Geometric families of 4-dimensional Galois representations with generically large images

Luis Dieulefait

Núria Vila

Dept. d'Álgebra i Geometria, Universitat de Barcelona;

Gran Via de les Corts Catalanes 585; 08007 - Barcelona; Spain.

e-mail: ldieulefait@ub.edu; vila@mat.ub.es.

Running head: Four-dimensional Galois representations

Author to send proofs:

Núria Vila
Dept. d'Álgebra i Geometria, Universitat de Barcelona;
Gran Via de les Corts Catalanes 585; 08007 - Barcelona; Spain.
vila@mat.ub.es.

Abstract

We study compatible families of four-dimensional Galois representations constructed in the étale cohomology of a smooth projective variety. We prove a theorem asserting that the images will be generically large if certain conditions are satisfied. We only consider representations with coefficients in an imaginary quadratic field. We apply our result to an example constructed by Jasper Scholten (see [Sc]), obtaining a family of linear groups and one of unitary groups as Galois groups over $\mathbb{Q}$.

1   Introduction

In this article we study geometric compatible families of four-dimensional Galois representations with coefficients in a quadratic imaginary field K and their images. We will assume that the representations have four different Hodge-Tate weights.

After introducing some of the main tools, we study all possible types of residual image and show that, under certain conditions, the residual images, and the images themselves, of these geometric representations are "as large as possible" for almost every prime (i.e., for all but finitely many primes): a linear or a unitary group, depending on the decomposition type of the prime in K. Then, we do explicit computations for one example constructed by J. Scholten and we bound the finite exceptional set, i.e., the set of primes where the image is not "as large as possible" with an explicit small density set of primes. Furthermore, we prove that this example verifies all conditions of the result of the previous section and, as a consequence, for this example the images are "as large as possible" for almost every prime. We stress the consequences of these results for Inverse Galois Theory.

Results of generically large images similar to the one obtained in this article were obtained by the authors in [D-V] for compatible families of three-dimensional Galois representations.

The first large image result of this kind was obtained by Serre for the case of elliptic curves without Complex Multiplication (cf. [S1]). Serre also obtained a large image result for the four-dimensional Galois representations attached to abelian surfaces with trivial endomorphism ring (cf. [S2]), in his result he considers only symplectic representations (he can do this by reducing to

2

the case of principally polarized abelian varieties), while we will work with four-dimensional representations which are not symplectic. The first author also proved a large image result for the case of four-dimensional Galois representations attached to genus two Siegel modular forms (cf. [D1]), but in that case he also restricts to symplectic representations.

2 Geometric representations

A compatible family of geometric four-dimensional λ-adic Galois representations was constructed by Scholten (see [Sc]), we will recall this construction in section 7. In this section we will describe a fairly general class of compatible families of 4-dimensional Galois representations containing the one constructed by Scholten, all the results in this article hold for any family in this class.

Let us start with a smooth projective variety S defined over $\mathbb{Q}$ and we consider the action of $G_{\mathbb{Q}}$ on the groups $H^3(S)_\ell := H^3_{et}(S_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$.

Let N be the product of the primes of bad reduction of S. This gives a compatible family of Galois representations unramified at primes not dividing N.

Let K be an imaginary quadratic field, which for simplicity (and according to the construction in section 7) we will assume from now on to be $\mathbb{Q}(\zeta)$, where ζ is a cubic root of unity.

Assume that the Galois representations on $H^3(S)_\ell \otimes_{\mathbb{Q}_\ell} K_\ell$ are all reducible, and they contain as subrepresentations a compatible family $\{\sigma_{S,\lambda}\}$ of λ-adic 4-dimensional Galois representations, with λ prime in K dividing ℓ. For this family it also holds that the ramification set is (at most) the set of prime factors of N. We will assume that the traces at Frobenius elements of these 4-dimensional Galois representations, which lie all in K, generate K (i.e., that they are not all rational integers). We will also assume that these representations have four different Hodge-Tate numbers: $\{0, 1, 2, 3\}$, and that the determinant of each $\sigma_{S,\lambda}$ is just χ^6, where χ denotes the ℓ-adic cyclotomic character (this last condition can always be satisfied after twisting by a suitable Dirichlet character unramified outside N).

Thus, the families we are considering can be shortly described as Galois representations associated to a pure motive defined over $\mathbb{Q}$ with coefficients in a quadratic field K and Hodge-Tate numbers $\{0, 1, 2, 3\}$ (purity follows from Deligne's proof of the Weil's conjectures), with the simplifying assumptions that $K = \mathbb{Q}(\zeta)$ and the determinants are just χ^6.

From the compacity of $G_{\mathbb{Q}}$ and the continuity of the representations $\sigma_{S,\lambda}$, it follows that (after a suitable conjugation) we can assume that the images are contained in $\mathrm{GL}(4, \mathcal{O}_\lambda)$, where $\mathcal{O}$ denotes the ring of integers of K. This implies that we can consider the residual representations $\bar{\sigma}_{S,\lambda}$ with values in $\mathrm{GL}(4, \mathbb{F}_\lambda)$, obtained by composing $\sigma_{S,\lambda}$ with the naive map "reduction modulo λ".

Observe that from purity it follows that if we denote by a_p the trace of the characteristic polynomial of $\sigma_{S,\lambda}(\mathrm{Frob}\ p)$ for any prime p of good reduction (and $p \neq \ell$), it holds: $|a_p| \leq 4p^{3/2}$.

The characteristic polynomials of $\sigma_{S,\lambda}(\mathrm{Frob}\ p)$ $(p \nmid \ell N)$ will be denoted

$$x^4 - a_p x^3 + b_p x^2 - p^3 \bar{a}_p x + p^6 \qquad (*)$$

The particular form of these polynomials follows, as is well known, from purity. Recall that the values a_p generate K.
We will also assume (once again, this property holds in Scholten example) that all quadratic coefficients b_p are rational integers.

