cos A\,, \tag{17b}$$

where

$$R = -\frac{2e}{E + m\cos A}\,, \qquad \sin A = \frac{e}{\nu}\,, \qquad \cos A = \sqrt{1 - \frac{e^2}{\nu^2}}\,; \tag{18a}$$
$$R = -\frac{e + \nu\sin A}{2E}\,, \qquad \cos A = \frac{E}{m}\,, \qquad \sin A = \sqrt{1 - \frac{E^2}{m^2}}\,. \tag{18b}$$

Let us turn again to radial equations in presence of Coulomb potential

$$(\frac{d}{dr} + \frac{\nu}{r})\,f + (E + \frac{e}{r} + m)\,g = 0\,,$$
$$(\frac{d}{dr} - \frac{\nu}{r})\,g - (E + \frac{e}{r} - m)\,f = 0\,. \tag{19}$$

Excluding the function g, one gets

$$\frac{d^2 f}{dr^2} + \frac{e}{r(Er + e + mr)}\,\frac{df}{dr} + \Big[\frac{e(e^2 - \nu^2)}{r^2(Er + e + mr)}$$

$$+\frac{E\,(3e^2-\nu^2)-\nu\,(m+E)+m\,(e^2-\nu^2)}{r\,(Er+e+mr)}$$

$$+\frac{e\,(E+m)\,(3E-m)}{Er+e+mr}+\frac{r(E-m)(E+m)^2}{Er+e+mr}\Big]\,f=0\,. \tag{20}$$

After changing the variable

$$x=-\frac{(E+m)\,r}{e}\,, \tag{21}$$

eq. (20) takes the form

$$x\,\frac{d^2f}{dx^2}-\frac{1}{x-1}\,\frac{df}{dx}+\left[\frac{e^2(Ex-mx-2E)}{E+m}+\frac{e^2-\nu^2}{x}-\frac{\nu}{x-1}\right]f=0\,. \tag{22}$$

Separating two factors

$$f(x)=x^A e^{Cx}F(x)\,,$$

for F one derives

$$\frac{d^2F}{dx^2}+\left(2C+\frac{2A+1}{x}-\frac{1}{x-1}\right)\frac{dF}{dx}+\Big[C^2+\frac{e^2(E-m)}{E+m}$$

$$+\frac{A^2+e^2-\nu^2}{x^2}+\frac{A+C+2AC-2Ee^2/(E+m)+\nu}{x}-\frac{A+C+\nu}{x-1}\Big]\,F=0\,. \tag{23}$$

When A,C are taken as (bound states are of the prime interest)

$$C^2+\frac{e^2(E-m)}{E+m}=0\qquad\Rightarrow\qquad C=+e\sqrt{\frac{m-E}{m+E}}\,,$$

$$A^2+e^2-\nu^2=0\qquad\Rightarrow\qquad A=+\sqrt{\nu^2-e^2}\,, \tag{24}$$

eq. (23) becomes simpler

$$\frac{d^2F}{dx^2}+\left(2C+\frac{2A+1}{x}-\frac{1}{x-1}\right)\frac{dF}{dx}$$

$$+\left[\frac{A+C+\nu+2AC-2Ee^2/(E+m)}{x}-\frac{A+C+\nu}{x-1}\right]F=0\,, \tag{25}$$

which is the confluent Heun equation for $F(\alpha,\beta,\gamma,\delta,\eta;x)$

$$\frac{d^2}{dz^2}F+\left(a+\frac{\beta+1}{z}+\frac{\gamma+1}{z-1}\right)\frac{dF}{dz}$$

$$+\left(\frac{1}{2}\,\frac{a+a\gamma+\beta+\beta\gamma+\gamma+2\delta+2\eta}{z-1}+\frac{1}{2}\,\frac{a\beta+a-\beta\gamma-\beta-\gamma-2\eta}{z}\right)F=0 \tag{26a}$$

with parameters determined by

$$a = 2C = +2e\,\sqrt{\frac{m-E}{m+E}}\,, \qquad \beta = 2A = +2\,\sqrt{\nu^2 - e^2}\,,$$

$$\gamma = -2\,, \qquad \delta = -\frac{2Ee^2}{E+m}\,, \qquad \eta = 1 - \nu + \frac{2Ee^2}{E+m}\,. \tag{26b}$$

