Compact Asymptotic Center and Common Fixed Point in Strictly Convex Banach Spaces

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Abstract

In this paper, we present some common fixed point theorems for a commuting pair of mappings, including a generalized nonexpansive singlevalued mapping and a generalized nonexpansive multivalued mapping in strictly convex Banach spaces. The results obtained in this paper extend and improve some recent known results.

Key Words: common fixed point, generalized nonexpansive mapping, strictly convex Banach space, asymptotic center

1 Introduction

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [1] and Nadler [2]. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. Using the Edelstein's method of asymptotic centers, Lim [3] proved the existence of fixed points for multivalued nonexpansive mappings on uniformly convex Banach spaces. Kirk and Massa [4] extended Lim's theorem to Banach spaces for which the asymptotic center of a bounded sequence in a bounded closed convex subset is nonempty and compact.

On the other hand, in 2008, Suzuki [5] introduced a condition on mappings, called condition (C), which is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. He then proved some fixed point and convergence theorems for such mappings. Motivated by this result, J. Garcia-Falset, E. Llorens-Fuster and T. Suzuki in [6], introduced two kinds of generalization for the condition (C) and studied both the existence of fixed points and their asymptotic behavior. Very recently, the current authors used a modified condition for multivalued mappings, and proved some fixed point theorems for multivalued mappings satisfying this condition in Banach spaces [7, 8], as well as in CAT(0) spaces [9].

Let D be a nonempty subset of a strictly convex Banach space X, and let $T : D \to KC(D)$ be a generalized nonexpansive multivalued mapping in the sense of Garcia-Falset et al., that is, a mapping satisfying the conditions (E) and (C_λ) (see Definitions 2.9 and 2.10). Moreover, let $t : D \to D$ be a single valued mapping satisfying the conditions (E) and (C_λ) of definitions 2.2 and 2.3. We call t a single-valued generalized nonexpansive mapping. We assume that t and T commute, and that the asymptotic center of a bounded sequence in the fixed point set of t is nonempty and compact. The main result of this paper says that t and T have a common fixed point (see Theorem 3.5). Our result improves a number of known results; including that of Suzuki [5], Garcia et al. [6], Kirk and Massa [4], Dhompongsa et al. [10], and of Lim [3].

2 Preliminaries

Let X be a Banach space. X is said to be strictly convex if $\|x+y\| < 2$ for all $x, y \in X$, $\|x\| = \|y\| = 1$ and $x \neq y$. We recall that a Banach space X is said to be *uniformly convex in every direction* (UCDE, for short) provided that for every $\epsilon \in (0, 2]$ and $z \in X$ with $\|z\| = 1$, there exists a positive number δ (depending on ϵ and z) such that for all $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $x - y \in \{tz : t \in [-2, -\epsilon] \cup [\epsilon, 2]\}$ we have $\|x + y\| \leq 2(1 - \delta)$. X is said to be *uniformly convex* if X is UCED and $inf\{\delta(\epsilon, z) : \|z\| = 1\} > 0$ for all $\epsilon \in (0, 2]$. It is obvious that uniformly convexity implies UCED, and UCED implies strictly convexity.
The following definition is due to Susuki [5].

Definition 2.1. *([5]) Let T be a mapping on a subset D of a Banach space X. T is said to satisfy condition (C) if*

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|, \quad x, y \in E.$$

Example ([5]). Define a mapping T on [0,3] by

$$T(x) = \begin{cases} 0, & x \neq 3 \\ 1, & x = 3. \end{cases}$$

Then T satisfies the condition (C), but T is not continuous, and hence T is not nonexpansive.

In [6], J. Garcia-Falset et al. introduced two generalizations of the condition (C) in a Banach space:

Definition 2.2. *Let T be a mapping on a subset D of a Banach space X and $\mu \geq 1$. T is said to satisfy condition (E_μ) if*

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|, \quad x, y \in D.$$

We say that T satisfies condition (E) whenever T satisfies the condition (E_μ) for some $\mu \geq 1$.