Remark: We require that the traces at Frobenius elements generate K, an imaginary quadratic field, because in the opposite case, if all traces are real, the representations would be self-dual, thus for all primes λ the images would not be "as large as possible" (in fact, the images would be contained in an orthogonal group). Observe that even with the assumption that the traces generate K, the particular form of the characteristic polynomials in $(*)$ (together with Cebotarev density theorem) imply that for any prime λ inert in K the image of $\sigma_{S,\lambda}$ will be contained in the unitary group $\mathrm{GU}(4, \mathbb{Z}_\ell)$.

By conductor of the representations $\sigma_{S,\lambda}$, $\mathrm{cond}(\sigma_{S,\lambda})$, we mean the prime-to-$\ell$ part of the Artin conductor. It is known that the conductor of the representations $\sigma_{S,\lambda}$ is bounded independently of ℓ, because the representations are

constructed from a smooth projective variety. This follows from Deligne's corollary to the results of de Jong (cf. [Be], Proposition 6.3.2). We will denote by $c := \text{l.c.m.}\{\text{cond}(\sigma_{S,\lambda})\}_\lambda$, this finite uniform bound, and we will call it "conductor of the family of representations".

3 Classification of subgroups of $\mathrm{GL}(4, q)$

The classification of the maximal subgroups $\mathcal{G}$, up to conjugation, of $\mathrm{GL}(4, p^t)$ can be derived from the results in Kleidman and Liebeck (cf. [K-L]), as done in [K] and in [N-P], pag. 592 (see also [A4-EMS], pags. 173 to 177). We will only consider the case $t = 1$ or 2. In the following proposition, we are also using, for $t = 2$, the classification of maximal proper subgroups of $\mathrm{GU}(4, p^t)$ (also done in [K]).

Proposition 3.1 *Let $\mathcal{G}$ be a subgroup of $\mathrm{GL}(4, p^t)$ with $t = 1$ or 2, $p > 3$, and suppose that $\mathcal{G}$ does not contain $\mathrm{SL}(4, p^t)$. Let Z be the subgroup of scalar matrices contained in $\mathcal{G}$. Then at least one of the following holds:*
1) $\mathcal{G}$ is reducible;
2) $\mathcal{G}$ is imprimitive, that is, conjugate either to (a) a subgroup of the group $\mathrm{GL}(1, p^t) \wr S_4$ of monomial matrices, or to (b) a subgroup of $\mathrm{GL}(2, p^t) \wr C_2$ (here C_2 denotes a group of order 2);
3) $\mathcal{G}$ is conjugate to a subgroup of the semidirect product of $\mathrm{GL}(2, p^{2t})$ by C_2 acting by field automorphisms;
4) (only for $t = 2$) $\mathcal{G}$ is conjugate to a subgroup of $\mathbb{F}_{p^2}^* \cdot \mathrm{GL}(4, p)$;
5) $\mathcal{G}$ normalizes an extraspecial group of order 2^5 and $\mathcal{G}/Z$ is isomorphic to a subgroup of the split extension $2^4 \cdot S_5$;
6) $\mathcal{G}$ is conjugate to a subgroup of $\mathrm{GO}^+(4, p^t)$, $\mathrm{GO}^-(4, p^t)$ or $\mathrm{GSp}(4, p^t)$;
7) $\mathcal{G}$ is absolutely irreducible and primitive, and $\mathcal{G}/Z$ is isomorphic to one of the following: A_5, S_5, A_6, S_6, A_7, $\mathrm{PSL}(2, 7)$ or $\mathrm{PSp}(4, 3)$;
8) (only for $t = 2$) $\mathcal{G}$ is conjugate to a subgroup of $\mathrm{GU}(4, p)$.
Furthermore, if $t = 2$, $\mathcal{G}$ is contained in $\mathrm{GU}(4, p)$ (as in case (8)) and $\mathcal{G}$ does not fall in any of cases (1) to (7), then $\mathrm{SU}(4, p) \subseteq \mathcal{G} \subseteq \mathrm{GU}(4, p)$.

4   Comparison of Étale and Crystalline cohomologies and the action of inertia

As a consequence of the result of Fontaine-Messing in [F-M] giving a (canonical, functorial) isomorphism between the étale cohomology and the crystalline cohomology for a variety S having good reduction at ℓ and with $\dim(S) < \ell$, the following theorem, conjectured by Serre in [S2], holds:

Theorem 4.1 *Let V be a $G_\mathbb{Q}$-stable lattice in $H_{et}^m(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ and $\tilde{V}$ the semi-simplification of $V/\ell V$ with respect to the action of $I = I_\ell$. Then, if $\dim(S) < \ell$ and S has good reduction at ℓ, the action of I on each irreducible component M of $\tilde{V}$ is given by a character $\Psi : I_t \to \mathbb{F}_q^*$ with $q = \ell^n, n = \dim M$, such that*

$$\Psi = \Psi_f^{d_0 + \ell d_1 + \ldots + \ell^{n-1} d_{n-1}}$$

with Ψ_f fundamental character of level n and the exponents verify: $0 \le d_i \le m$.

Moreover, since we know that the Hodge-Tate numbers of our family of four-dimensional representations are $\{0, 1, 2, 3\}$, if $\ell > 3$ we can be more specific (cf. [F-L], theorem 5.3): the digits d_i appearing in the exponents above (for M ranging over all irreducible components) must agree with the Hodge-Tate numbers. For example, if $\tilde{V}$ has, with respect to the action of I_ℓ, an irreducible component M_1 of dimension one and another M_2 of dimension three, then the action on M_1 is given by χ^a, and the action on M_2 by $\psi_3^{b + c\ell + d\ell^2}$, and it must hold: $\{a, b, c, d\} = \{0, 1, 2, 3\}$.