The known condition to reach polynomials is

$$\delta = -a\,\left(n + \frac{\gamma+\beta+2}{2}\right)\,. \tag{27a}$$

It results in

$$-\frac{2Ee^2}{E+m} = -2e\,\sqrt{\frac{m-E}{m+E}}\,(\,n + \sqrt{\nu^2-e^2}\,)\,, \tag{27b}$$

or

$$\frac{Ee}{\sqrt{m^2-E^2}} = N\,, \qquad N = n + \sqrt{\nu^2-e^2}\,; \tag{27c}$$

from whence it follows

$$E = \frac{m}{\sqrt{1+e^2/N^2}}\,. \tag{28}$$

It is the exact energy spectrum for hydrogen atom in the Dirac theory.

Supplement A. Coulomb problem in spaces of constant curvature

In the Lobachevsky space H_3, the problem of Dirac particle in Coulomb field reduces to a radial system (let us specify the case $\delta = 1$)

$$(\frac{d}{d\beta} + \frac{\nu}{\sinh\beta}\,)\,f \;+\; (E + \frac{e}{\tanh\beta} \;+\; m)\,g \;=\; 0\,,$$

$$(\frac{d}{d\beta} \;-\; \frac{\nu}{\sinh\beta}\,)\,g \;-\; (E + \frac{e}{\tanh\beta} \;-\; m)\,f \;=\; 0\,. \tag{A.1a}$$

From this it follows a second order equation for f

$$\frac{d^2f}{d\beta^2} + \frac{e}{\sinh\beta}\frac{1}{[e\cosh\beta + (E+m)\sinh\beta]}\,\frac{df}{d\chi}$$

$$+\left[(E + \frac{e}{\tanh\beta})^2 - m^2 - \frac{\nu^2 + \nu\cosh\beta}{\sinh^2\beta} + \frac{\nu}{\sinh^2\beta}\;\frac{e}{e\cosh\beta + (E+m)\sinh\beta}\right] f = 0\,. \tag{A.1b}$$

Let us apply a linear transformation with $a(\beta)b(\beta) - c(\beta)d(\beta) = 1$:

$$f(\beta) = a\;F(\beta) + c\;G(\beta)\;, \qquad g(\beta) = d\;F(\beta) + b\;G(\beta)\;,$$

$$F(\beta) = b\;f(\beta) - c\;g(\beta)\;, \qquad G(\beta) = -d\;f(\beta) + a\;g(\beta)\;. \tag{A.2}$$

After combining eqs. $(A.1a)$ in the same way as in the flat space model we obtain

$$\left[\frac{d}{d\beta} - b'a + c'd + \frac{\nu(ba+cd)}{\sinh\beta} + (E + \frac{e}{\tanh\beta} + m)bd + (E + \frac{e}{\tanh\beta} - m)ca\right] F$$
$$= \left[b'c - bc' - \frac{\nu}{\sinh\beta}2bc - (E + \frac{e}{\tanh\beta} + m)b^2 - (E + \frac{e}{\tanh\beta} - m)c^2\right] G\,,$$
$$\left[\frac{d}{d\beta} + d'c - a'b - \frac{\nu(dc+ab)}{\sinh\beta} - (E + \frac{e}{\tanh\beta} + m)\, bd - (E + \frac{e}{\tanh\beta} - m)ca\right] G$$
$$= \left[-d'a + da' + \frac{\nu}{\sinh\beta}2ad + (E + \frac{e}{\tanh\beta} + m)d^2 + (E + \frac{e}{\tanh\beta} - m)a^2\right] F\,. \tag{A.3}$$