Definition 2.3. *Let T be a mapping on a subset D of a Banach space X and $\lambda \in (0,1)$. T is said to satisfy condition (C_λ) if*

$$\lambda\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|, \quad x, y \in D.$$

Notice that if $0 < \lambda_1 < \lambda_2 < 1$ then the condition (C_{λ_1}) implies the condition (C_{λ_2}). The following example shows that the class of mappings satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$ is broader than the class of mappings satisfying the condition (C).

Example ([6]). For a given $\lambda \in (0,1)$ define a mapping T on [0,1] by

$$T(x) = \begin{cases} \frac{x}{2}, & x \neq 1 \\ \frac{1+\lambda}{2+\lambda}, & x = 1. \end{cases}$$

Then the mapping T satisfies condition (C_λ) but it fails condition $(C_{\lambda'})$ whenever $0 < \lambda < \lambda'$. Moreover T satisfies condition (E_μ) for $\mu = \frac{2+\lambda}{2}$.

Theorem 2.4. *([6]) Let D be a nonempty bounded convex subset of a Banach space X. Let $T : D \to D$ satisfy the condition (C_λ) on D for some $\lambda \in (0,1)$. For $r \in [\lambda, 1)$ define a sequence $\{x_n\}$ in D by taking $x_1 \in D$ and*

$$x_{n+1} = rT(x_n) + (1-r)x_n \qquad for \quad n \geq 1,$$

then $\{x_n\}$ is an approximate fixed point sequence for T, that is

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0.$$

Lemma 2.5. *([6]) Let T be a mapping defined on a closed subset D of a Banach space X. Let T be a single valued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. Then Fix(T) is closed. Moreover, if X is strictly convex and D is convex, then Fix(T) is also convex.*

Let D be a nonempty subset of X Banach space X. for $x \in X$ denote

$$dist(x, D) = inf\{\| x - z \| : z \in D\}.$$

We denote by $CB(D)$ and $KC(D)$ the collection of all nonempty closed bounded subsets, and nonempty compact convex subsets of D respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) := \max\{\sup_{x \in A} dist(x, B), \sup_{y \in B} dist(y, A)\},$$

for all $A, B \in CB(X)$.

Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$.

It is obvious that if D is a convex subset of strictly convex Banach space X, then for $x \in X$, if there exist $y, z \in D$ such that

$$\| x - y \| = dist(x, D) = \| x - z \|$$

then $y = z$.

Definition 2.6. *A multivalued mapping $T : X \to CB(X)$ is said to be nonexpansive provided that*

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in X.$$

We state Suzuki's condition for multivalued mappings as follows:

Definition 2.7. *A multivalued mapping $T : X \to CB(X)$ is said to satisfy the condition (C) provided that*

$$\frac{1}{2} dist(x, Tx) \leq \|x - y\| \implies H(Tx, Ty) \leq \|x - y\|, \quad x, y \in X.$$

We now state the multivalued analogs of the conditions (E) and (C_λ) in the following manner (see also [9]):

Definition 2.8. *A multivalued mapping* $T : X \to CB(X)$ *is said to satisfy condition* (E_μ) *provided that*

$$dist(x, Ty) \leq \mu dist(x, Tx) + \|x - y\|, \quad x, y \in X.$$

We say that T *satisfies condition (E) whenever* T *satisfies* (E_μ) *for some* $\mu \geq 1$.

Definition 2.9. *A multivalued mapping* $T : X \to CB(X)$ *is said to satisfy condition* (C_λ) *for some* $\lambda \in (0, 1)$ *provided that*

$$\lambda dist(x, Tx) \leq \|x - y\| \implies H(Tx, Ty) \leq \|x - y\|, \quad x, y \in X.$$

Lemma 2.10. *Let* $T : X \to CB(X)$ *be a multivalued nonexpansive mapping, then* T *satisfies the condition* (E_1).