Now that we know that the exponents of the action of I_ℓ through fundamental characters are fixed, observe that there are many possibilities for this action since it will depend on the dimension of the irreducible components, and also on permutations of the four exponents. Nevertheless, it is easy to see that in any case the image of inertia can not be too small. More precisely, an easy computation gives the following:

Lemma 4.2 *Consider a four-dimensional crystalline representation σ_λ with Hodge-Tate numbers $\{0, 1, 2, 3\}$. Then, if $\ell > 3$, the exponents of the fundamental characters giving the action of the (tame) inertia subgroup I_ℓ of the*

residual representation $\bar{\sigma}_\lambda$ are also $\{0, 1, 2, 3\}$. Let $P_I := \mathbb{P}(\bar{\sigma}_\lambda|_{I_\ell}^{s.s.})$ be the projectivization of the image of I_ℓ. Then P_I is a cyclic group with order greater than or equal to $\ell - 1$.

5 Determination of the images of the geometric Galois representations

We will consider a geometric compatible family of four-dimensional Galois representations $\sigma_{S,\lambda}$ as the one described in section 2.

We will prove that the image of the residual Galois representation is a linear group for almost every prime decomposing in $\mathbb{Q}(\zeta)$, i.e., $\ell \equiv 1 \pmod 3$, and a unitary group for almost every inert prime $\ell \equiv 2 \pmod 3$, whenever certain general conditions that we will describe later are satisfied.

5.1 Reducible representations

5.1.1 With a one dimensional component

Recall that N denotes the product of the primes of bad reduction of S. Suppose that ℓ is a prime not in S, $\ell > 3$, such that $\bar{\sigma}_{S,\lambda}$ is reducible (not necessarily over the residue field $\mathbb{F}_\lambda$) with a one dimensional component. In this case there is a character μ unramified outside $N\ell$ with image in $\bar{\mathbb{F}}_\lambda^*$ such that $\mu(p)$ is a root of $x^4 - a_p x^3 + b_p x^2 - p^3 \bar{a}_p x + p^6$, for every $p \nmid \ell N$.

Let $I := I_\ell$. Using the description of $\bar{\sigma}_{S,\lambda}|_I$ we know that $\mu = \chi^i \varepsilon$, with χ the mod ℓ cyclotomic character, $i = 0, 1, 2, 3$ and ε a character unramified outside N. Clearly, $\mathrm{cond}(\varepsilon)|\mathrm{cond}(\bar{\sigma}_{S,\lambda})|c$.

Using the description of $\bar{\sigma}_{S,\lambda}|_I$ we can also see that reducibility must necessarily occur over $\mathbb{F}_\lambda$. If not, there should be (at least) two one dimensional components, being each conjugate of the other by some element of $\mathrm{Gal}(\bar{\mathbb{F}}_\lambda/\mathbb{F}_\lambda)$. But on the other hand, lemma 4.2 implies that the restriction of these two one-dimensional components to I must give two different powers χ^i and χ^j of the cyclotomic character; if they where Galois conjugated we would conclude that $i = j$ which is a contradiction.

Thus, the character ϵ has image contained in $\mathbb{F}_\lambda^*$.

At this point, we are led to introduce the following:

Condition 1): *Assume that there exists a prime p not in S such that none of the roots of $Pol_p(x)$, the characteristic polynomial of $\sigma_{S,\lambda}(\text{Frob } p)$, is a number of the form ηp^i, where η denotes an arbitrary root of unity and $i \in \{0, 1, 2, 3\}$.*

Under condition 1, we prove that there is a finite number of primes λ such that the residual mod λ representation is reducible with a one dimensional component $\mu_\lambda = \varepsilon_\lambda \chi^i$.

Assume on the contrary that this is the case for an infinite set of primes Λ. Since the characters ε_λ appearing in the one dimensional components of all these residual representations will have their conductors uniformly bounded by c, we can assume (by restricting to some infinite subset Λ' of Λ) that there is a fixed character ϵ of conductor dividing c with values in $\mathbb{C}^*$ such that the reduction of ϵ modulo λ gives ε_λ, for every $\lambda \in \Lambda'$. Of course, we can also assume (shrinking Λ' if necessary) that the exponent of the cyclotomic character in μ_λ is independent of $\lambda \in \Lambda'$.

Now take p a prime as in condition 1), and if necessary delete p from Λ'. Then, for every $\lambda \in \Lambda'$ the polynomial $Pol_p(x)$ (which is independent of λ) when reduced modulo λ has the reduction of $\epsilon(p)p^i$ modulo λ as a root.

Let d be the order of $\epsilon(p)$. Then the above congruences implies that the resultant of the polynomials $Pol_p(x)$ and $x^d - p^{id}$ is divisible by λ, for every λ in Λ': Thus, since this set of primes is infinite, we conclude that the resultant is 0, and that the polynomial $Pol_p(x)$ has a root of the form: $p^i \cdot$ root of unity, contrary to condition 1). This proves the finitude of the primes in the residual reducible case with one dimensional component, assuming condition 1).

5.1.2   With two two-dimensional irreducible components

This case can be treated in a similar way that the previous one, but considering the exterior power $\rho_\lambda := \wedge^2(\sigma_{S,\lambda})$. The determinant of the two-dimensional irreducible components of $\bar\sigma_{S,\lambda}$ give rise to one dimensional components of $\bar\rho_\lambda$.

Using the information of the action of I given in lemma 4.2, we see that a one dimensional component of $\bar\rho_\lambda$ must be of the form $\varepsilon \chi^i$, with the conductor of ε dividing c and the exponent $i \in \{1, 2, 3, 4, 5\}$. Again, using the restriction to I one can easily see that the irreducible components can not be conjugated to each other, and so they must necessarily be defined over $\mathbb{F}_\lambda$. In particular,

8

the image of ε must be contained in $\mathbb{F}_\lambda^*$.

Remark: Observe that, since we have shown that reducibility must necessarily occur over the field of definition, this eliminates case 3) of the classification given in section 3.

The natural condition to control this case is the following:

Condition 2): *Assume that there exists a prime p not in S such that none of the roots of $Q_p(x)$, the characteristic polynomial of $\wedge^2(\sigma_{S,\lambda})(\text{Frob } p)$, is a number of the form ηp^i, where η denotes an arbitrary root of unity and $i \in \{1, 2, 3, 4, 5\}$.*

As before, if we assume that this case occurs for infinitely many primes, we get congruences modulo infinitely many primes between $Q_p(x)$ and $x^d - p^{id}$, where $i \in \{1, 2, 3, 4, 5\}$ is fixed and d is a fixed divisor of $\varphi(c)$ (here φ denotes Euler function), thus we conclude that these two polynomials have a common root. For a prime p verifying condition 2) this gives a contradiction.