When the above transformation is orthogonal

$$S = \begin{vmatrix} a & c \\ d & b \end{vmatrix} = \begin{vmatrix} \cos A/2 & \sin A/2 \\ -\sin A/2 & \cos A/2 \end{vmatrix}\,, \tag{A.4}$$

eqs. (A.3) become simpler

$$\left(\frac{d}{d\beta} + \frac{\nu\cos A}{\sinh\beta} - m\ \sin A\right) F = \left(-\frac{A'}{2} - \frac{e\cosh\beta + \nu\sin A}{\sinh\beta} - E - m\cos A\right) G\,,$$
$$\left(\frac{d}{d\beta} - \frac{\nu\cos A}{\sinh\beta} + m\ \sin A\right) G = \left(+\frac{A'}{2} + \frac{e\cosh\beta - \nu\sin A}{\sinh\beta} + E - m\cos A\right) F\,. \tag{A.5}$$

Let ut translate equations $(A.1a)$ to a new variable

$$\tanh\frac{\beta}{2} = z\,;$$

eqs. $(A.1a)$ will take the form

$$\frac{d}{dz}f + \frac{\nu}{z}f - \left(\frac{2(E+m)}{z^2-1} + \frac{e(z^2+1)}{z(z^2-1)}\right) g = 0\,,$$
$$\frac{d}{dz}g - \frac{\nu}{z}g + \left(\frac{2(E-m)}{z^2-1} + \frac{e(z^2+1)}{z(z^2-1)}\right) f = 0\,,$$

or differently

$$\frac{d}{dz}f + \frac{\nu}{z}f + \left(\frac{e}{z} + \frac{-E-e-m}{z-1} + \frac{E-e+m}{z+1}\right) g = 0\,,$$
$$\frac{d}{dz}g - \frac{\nu}{z}g - \left(\frac{e}{z} + \frac{-E-e+m}{z-1} + \frac{E-e-m}{z+1}\right) f = 0\,. \tag{A.6}$$

From (A.6) it follows a second order differential equation for f

$$\frac{d^2f}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z+1} - 2\,\frac{ez+E+m}{ez^2 + 2(E+m)\,z + e}\right]\frac{df}{dz}$$

$$
+\left[2\,\frac{2Ee^2-(E+m)\,\nu}{ez}+\frac{-(E+e)^2+m^2+\nu}{z-1}+\frac{(E-e)^2-m^2-\nu}{z+1}+\frac{e^2-\nu^2}{z^2}\right.
$$

$$
\left.+\frac{(E+e)^2-m^2}{(z-1)^2}+\frac{(E-e)^2-m^2}{(z+1)^2}+\frac{2\,\nu\,[ez(E+m)+2\,(E+m)^2-e^2]}{e\,[\,ez^2+2(E+m)\,z+e\,]}\right] f=0\,. \tag{A.7a}
$$

It is convenient to use shortening notation

$$
\frac{E+m}{e}=\sigma\,.
$$

Eq. $(A.7a)$ has 6 singular points

$$
0\,,\qquad \infty\,,\qquad \pm 1,
$$

$$
z_{1,\,2}=-\sigma\pm\sqrt{\sigma^2-1}\,,\qquad (z_1 z_2=1\,,\;\; z_1+z_2=-2\sigma)\,,
$$

and it reads

$$
\frac{d^2 f}{dz^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right]\frac{df}{dz}
$$

$$
+\left[\frac{4Ee-2\sigma\,\nu}{z}-\frac{(E+e)^2-m^2-\nu}{z-1}+\frac{(E-e)^2-m^2-\nu}{z+1}+\frac{e^2-\nu^2}{z^2}\right.
$$

$$
\left.+\frac{(E+e)^2-m^2}{(z-1)^2}+\frac{(E-e)^2-m^2}{(z+1)^2}+\frac{A}{z-z_1}+\frac{B}{z-z_2}\right] f=0\,, \tag{A.7b}
$$

where

$$
2\nu\frac{\sigma\,z+2\,\sigma^2-1}{(z-z_1)\,(z-z_2)}=\frac{A}{z-z_1}+\frac{B}{z-z_2}\,,
$$

$$
A=2\nu\frac{\sigma z_1+2\sigma^2-1}{z_1-z_2}\,,\qquad B=2\nu\frac{\sigma z_2+2\sigma^2-1}{z_2-z_1}\,. \tag{A.8a}
$$