We now provide an example of a generalized nonexpansive multivalued mapping satisfying the conditions (C_λ) and (E) which is not a nonexpansive multivalued mapping (for details, see [9]).

Example. We consider a multivalued mapping T on $[0, 5]$ given by

$$T(x) = \begin{cases} [0, \frac{x}{5}], & x \neq 5 \\ \{1\} & x = 5. \end{cases}$$

This mapping has the required properties. Finally we recall the following lemma from [11].

Lemma 2.11. *Let* $\{z_n\}$ *and* $\{w_n\}$ *be two bounded sequences in a Banach space* X*, and let* $0 < \lambda < 1$*. If for every natural number* n *we have* $z_{n+1} = \lambda w_n + (1-\lambda) z_n$ *and* $\|w_{n+1} - w_n\| \leq \|z_{n+1} - z_n\|$*, then* $\lim_{n\to\infty} \|w_n - z_n\| = 0$.

3 Common fixed point

Let D be a nonempty bounded closed convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X, we use $r(x, \{x_n\})$ and $A(D, \{x_n\})$

to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in D, respectively, i.e.

$$r(D,\{x_n\}) = \inf\{\limsup_{n\to\infty} \|x_n - x\| : x \in D\},$$

$$A(D,\{x_n\}) = \{x \in D : \limsup_{n\to\infty} \|x_n - x\| = r(D,\{x_n\})\}.$$

Obviously, the convexity of E implies that $A(D,\{x_n\})$ is convex. It is known that in a UCED Banach space X, the asymptotic center of a sequence with respect to a weakly compact convex set is a singleton; the same is true for a sequence in a bounded closed convex subset of uniformly convex Banach space X [12].

Definition 3.1. *A bounded sequence $\{x_n\}$ is said to be* regular *with respect to D if for every subsequence $\{x'_n\}$ we have*

$$r(D,\{x_n\}) = r(D,\{x'_n\});$$

further, $\{x_n\}$ is called asymptotically uniform relative to D if

$$A(D,\{x_n\}) = A(D,\{x'_n\}).$$

The following lemma was proved by Goebel and Lim.

Lemma 3.2. *(see [13] and [3]). Let $\{x_n\}$ be a bounded sequence in X and let D be a nonempty closed convex subset of X.*

(i) then there exists a subsequence of $\{x_n\}$ which is regular relative to D.

(ii) if D is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to D.

Theorem 3.3. *Let D be a nonempty closed convex bounded subset of a Banach space X. Let $t : D \to D$ be a single valued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. Suppose the asymptotic center* *relative D of each sequence in D is nonempty and compact. Then T has a fixed point.*

Proof. By Theorem 2.4, there exists a sequence $\{x_n\}$ in D such that

$$\lim_{n\to\infty} dist(x_n, Tx_n) = 0.$$

Since $A(D, \{x_n\})$ is nonempty convex and compact, by invoking Theorem 2.4 again, there exists a sequence $\{z_n\}$ such that $\lim_{n\to\infty} dist(z_n, Tz_n) = 0$. By passing to subsequence we can assume that $\lim_{n\to\infty} z_n = z$. By condition E, there exists $\mu > 1$ such that

$$\|z_n - Tz\| \leq \mu\|z_n - Tz_n\| + \|z_n - z\|.$$

Taking limit in the above inequality we obtain $Tz = z$. □

Definition 3.4. *Let D be a nonempty closed convex bounded subset of a Banach space X, and let $t : D \to X$ and $T : D \to CB(X)$ be two mappings. Then t and T are said to be commuting if for every $x, y \in D$ such that $x \in Ty$ and $ty \in D$, we have $tx \in Tty$.*

We now state and prove the main result of this paper.