5.2 Imprimitive irreducible case

5.2.1 Monomial Case

Consider first the monomial case: the image of $\bar{\sigma}_{S,\lambda}$ is a group G having as normal subgroup the image H of a one-dimensional representation, with quotient $G/H := U$ a subgroup of S_4.

Observe that since the largest cicle in S_4 has order 4 and the projectivization of the image of the inertia subgroup I is cyclic of order at least $\ell - 1$ (by lemma 4.2), if $\ell > 5$ the quotient G/H gives an extension of $\mathbb{Q}$ with Galois group U unramified at ℓ, thus ramifying only at primes in N.

Remark: In [D-V] a similar situation was solved: the case of monomial three-dimensional Galois representations.

We divide in two cases:

a) U contains odd permutations

b) $U \subseteq A_4$

a) In this case, U has a quotient isomorphic to C_2, an element of U corre-

sponds to an odd permutation if and only if its image through this quotient is -1. This gives us a quadratic character ψ corresponding to a quadratic extension of $\mathbb{Q}$ unramified outside N, which is a quotient of $\bar{\sigma}_{S,\lambda}$. An element g of the Galois group of $\mathbb{Q}$ has $\psi(g) = -1$ if and only if the matrix $\bar{\sigma}_{S,\lambda}(g)$ becomes diagonal after an odd permutation of its columns. It is easy to see that all such matrices have a property that can be read in their characteristic polynomials $Pol_g(x)$ (as a polynomial with coefficients in $\mathbb{F}_\lambda$): $Pol_g(x)$ has (at least) one pair of opposite roots $\alpha, -\alpha$. Taking $g = \text{Frob } p$, $p \nmid N\ell$, we now translate this property in terms of the coefficients of $Pol_p(x)$:

$$p^3(a_p^2 + \bar{a}_p^2) \equiv a_p \bar{a}_p b_p \pmod{\lambda} \qquad (*)$$

(in general, for a polynomial $x^4 + t_1 x^3 + t_2 x^2 + t_3 x + t_4$, the property of having two opposite roots gives the relation: $t_3^2 + t_4 t_1^2 = t_1 t_2 t_3$).

Thus, we have seen that if the image of $\bar{\sigma}_{S,\lambda}$ falls in case a), and $\ell > 5$, there should exist a quadratic character ψ unramified outside N such that for every prime $p \nmid N\ell$ with $\psi(p) = -1$, the relation $(*)$ is satisfied. Therefore, using the principle that an equality of two algebraic integers is equivalent to infinitely many congruences:

$$A = B \iff A \equiv B \pmod{\lambda}, \text{for infinitely many prime ideals } \lambda$$

we just have to impose the following condition to make sure that case a) can only occur for finitely many primes λ:

Condition 3): *For every quadratic character ψ unramified outside N there exists a prime $p \nmid N$ with $\psi(p) = -1$ and*

$$p^3(a_p^2 + \bar{a}_p^2) \neq a_p \bar{a}_p b_p.$$

b) In this case, we use the fact that A_4 has a normal subgroup T isomorphic to $C_2 \times C_2$ with quotient $A_4/T \cong C_3$. Since U is a subgroup of A_4, if we denote $T' = T \cap U$, T' is a normal subgroup of U with quotient $U/T' \cong C_3$ or $T' = U$. We will assume that we are not in the case $T' = U$, because in this case the representation would either be reducible, or with image contained in $GL(2, p^t) \wr C_2$, a case that we will consider later.

Thus, we have a C_3 extension which is a quotient of our representation: this gives us a cubic character ϕ corresponding to a cubic extension of $\mathbb{Q}$

10

unramified outside N. An element g of the Galois group of $\mathbb{Q}$ has $\phi(g) \neq 1$ if and only if the matrix $\bar{\sigma}_{S,\lambda}(g)$ becomes diagonal after a permutation $P \in U$ of its columns such that the projection $\bar{P} \in U/T' \cong C_3$ is not the identity. It is easy to see that all such matrices have a property that can be read in their characteristic polynomials: they have three roots α, β, γ such that $\alpha + \beta + \gamma = 0$.

Taking $g = \operatorname{Frob} p$, this condition translates into the following restriction for the coefficients of $Pol_p(x)$:

$$a_p^2 b_p + p^6 \equiv p^3 \bar{a}_p a_p \pmod{\lambda} \qquad (**)$$

(in general, for a polynomial $x^4 + t_1 x^3 + t_2 x^2 + t_3 x + t_4$, the property of having three roots whose sum is 0 gives the relation: $t_1^2 t_2 + t_4 = t_3 t_1$).

Thus, if the image of the residual mod λ representation falls in case b) and $\ell > 5$, there is a cubic character ϕ unramified outside N such that for every prime $p \nmid N\ell$ with $\phi(p) \neq 1$, the relation $(**)$ holds.

Obviously, the right condition that must be imposed to guarantee that this case can only occur for finitely many primes is the following:

Condition 4): *For every cubic character ϕ unramified outside N there exists a prime $p \nmid N$ with $\phi(p) \neq 1$ and*

$$a_p^2 b_p + p^6 \neq p^3 \bar{a}_p a_p.$$

Observe that since the right hand side is a rational integer, it is enough for this to hold to have $a_p^2 b_p \notin \mathbb{R}$.

5.2.2 Case 2)b) in the classification

This case already appeared in the study of symplectic representations done in [D1] and [D2], the required condition that the image of inertia must be not too small holds for every $\ell > 3$ by lemma 4.2. Let us recall the result: if the image of $\bar{\sigma}_{S,\lambda}$ falls in this case and $\ell > 3$, then there is a quadratic character μ unramified outside N such that for every prime $p \nmid N\ell$ with $\mu(p) = -1$, it holds:

$$a_p \equiv 0 \pmod{\lambda}.$$

Observe that the situation is similar to the case of "dihedral image" of two-dimensional Galois representations, studied by Serre and Ribet.