Let us introduce notation

$$
C=(E+e)^2-m^2\,,\qquad D=(E-e)^2-m^2\,,\qquad 4Ee=C-D\,, \tag{A.8b}
$$

then eq. $(A.7b)$ can be presented shorter

$$
\frac{d^2 f}{dz^2}+\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right)\frac{df}{dz}
$$

$$
+\left(\frac{C-D-2\sigma\,\nu}{z}-\frac{C-\nu}{z-1}+\frac{D-\nu}{z+1}+\frac{e^2-\nu^2}{z^2}\right.
$$

$$
\left.+\frac{C}{(z-1)^2}+\frac{D}{(z+1)^2}+\frac{A}{z-z_1}+\frac{B}{z-z_2}\right) f=0\,. \tag{A.9}
$$

Behavior of its solutions near 3 singular points $0,+1,-1$ is given by

$$x=+1\,, \qquad f''+\frac{1}{x-1}f'+C\,f=0\,, \qquad f\sim z^{\alpha}\,,\ \alpha=\pm\sqrt{-C}\,;$$

$$x=-1\,, \qquad f''+\frac{1}{x+1}f'+D\,f=0\,, \qquad f\sim z^{\beta}\,,\ \beta=\pm\sqrt{-D}\,;$$

$$x=0\,, \qquad f''+\frac{1}{x}f'+(e^2-\nu^2)f=0\,, \qquad f\sim z^{M}\,,\ M=\pm\sqrt{\nu^2-e^2}\,. \tag{A.10a}$$

Besides, near the singular point $z=\infty$ we have

$$x=\infty\,, \qquad f''+\frac{1}{x}f'+(\frac{A+B-2\sigma\nu}{x}+\frac{C+D+e^2-\nu^2}{x^2})\;f=0\,,$$

remembering that $A+B-2\sigma\nu=0$ we get

$$a\sim x^{N}\,, \qquad N=\pm\sqrt{C+D+e^2-\nu^2}\,. \tag{A.10b}$$

Let us search solutions in the form

$$f=x^{M}(z-1)^{\alpha}(z+1)^{\beta}\varphi\,;$$

eq. (A.9) leads ro

$$\frac{d^2\varphi}{dz^2}+\left[\frac{2M+1}{z}+\frac{2\alpha+1}{z-1}+\frac{2\beta+1}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right]\frac{d\varphi}{dz}$$
$$+\left[\frac{M^2+e^2-\nu^2}{z^2}+\frac{\alpha^2+C}{(z-1)^2}+\frac{\beta^2+D}{(z+1)^2}\right.$$
$$+\frac{C-D-(\alpha-\beta)\,(2M+1)-2\sigma(\nu+M)}{z}$$
$$+\frac{M+\alpha/2+\beta/2-C+\nu+2M\alpha+\alpha\beta}{z-1}$$
$$-\frac{M+\alpha/2+\beta/2-D+\nu+2M\beta+\alpha\beta}{z+1}$$
$$\left.+\frac{1}{z-z_1}\left(A-\frac{\alpha}{z_1-1}-\frac{\beta}{z_1+1}-\frac{M}{z_1}\right)+\frac{1}{z-z_2}\left(B-\frac{\alpha}{z_2-1}-\frac{\beta}{z_2+1}-\frac{M}{z_2}\right)\right]\varphi=0\,. \tag{A.11a}$$

Requiring

$$M^2=\pm\sqrt{\nu^2-e^2}\,, \qquad \alpha=\pm\sqrt{-C}\,, \qquad \beta=\pm\sqrt{-D}\,,$$

we arrive at a differential equation with six singular points $(0,\infty,+1,-1,z_1,z_2)$

$$\frac{d^2\varphi}{dz^2}+\left[\frac{2M+1}{z}+\frac{2\alpha+1}{z-1}+\frac{2\beta+1}{z+1}-\frac{1}{z-z_1}-\frac{1}{z-z_2}\right]\frac{d\varphi}{dz}$$