We impose the following condition to ensure that this case does not occur

for infinitely many primes:

Condition 5): *For every quadratic character μ unramified outside N there exists a prime $p \nmid N$ with $\mu(p) = -1$ and $a_p \neq 0$.*

5.3 Orthogonal and Symplectic groups

If the image is contained in an orthogonal or symplectic group (case 6 of the classification) we use the following common property of these groups: the roots of the characteristic polynomial of any matrix come in reciprocal pairs. In particular, if we take $Pol_p(x)$ the characteristic polynomial of $\bar{\sigma}_{S,\lambda}(\text{Frob } p)$ for an unramified prime p, its roots will be $\alpha, \beta, c/\alpha, c/\beta$, with $c = \pm p^3$. This translates into the following condition on the coefficients of $Pol_p(x)$: for every $p \nmid N\ell$

$$\bar{a}_p \equiv \pm a_p \pmod{\lambda}.$$

We impose the following condition to ensure that this case does not occur for infinitely many primes:

Condition 6): *There exists a prime $p \nmid N$ such that: $a_p \neq \pm\bar{a}_p$.*

Observe that this is equivalent to impose that $a_p^2 \notin \mathbb{R}$.

5.4 Case 4) in the classification

We apply the same technique used to treat the corresponding case for symplectic representations in [D1]. In this case, for any prime $p \nmid N\ell$, there should exist a constant $c \in \mathbb{F}_{\ell^4}^*$ such that the reduction mod λ of the polynomial:

$$x^4 - ca_p x^3 + c^2 b_p x^2 + c^3 p^3 \bar{a}_p x + c^4 p^4$$

a priori with coefficients in $\mathbb{F}_\lambda$, has coefficients in $\mathbb{F}_\ell$. This implies in particular that:

$$c^2 b_p \in \mathbb{F}_\ell; \quad c^2 a_p^2 \in \mathbb{F}_\ell$$

If we impose $\ell \nmid b_p$, (recall that, by assumption, $b_p \in \mathbb{Z}$), we conclude first that $c^2 \in \mathbb{F}_\ell^*$, and then: $a_p^2 \in \mathbb{F}_\ell$.
Thus, any prime ℓ falling in this case must verify: $\ell \mid b_p$ or $a_p^2 \in \mathbb{F}_\ell$.

Clearly, the following condition implies that this case can only occur for finitely many primes:

Condition 7): *There exists a prime $p \nmid N$ such that $b_p \neq 0$ and a_p^2 generates $\mathbb{Q}(\zeta)$.*

5.5 Exceptional cases

The case of "exceptional groups", i.e., those listed as cases 5) and 7) in the classification, can be discarded, for ℓ sufficiently large, just observing that thanks to lemma 4.2 we know that the projective image of inertia gives a subgroup of the image of order at least $\ell - 1$ (see [D-V] and [D1] for a similar argument). Computing the maximal order of an element in each of the (projectivizations of the) exceptional groups, we easily see that for $\ell > 11$ these cases can not occur.

5.6 Unitariness

Recall that we are considering representations which are pure (i.e., all roots of every characteristic polynomial of Frobenius have the same absolute value). This already reflected in the coefficients of the characteristic polynomial, and it also implies that for primes λ inert in $\mathbb{Q}(\zeta)$ the image of the representation is contained in the unitary group $\mathrm{GU}(4, \mathbb{Z}_\ell)$. Thus, considering residual representations, it is clear that their images, for every prime inert in $\mathbb{Q}(\zeta)$, are contained in $\mathrm{GU}(4, \ell)$.

5.7 Conclusion

Using the classification given in section 3, we conclude that any compatible family of representations of $G_{\mathbb{Q}}$ with the properties described in section 2 (geometric, pure, with Hodge-Tate weights $\{0, 1, 2, 3\}$, coefficients in $\mathbb{Q}(\zeta)$, quadratic coefficients in $\mathbb{Z}$ and determinant χ^6) will have residual image "as large as possible" for almost every prime, i.e., a general linear group or a unitary group depending on the decomposition type of the prime in $\mathbb{Q}(\zeta)$, as long as conditions 1) to 7) are satisfied. Using a lemma of Serre (cf. [S], [S2]), this also gives "maximality" of the image of the λ-adic representations

for these primes. For simplicity, we state the result in terms of the images of the projectivizations $P(\sigma_\lambda)$.

Theorem 5.1 *Let $\{\sigma_\lambda\}$ be a compatible family of geometric pure 4-dimensional Galois representations as those described in section 2, with coefficients in $\mathbb{Q}(\zeta)$. Then, if the explicit conditions 1), 2),...7) given in this section are satisfied, the image of σ_λ is "as large as possible" for almost every prime, i.e., the image of its projectivization $P(\sigma_\lambda)$ satisfies:*

$$\text{Image}(P(\sigma_\lambda)) \supseteq \text{PSL}(4, \mathbb{Z}_\ell)$$

if ℓ splits in $\mathbb{Q}(\zeta)$, and

$$\text{Image}(P(\sigma_\lambda)) = \text{PSU}(4, \mathbb{Z}_\ell)$$

if ℓ is inert in $\mathbb{Q}(\zeta)$.