$$+\left[\frac{C-D-(\alpha-\beta)\,(2M+1)-2\sigma(\nu+M)}{z}\right.$$
$$+\frac{M+\alpha/2+\beta/2-C+\nu+2M\alpha+\alpha\beta}{z-1}$$
$$-\frac{M+\alpha/2+\beta/2-D+\nu+2M\beta+\alpha\beta}{z+1}$$
$$+\frac{1}{z-z_1}\left(A-\frac{\alpha}{z_1-1}-\frac{\beta}{z_1+1}-\frac{M}{z_1}\right)$$
$$\left.+\frac{1}{z-z_2}\left(B-\frac{\alpha}{z_2-1}-\frac{\beta}{z_2+1}-\frac{M}{z_2}\right)\right]\varphi=0\;. \tag{A.11b}$$

Let us again try to perform a linear transformation of the type (A.2) – which results in

$$\left[\frac{d}{dz}-b'a+c'd+\frac{\nu}{z}\,(ba+cd)+\left(\frac{e}{z}+\frac{-E-e-m}{z-1}+\frac{E-e+m}{z+1}\right)\;bd\right.$$
$$\left.+\left(\frac{e}{z}+\frac{-E-e+m}{z-1}+\frac{E-e-m}{z+1}\right)\;ca\;\right]F$$
$$=\left[b'c-bc'-\frac{\nu}{z}\;2bc-\left(\frac{e}{z}+\frac{-E-e-m}{z-1}+\frac{E-e+m}{z+1}\right)b^2\right.$$
$$\left.-\left(\frac{e}{z}+\frac{-E-e+m}{z-1}+\frac{E-e-m}{z+1}\right)c^2\;\right]G\;,$$
$$\left[\frac{d}{dz}+d'c-a'b-\frac{\nu}{z}\,(dc+ab)-\left(\frac{e}{z}+\frac{-E-e-m}{z-1}+\frac{E-e+m}{z+1}\right)\;bd\right.$$
$$\left.-\left(\frac{e}{z}+\frac{-E-e+m}{z-1}+\frac{E-e-m}{z+1}\right)\;ca\right]G$$
$$=\left[-d'a+da'+\frac{\nu}{z}\;2ad+\left(\frac{e}{z}+\frac{-E-e-m}{z-1}+\frac{E-e+m}{z+1}\right)d^2\right.$$
$$\left.+\left(\frac{e}{z}+\frac{-E-e+m}{z-1}+\frac{E-e-m}{z+1}\right)\;a^2\right]F\;. \tag{A.12}$$

When the transformation is orthogonal, eqs. (A.12) become simpler

$$\left[\frac{d}{dz}+\frac{\nu}{z}\;\cos A-\left(\frac{e}{z}+\frac{-E-e-m}{z-1}+\frac{E-e+m}{z+1}\right)\;\frac{\sin A}{2}\right.$$
$$\left.+\left(\frac{e}{z}+\frac{-E-e+m}{z-1}+\frac{E-e-m}{z+1}\right)\;\frac{\sin A}{2}\right]F$$
$$=\left[-\frac{A'}{2}-\frac{\nu}{z}\;\sin A-\left(\frac{e}{z}+\frac{-E-e-m}{z-1}+\frac{E-e+m}{z+1}\right)\cos^2\frac{A}{2}\right.$$
$$\left.-\left(\frac{e}{z}+\frac{-E-e+m}{z-1}+\frac{E-e-m}{z+1}\right)\;\sin^2\frac{A}{2}\right]G\;,$$

$$\Big[\frac{d}{dz} - \frac{\nu}{z}\ \cos A + \Big(\frac{e}{z} + \frac{-E-e-m}{z-1} + \frac{E-e+m}{z+1}\Big) \frac{\sin A}{2}$$

$$- \Big(\frac{e}{z} + \frac{-E-e+m}{z-1} + \frac{E-e-m}{z+1}\Big)\ \frac{\sin A}{2}\Big] G$$

$$= \Big[+\frac{A'}{2} - \frac{\nu}{z}\ \sin A + \Big(\frac{e}{z} + \frac{-E-e-m}{z-1} + \frac{E-e+m}{z+1}\Big) \sin^2 \frac{A}{2}$$

$$+ \Big(\frac{e}{z} + \frac{-E-e+m}{z-1} + \frac{E-e-m}{z+1}\Big)\ \cos^2 \frac{A}{2}\Big]\ F\ .$$

After simple algebraic manipulation we arrive at the system

$$\Big(\frac{d}{dz} + \frac{\nu\cos A}{z}\ + \frac{m\sin A}{z-1} - \frac{m\sin A}{z+1}\Big) F$$

$$= \Big(-\frac{A'}{2} - \frac{e+\nu\sin A}{z} - \frac{-E-e-m\cos A}{z-1} - \frac{E-e+m\cos A}{z+1}\Big) G\ ,$$

$$\Big(\frac{d}{dz} - \frac{\nu\cos A}{z} - \frac{m\sin A}{z-1} + \frac{m\sin A}{z+1}\Big) G$$

$$= \Big(+\frac{A'}{2} + \frac{e-\nu\sin A}{z} + \frac{-E-e+m\cos A}{z-1} + \frac{E-e-m\cos A}{z+1}\Big) F\ . \qquad (A.13)$$

In the limit of flat space model, $z << 1$, they lead to the known formulas (1.5).

Let us try the variant (see Section **1.10**)

$$\sin A = e/\nu\ ; \qquad (A.14a)$$

in this case eqs. (A.13) read

$$\Big(\frac{d}{dz} + \frac{\nu\cos A}{z} + \frac{m\sin A}{z-1} - \frac{m\sin A}{z+1}\Big) F$$

$$= \Big(\ -\frac{2\nu}{z}\sin A - \frac{-E-e-m\cos A}{z-1} - \frac{E-e+m\cos A}{z+1}\Big) G\ ,$$

$$\Big(\ \frac{d}{dz} - \frac{\nu\cos A}{z} - \frac{m\sin A}{z-1} + \frac{m\sin A}{z+1}\Big) G = \frac{(2m\cos A - 2E) - 2ez}{(z-1)(z+1)}\ F\ . \qquad (A.14b)$$

Excluding the variable F, we get

$$\Big(\frac{d}{dz} + \frac{\nu\cos A}{z} + \frac{m\sin A}{z-1} - \frac{m\sin A}{z+1}\Big) \frac{(z-1)(z+1)}{(2m\cos A - 2E) - 2ez}$$

$$\times \Big(\frac{d}{dz} - \frac{\nu\cos A}{z} - \frac{m\sin A}{z-1} + \frac{m\sin A}{z+1}\Big) G$$

$$= \Big(-\frac{2\nu}{z}\sin A - \frac{-E-e-m\cos A}{z-1} - \frac{E-e+m\cos A}{z+1}\Big) G\ ,$$

$$(A.14c)$$

from whence it follows

$$\frac{d^2}{dz^2}G + \left(\frac{d}{dz}\ln\frac{(z-1)(z+1)}{(2m\cos A - 2E) - 2ez}\right)\left(\frac{d}{dz} - \frac{\nu\cos A}{z} - \frac{m\sin A}{z-1} + \frac{m\sin A}{z+1}\right)G$$
$$-\left(\frac{\nu}{z}\cos A + \frac{m\sin A}{z-1} - \frac{m\sin A}{z+1}\right)^2 G - \left(\frac{\nu\cos A}{z} + \frac{m\sin A}{z-1} - \frac{m\sin A}{z+1}\right)' G$$
$$+\frac{2m\cos A - 2E - 2ez}{(z-1)(z+1)}\left(\frac{2\nu}{z}\sin A + \frac{-E-e-m\cos A}{z-1} + \frac{E-e+m\cos A}{z+1}\right)G = 0.$$