Remark: In the above result, in the case of splitting primes, let us give the exact value of the image of the projective residual representation $P(\bar{\sigma}_\lambda)$ according to the restriction imposed by $\det(\bar{\sigma}_\lambda) = \chi^6$, a square in $\mathbb{F}_\ell$. This depends on the value of ℓ modulo 4 (because this determines whether or not the order of $\mathbb{F}_\ell^*$ is divisible by 4), so we easily see that for split primes with image "as large as possible" we have:

$$\text{Image}(P(\bar{\sigma}_\lambda)) = \text{PSL}(4, \mathbb{F}_\ell)$$

if $\ell \equiv 3 \pmod 4$, and

$$\text{Image}(P(\bar{\sigma}_\lambda)) = \{x \in \text{PGL}(4, \mathbb{F}_\ell) \ : \ \det(x) \in (\mathbb{F}_\ell^*)^2\}$$

if $\ell \equiv 1 \pmod 4$

6 Explicit determination of the finite exceptional set

Assuming that the value c of the conductor of a geometric family is known, the discussion of the previous section not only gives a set of conditions guaranteeing generically large image, but also an algorithm to explicitly bound the finite set of exceptional primes (for a family verifying the 7 conditions).

Let us write this algorithm, which is just a recollection of the results obtained in the different subsections of section 5. From now on we assume that $\ell \nmid N$, $\ell > 11$. At each step, only finitely many exceptional primes are detected: this follows from the analysis done in the previous section, under the assumption that the family of representations verify all conditions 1) to 7).

- Step 1) Take a few primes $\{p_r\}$ with $Pol_{p_r}(x)$ as in condition 1). For each of them compute the resultant $R_{r,i}$ of the pair of polynomials:

$$Pol_{p_r}(x), x^d - p_r{}^{id}$$

with $i = 0, 1, 2$ and d equal to the order of p_r in $(\mathbb{Z}/c\mathbb{Z})^*$. If ℓ does not divide the resultants $R_{r,0}, R_{r,1}$ and $R_{r,2}$ for one of these primes p_r (and $\ell \neq p_r$) then ℓ is not exceptional for step 1).

- Step 2) Take a few primes $\{q_r\}$ with $Q_{q_r}(x)$ as in condition 2). For each of them compute the resultant $R_{r,i}$ of the pair of polynomials:

$$Q_{q_r}(x), x^d - q_r{}^{id}$$

with $i = 1, 2, 3, 4, 5$ and d equal to the order of q_r in $(\mathbb{Z}/c\mathbb{Z})^*$. If ℓ does not divide the resultants $R_{r,1}, R_{r,2}, ... R_{r,5}$ for one of these primes q_r (and $\ell \neq q_r$) then ℓ is not exceptional for step 2).

- Step 3) For every quadratic character ψ unramified outside N take a few primes $\{p_r\}$ such that $\psi(p_r) = -1$ for all of them and they also satisfy the rest of condition 3). Then, if for some of these p_r, the prime ℓ does not divide:

$$p_r^3(a_{p_r}^2 + \bar{a}_{p_r}^2) - a_{p_r}\bar{a}_{p_r}b_{p_r}$$

and $\ell \neq p_r$, the prime ℓ is not exceptional "with respect to ψ" for step 3). Repeating the process for the finitely many possible ψ, we compute all exceptional primes for step 3).

- Step 4) For every cubic character ϕ unramified outside N take a few primes $\{p_r\}$ such that $\phi(p_r) \neq 1$ for all of them and they also satisfy

15

the rest of condition 4). Then, if for some of these p_r, the prime ℓ does not divide:

$$a_{p_r}^2 b_{p_r} + p_r^6 - \bar{a}_{p_r} a_{p_r}$$

and $\ell \neq p_r$, the prime ℓ is not exceptional "with respect to ϕ" for step 4). Repeating the process for the finitely many possible ϕ, we compute all exceptional primes for step 4).

- Step 5) For every quadratic character ψ unramified outside N take a few primes $\{p_r\}$ such that $\psi(p_r) = -1$ for all of them and they also satisfy the rest of condition 5) (i.e., $p_r \nmid N$ and $a_{p_r} \neq 0$). Then, if for some of these p_r, the prime ℓ does not divide a_{p_r} and $\ell \neq p_r$, the prime ℓ is not exceptional "with respect to ψ" for step 5). Repeating the process for the finitely many possible ψ, we compute all exceptional primes for step 5).

- Step 6) Take a few primes $\{p_r\}$ with a_{p_r} as in condition 6), i.e., $p_r \nmid N$, $a_{p_r}^2 \notin \mathbb{R}$. Then, if for some of these p_r, the prime ℓ does not divide $a_{p_r}^2 - \bar{a}_{p_r}^2$ and $\ell \neq p_r$, the prime ℓ is not exceptional for step 6).

- Step 7) Take a few primes $\{p_r\}$ verifying condition 7). Then, if for some of these p_r the prime ℓ verifies: $\ell \nmid b_{p_r}$, $a_{p_r}^2 \notin \mathbb{F}_\ell$ and $\ell \neq p_r$, the prime ℓ is not exceptional for step 7).

Thus, we bound the set of exceptional primes with the finite set of primes which are exceptional at some step of the algorithm, or divide N, or are smaller than 13.

Remark: One can expect that by sufficiently enlarging the set of "auxiliary primes" $\{p_r\}$ at each step, one can get a bounding set which contains only the true exceptional primes, and the primes dividing N or smaller than 13. The problem is that the size of such a "sufficiently large" auxiliary set may be (according to the existing effective versions of Cebotarev density theorem) too large to be suitable for computations.

7 Explicit computations on the example

We will now apply the former results to the non-selfdual 4-dimensional ℓ-adic Galois representation given by Scholten (see [Sc]): we check that the image is "as large as possible" for almost every prime, but since the value of the conductor of the family is unknown, we can not give a finite bound for the set of exceptional primes. Instead, we can bound the set of exceptional primes by a small density set of primes by a method already applied in [D-V].

7.1 The example

First, we summarize the construction of a compatible family of geometric four dimensional λ-adic Galois representation given by Scholten (see [Sc]). This family belongs to the class of families discussed in section 2 (in fact, it inspired the definitions therein).

The example is constructed using a subspace of $H^3_{et}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$, where S is a threefold obtained from a base change fibre product of two explicitly given elliptic surfaces.

Let X and X' be the elliptic surfaces given by the Weierstrass equations:

$$X: \quad y^2 = x^3 - 27(t^2 + t + 1)^2 x + 18(3t^2 + 3t - 1)(t^2 + t + 1)^2,$$

$$X': \quad y^2 = x^3 - 3(3t + 2)(243t^3 + 486t^2 + 324t + 80)x - 39366t^6 - 157464t^5$$

$$-262440t^4 - 235224t^3 - 120528t^2 - 33696t - 4048.$$

Let $\mathcal{E}$ and $\mathcal{E}'$ be the Kodaira-Néron models of the cubic base change of the elliptic surfaces $X \to \mathbb{P}^1_t$ and $X' \to \mathbb{P}^1_t$ over the Galois cover $\varphi : \mathbb{P}^1_s \to \mathbb{P}^1_t$ given by

$$t = \varphi(s) = \frac{s^3 - 3s^2 + 1}{3s^2 - 3s}.$$

The Galois group of φ has order 3 and it is generated by $\phi : s \mapsto (s-1)/s$. Let S denote the blowup at the singular points on $\mathcal{E} \times_{\mathbb{P}^1} \mathcal{E}'$. From the construction, the threefold S has bad reduction at the primes dividing $N = 6$ and ϕ induces an automorphism on S defined over $\mathbb{Q}$. Let ζ denote a fixed primitive third root of unity. The automorphism ϕ induces a $G_{\mathbb{Q}}$-invariant linear automorphism on the ℓ-adic cohomology, also denoted by ϕ. Since ϕ

has order 3, the ℓ-adic $G_\mathbb{Q}$-module $H^3(S)_\ell := H^3_{et}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ decomposes as a sum of three eigenspaces over $\mathbb{Q}_\ell(\zeta)$. Let W_λ denote the ζ-eigenspace of the automorphism ϕ restricted to $H^3(S)_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(\zeta)$. In [Sc] it is shown, assuming a statement about the generators of the $H^2_{et}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$, that the ζ-eigenspace W_λ has dimension 4 and that

$$\dim H^{3,0}(W_\lambda) = \dim H^{2,1}(W_\lambda) = \dim H^{1,2}(W_\lambda) = \dim H^{0,3}(W_\lambda) = 1.$$

So, we have a compatible family of λ-adic four dimensional Galois representations σ'_λ on W_λ, a $\mathbb{Q}(\zeta)_\lambda$ vector space, with four different Hodge-Tate numbers: $\{0, 1, 2, 3\}$. In general σ'_λ has to be twisted by a Dirichlet character ϵ to obtain $\epsilon \otimes \sigma'_\lambda = \sigma_\lambda$ with $\det(\sigma_\lambda) = \chi^6$.

This is the compatible family of geometric Galois representations we will consider:

$$\sigma_{S,\lambda} : G_\mathbb{Q} \to \mathrm{GL}(W_\lambda).$$

On the other hand, we put again $a_p := \mathrm{trace}(\sigma_{S,\lambda}(\mathrm{Frob}\ p)) \in \mathbb{Q}(\zeta)$ for every $p \nmid \ell 6$, we have $|a_p| \le 4p^{3/2}$. These traces can be determined using Lefschetz trace formula and counting points of S over finite fields. The characteristic polynomials of $\sigma_{S,\lambda}(\mathrm{Frob}\ p)$ will be

$$x^4 - a_p x^3 + b_p x^2 - p^3 \bar{a}_p x + p^6.$$

For all primes $p \nmid \ell 6$, the quadratic coefficients b_p are rational integers. According with [Sc] we list the first characteristic polynomials:

$5 : x^4 + (\overline{13 + 10\zeta})x^3 - 5x^2 + 5^3(13 + 10\zeta)x + 5^6$
$7 : x^4 + (\overline{7 + 3\zeta})x^3 - 189x^2 + 7^3(7 + 3\zeta)x + 7^6$
$11 : x^4 + (\overline{21 + 2\zeta})x^3 + 517x^2 + 11^3(21 + 2\zeta)x + 11^6$
$13 : x^4 + (\overline{70 + 77\zeta})x^3 - 1742x^2 + 13^3(70 + 77\zeta)x + 13^6$
$17 : x^4 + (\overline{87 + 63\zeta})x^3 - 1802x^2 + 17^3(87 + 63\zeta)x + 17^6$
$19 : x^4 - (\overline{8 + 81\zeta})x^3 - 4275x^2 - 19^3(8 + 81\zeta)x + 19^6$
$23 : x^4 + (\overline{129 + 33\zeta})x^3 + 14536x^2 + 23^3(129 + 33\zeta)x + 23^6$
$29 : x^4 + (\overline{186 + 86\zeta})x^3 + 16936x^2 + 29^3(186 + 86\zeta)x + 29^6$

Remark: In this example, we have twisted by a cubic character

$$\epsilon : G_\mathbb{Q} \to C_3$$

of conductor 9 to obtain $\det(\sigma_{S,\lambda}) = \chi^6$, χ the ℓ-adic cyclotomic character.

7.2 Determination of the images of $\sigma_{S,\lambda}$

Using the 8 characteristic polynomials listed above it is easy to check that the family of Galois representations $\sigma_{S,\lambda}$ satisfies the conditions of theorem 5.1. One easily checks that the characteristic polynomial at 5 satisfies conditions 1) and 2) (all roots and products of two roots of this characteristic polynomial generate non-abelian extensions of $\mathbb{Q}$). We consider all quadratic fields unramified outside 6 and we check that conditions 3) and 5) are satisfied for some prime p inert in each of them. The same also holds for cubic characters and condition 4). Conditions 6) and 7) are easily checked . Thus, we conclude:

Theorem 7.1 *The images of the 4-dimensional Galois representations $\sigma_{S,\lambda}$ are "as large as possible" for almost every prime, i.e., for almost every prime λ:*

$$\mathrm{Image}(P(\sigma_{S,\lambda})) \supseteq \mathrm{PSL}(4, \mathbb{Z}_\ell)$$

if ℓ splits in $\mathbb{Q}(\zeta)$ ($\ell \equiv 1 \pmod 3$), and

$$\mathrm{Image}(P(\sigma_{S,\lambda})) = \mathrm{PSU}(4, \mathbb{Z}_\ell)$$

if ℓ is inert in $\mathbb{Q}(\zeta)$ ($\ell \equiv 2 \pmod 3$).