It is convenient to introduce a special designation for the singular point $z_0 = (m\cos A - E)/e$, then

$$\frac{d^2}{dz^2}G + \left(\frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_0}\right)\frac{d}{dz}$$
$$+\left(\frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_0}\right)\left(-\frac{\nu\cos A}{z} - \frac{m\sin A}{z-1} + \frac{m\sin A}{z+1}\right)G$$
$$-\left(\frac{\nu}{z}\cos A + \frac{m\sin A}{z-1} - \frac{m\sin A}{z+1}\right)^2 G + \left(\frac{\nu\cos A}{z^2} + \frac{m\sin A}{(z-1)^2} - \frac{m\sin A}{(z+1)^2}\right)G$$
$$-e\left(\frac{1-z_0}{z-1} + \frac{1+z_0}{z+1}\right)\left(\frac{2\nu}{z}\sin A + \frac{-E-e-m\cos A}{z-1} + \frac{E-e+m\cos A}{z+1}\right)G = 0\,. \qquad (A.14d)$$

With the use of notation

$$m\cos A = c\,, \qquad m\sin A = s\,, \qquad \nu\cos A \to \nu\,, \qquad (A.15)$$

and taking in mind identity $\nu\sin A = e$, eq. $(A.14d)$ is written shorter

$$\frac{d^2}{dz^2}G + \left(\frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_0}\right)\frac{d}{dz}$$
$$+\Bigg[\left(\frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_0}\right)\left(-\frac{\nu}{z} - \frac{s}{z-1} + \frac{s}{z+1}\right)$$
$$-\left(\frac{\nu}{z} + \frac{s}{z-1} - \frac{s}{z+1}\right)^2 + \left(\frac{\nu}{z^2} + \frac{s}{(z-1)^2} - \frac{s}{(z+1)^2}\right)$$
$$-e\left(\frac{1-z_0}{z-1} + \frac{1+z_0}{z+1}\right)\left(\frac{2e}{z} + \frac{-E-e-c}{z-1} + \frac{E-e+c}{z+1}\right)\Bigg]G = 0\,. \qquad (A.16)$$

Equation (A.16) reduces to

$$\frac{d^2}{dz^2}G + \left(\frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_0}\right)\frac{d}{dz}$$

$$
+\Big[\ \frac{1}{z}\frac{-\nu+4\nu s z_0-4z_0^2e^2}{z_0}+\frac{1}{z-z_0}\ \frac{z_0^2\nu+2z_0s-\nu}{z_0(1+z_0)(z_0-1)}-\frac{\nu(\nu-1)}{z^2}
$$

$$
+\frac{1}{z+1}\ (+\nu-s^2-2\nu s+e^2+2e^2z_0-ez_0c-ez_0E+\frac{s}{1+z_0})
$$

$$
+\frac{1}{z-1}\ (-\nu+s^2-2\nu s-e^2+2e^2z_0+ez_0c+ez_0E+\frac{s}{1-z_0})
$$

$$
+\frac{-s^2+(1-z_0)\,(e^2+ec+eE)}{(z-1)^2}+\frac{-s^2+(1+z_0)\,(e^2-ec-eE)}{(z+1)^2}\Big]\ G=0\ . \tag{A.17}
$$

Let us specify behavior of $G(z)$ near the singular points

$$
0, \qquad \infty, \qquad +1, \qquad -1\ .
$$

Near the point $z=0$ we have

$$
\frac{d^2}{dz^2}G+\frac{1}{z_0}\frac{d}{dz}G-\frac{\nu(\nu-1)}{z^2}G=0\ ;
$$

therefore G behaves in accordance with

$$
G\sim z^a, \qquad a(a-1)z^{a-2}+\frac{1}{z_0}+a\ z^{a-2}z-\nu(\nu-1)z^{a-2}=0 \qquad \Longrightarrow
$$

$$
a(a-1)=\nu(\nu-1)\ , \qquad a=\underline{+\nu,}\ 1-\nu\ ; \tag{A.18a}
$$

positive values $a=+\nu$ correspond to bound states.

Near the point $z=1$, we have

$$
\frac{d^2}{dz^2}G+\frac{1}{z-1}\frac{d}{dz}+\frac{-s^2+(1-z_0)\,(e^2+ec+eE)}{(z-1)^2}\ G=0\ ,
$$

that is

$$
G\sim(z-1)^M\ , \qquad M=\pm\sqrt{s^2-(1-z_0)\,(e^2+ec+eE)}\ ; \tag{A.18b}
$$

because $z=\tanh{(\beta/2)}$, the sign minus in ($A.18b$) corresponds to the bound states.