We can not bound the set of exceptional primes with an explicit finite set (using the algorithm in section 6) because the value of the conductor of the family is unknown. In order to obtain a partial similar result, we will proceed as in [D-V] and take a fake value for this conductor: $c = 27 \cdot 64$. Recall that the value of c is only needed in order to deal with the reducible cases in steps 1) and 2) of the algorithm (section 6), and as a matter of fact (see section 5.1) it is only needed in order to bound the conductors of the Dirichlet characters appearing in the residually reducible cases, which are all characters unramified outside 6 and valued in $\mathbb{F}_\lambda^*$.

Therefore, what we really need is a bound for the conductors of these Dirichlet characters, and using the fact that they take values in $\mathbb{F}_\lambda^*$ we can bound these conductors after excluding a suitable set of primes (the density of this set tends to 0 when the value for the fake conductor tends to infinity): this follows easily from the fact that $\mathbb{F}_\lambda^*$ is cyclic and has order $\ell - 1$ if ℓ is split in $\mathbb{Q}(\zeta)$ and $\ell^2 - 1$ otherwise. In particular, we easily check that the conductors of the Dirichlet characters are bounded by $27 \cdot 64$ if we exclude those primes

in the following sets:

1) Splitting primes, $\ell \equiv 1 \pmod 3$:

$$D_1 := \{\ell \ : \ \ell \equiv 1 \pmod{27}\} \cup \{\ell \ : \ \ell \equiv 1 \pmod{32}\}$$

2) Inert primes, $\ell \equiv 2 \pmod 3$:

$$D_2 := \{\ell \ : \ \ell \equiv -1 \pmod{27}\} \cup \{\ell \ : \ \ell \equiv \pm 1 \pmod{16}\}$$

Thus, from now on we only consider primes λ such that ℓ is not in $D_1 \cup D_2$, and this implies that we can apply the algorithm in section 6 taking $c = 27 \cdot 64$.

Remark: In terms of Dirichlet density, we are excluding $1/6$ of the splitting primes and $1/3$ of the inert primes.

We executed the algorithm in Pari GP, using only the 8 characteristic polynomials listed in the previous subsection, and we found no exceptional prime greater than 11.

Theorem 7.2 *The images of the 4-dimensional Galois representations $\sigma_{S,\lambda}$ are "as large as possible" (as described in the previous theorem) for every prime λ such that $\ell > 11$ and $\ell \notin D_1 \cup D_2$.*

If we consider the projectivizations of the residual representations, for those primes λ where the image is "as large as possible" (and, for the case of splitting primes, we also impose $\ell \equiv 3 \pmod 4$, according to the remark after theorem 5.1), we obtain as a corollary two families of classical groups over finite fields realized as Galois groups over $\mathbb{Q}$. These groups were not previously known to be Galois groups over $\mathbb{Q}$ (cf. [M-M], [V]).

Corollary 7.3 *Let ℓ be a prime greater than 11 not in $D_1 \cup D_2$. Then, the following groups are Galois groups of a finite extension of $\mathbb{Q}$ unramified outside 6ℓ:*
1) If $\ell \equiv 7 \pmod{12}$: $\mathrm{PSL}(4, \ell)$.
2) If $\ell \equiv 2 \pmod 3$: $\mathrm{PSU}(4, \ell)$.

References

[A4-EMS] Kostrikin, A.I., Shafarevich, I. R. (Eds), Algebra IV, Encyclopedia of Mathematical Sciences, vol. 37, Springer-Verlag, 1989.

[Be] Berthelot, P., Altérations de variétés algébriques [d'après A. J. de Jong], in "Séminaire Bourbaki", vol. 1995/96, exp. 815, 273-311; Astérisque, 1997.

[D1] Dieulefait, L., On the images of the Galois representations attached to genus 2 Siegel modular forms, J. Reine Angew. Math. **553** (2002), 183-200.

[D2] _____ , Explicit determination of the images of the Galois representations attached to abelian surfaces with $\text{End}(A) = \mathbb{Z}$, Experiment. Math. **11** (2002), 503-512.

[D-V] Dieulefait, L., Vila, N., On the images of modular and geometric three-dimensional Galois representations, Amer. J. of Math. **126** (2004), 335-361.

[F-L] Fontaine, J.M., Laffaille, G., Construction de représentations p-adiques, Ann. Scient. Éc. Norm. Sup., 4e série, t. **15** (1982), 547-608.

[F-M] Fontaine, J.M, Messing, W., p-adic periods and p-adic etale cohomology, in "Currents Trends in Arithmetical Algebraic Geometry (Arcata, Calif., 1985)"; Contemporary Mathematics **67** (1987), 179-207.

[K] Kleidman, P. B., The subgroup structure of some finite simple groups, Ph. D. Thesis, Cambridge, 1986.

[K-L] Kleidman, P., Liebeck, M., *The subgroup structure of the finite classical groups*, London Math. Soc. LNS 129, Combridge University Press, 1990.

[M-M] Malle, G., Matzat, B. H., *Inverse Galois Theory*, Springer-Verlag, 1999.

[N-P] Neumann, P. M., Praeger, C. E., A recognition algorithm for special linear groups, Proc. London Math. Soc. (3) **65** (1992), 555-603.

[Sc] Scholten, J., Mordell-Weil groups of elliptic surfaces and Galois representations, Ph. D. Thesis, Rijksuniversiteit Groningen, 2000.

[S] Serre, J-P., *Abelian ℓ-adic representations and elliptic curves*, Benjamin, 1968.

[S1] _____ , Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, Invent. Math. **15** (1972), 259-331.

[S2] —— , Oeuvres, vol. 4, 1-55, Springer-Verlag, 2000.

[V] Völklein, H., Braid group action via $GL_n(q)$ and $U_n(q)$ and Galois realizations, Israel J. Math. **82** (1993), 405-427.