In the same manner, consider the vicinity of the point $z=-1$:

$$
\frac{d^2}{dz^2}G+\frac{1}{z+1}\frac{d}{dz}+\frac{-s^2+(1+z_0)\,(e^2-ec-eE)}{(z+1)^2}\ G=0\ ,
$$

that is

$$
G\sim(z+1)^N\ , \qquad N=\pm\sqrt{s^2-(1+z_0)\,(e^2-ec-eE)}\ ; \tag{A.18c}
$$

because $z=\tanh{(\beta/2)}$, the singular point $z=-1$ is not physical one.

For brevity, let us introduce special notation for coefficients at the for singular terms

$$
\frac{K}{z-z_0}\ , \qquad \frac{K_0}{z}\ , \qquad \frac{K_-}{x-1}\ , \qquad \frac{K_+}{x+1}\ ,
$$

then eq. (A.17) reads shorter

$$\frac{d^2}{dz^2}G + \left(\frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{z-z_0} \right) \frac{d}{dz}G$$

$$+ \left[\frac{K_0}{z} + \frac{K}{z-z_0} - \frac{\nu(\nu-1)}{z^2} + \frac{K_+}{z+1} + \frac{K_-}{z-1} - \frac{M^2}{(z-1)^2} - \frac{N^2}{(z+1)^2} \right] G = 0 \,. \tag{A.19}$$

Let us introduce the following substitution

$$G = z^a (z-1)^M (z+1)^N \varphi \; ;$$

after simple calculation we arrive at

$$\begin{aligned} &\frac{d^2\varphi}{dz^2} + \left[\frac{2a}{z} + \frac{2m+1}{z-1} + \frac{2n+1}{z+1} - \frac{1}{z-z_0} \right] \frac{d\varphi}{dz} \\ &+ \left[\frac{a(a-1)-\nu(\nu-1)}{z^2} + \frac{m^2-M^2}{(z-1)^2} \right. \\ &+ \frac{n^2-N^2}{(z+1)^2} + \frac{1}{z-z_0} \left(K_0 - \frac{a}{z_0} - \frac{m}{z_0-1} - \frac{n}{z_0+1} \right) \\ &+ \frac{1}{z} \left(K + \frac{a[-2mz_0+1+2nz_0]}{z_0} \right) \\ &+ \frac{1}{z-1} \left(K_- + \frac{1}{2}(2a+n)(2m+1) + \frac{1}{2} \frac{m(z_0+1)}{z_0-1} \right) \\ &\left. + \frac{1}{z+1} \left(K_+ - \frac{1}{2}(2a+m)(2n+1) - \frac{1}{2} \frac{n(z_0-1)}{z_0+1} \right) \right] \varphi = 0 \,. \end{aligned} \tag{A.20a}$$

Requiring

$$a(a-1) = \nu(\nu-1) \; , \qquad m^2 = M^2 \; , \qquad n^2 = N^2 \; ,$$

we arrive at

$$\begin{aligned} &\frac{d^2\varphi}{dz^2} + \left[\frac{2a}{z} + \frac{2m+1}{z-1} + \frac{2n+1}{z+1} - \frac{1}{z-z_0} \right] \frac{d\varphi}{dz} \\ &+ \left[\frac{1}{z-z_0} \left(K_0 - \frac{a}{z_0} - \frac{m}{z_0-1} - \frac{n}{z_0+1} \right) + \frac{1}{z} \left(K + \frac{a(-2mz_0+1+2nz_0)}{z_0} \right) \right. \\ &+ \frac{1}{z-1} \left(K_- + \frac{1}{2}(2a+n)(2m+1) + \frac{1}{2} \frac{m(z_0+1)}{z_0-1} \right) \\ &\left. + \frac{1}{z+1} \left(K_+ - \frac{1}{2}(2a+m)(2n+1) - \frac{1}{2} \frac{n(z_0-1)}{z_0+1} \right) \right] \varphi = 0 \,. \end{aligned} \tag{A.20b}$$

In the Riemann space, the Kepler problem can be treated in a similar way.

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