Theorem 3.5. *Let D be a nonempty closed convex bounded subset of a strictly convex Banach space X, $t : D \to D$ be a single valued mapping, and $T : D \to KC(D)$ be a multivalued mapping. Assume that both mappings satisfy the conditions (E) and (C_λ) for some $\lambda \in (0, 1)$, and that t, T commute. If the asymptotic center relative $Fix(t)$ of each sequence in $Fix(t)$ is nonempty and compact, then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Proof. By Theorem 3.3 the mapping t has a nonempty fixed point set $Fix(t)$ which is a closed convex subset of X (by Lemma 2.5). We show that for

$x \in Fix(t)$, $Tx \cap Fix(t) \neq \emptyset$. To see this, let $x \in Fix(t)$, since t and T commute, we have $ty \in Tx$ for each $y \in Tx$. Therefore Tx is invariant under t for each $x \in Fix(t)$. Since Tx is a nonempty compact convex subset of a Banach space X, the asymptotic center in Tx of each sequence is nonempty and compact. Therefore by Theorem 3.3 we conclude that t has a fixed point in Tx and therefore $Tx \cap Fix(t) \neq \emptyset$ for $x \in Fix(t)$.

Now we find an approximate fixed point sequence for T in $Fix(t)$. Take $x_0 \in Fix(t)$, since $Tx_0 \cap Fix(t) \neq \emptyset$, we can choose $y_0 \in Tx_0 \cap Fix(t)$. Define

$$x_1 = (1 - \lambda)x_0 + \lambda y_0.$$

Since $Fix(t)$ is a convex set, we have $x_1 \in Fix(t)$. Let $y_1 \in T(x_1)$ be chosen in such a way that

$$\|y_0 - y_1\| = dist(y_0, T(x_1)).$$

We see that $y_1 \in Fix(t)$. Indeed, we have

$$\lambda\|y_0 - ty_0\| = 0 \leq \|y_0 - y_1\|.$$

Since t satisfies the condition (C), we get

$$\|y_0 - ty_1\| = \|ty_0 - ty_1\| \leq \|y_0 - y_1\|$$

which contradicts the uniqueness of y_1 as the unique nearest point of y_0 (note that $ty_1 \in Tx_1$). Similarly, put

$$x_2 = (1 - \lambda)x_1 + \lambda y_1,$$

again we choose $y_2 \in T(x_2)$ in such a way that

$$\|y_1 - y_2\| = dist(y_1, T(x_2)).$$

By the same argument, we get $y_2 \in Fix(t)$. In this way we will find a sequence $\{x_n\}$ in $Fix(t)$ such that

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n.$$

where $y_n \in T(x_n) \cap Fix(t)$ and

$$\|y_{n-1} - y_n\| = dist(y_{n-1}, T(x_n)).$$

Therefore for every natural number $n \geq 1$ we have

$$\lambda \|x_n - y_n\| = \|x_n - x_{n+1}\|$$

from which it follows that

$$\lambda\, dist(x_n, T(x_n)) \leq \lambda \|x_n - y_n\| = \|x_n - x_{n+1}\|, \quad n \geq 1.$$

Our assumption now gives

$$H(T(x_n), T(x_{n+1})) \leq \|x_n - x_{n+1}\|, \quad n \geq 1,$$

hence for each $n \geq 1$ we have

$$\begin{aligned} \|y_n - y_{n+1}\| = dist(y_n, T(x_{n+1})) &\leq H(T(x_n), T(x_{n+1})) \\ &\leq \|x_n - x_{n+1}\|. \end{aligned}$$

We now apply Lemma 2.11 to conclude that $\lim_{n\to\infty} \|x_n - y_n\| = 0$ where $y_n \in T(x_n)$. From Lemma 3.2 by passing to a subsequence, we may assume $\{x_n\}$ is regular asymptotically uniform relative to $Fix(t)$. Let $r = r(Fix(t), \{x_n\})$. Now, we show that $Tx \cap A(Fix(t), \{x_n\}) \neq \emptyset$ for $x \in A(Fix(t), \{x_n\})$. The compactness of $T(x_n)$ implies that for each n we can take $y_n \in T(x_n)$ such that

$$\|x_n - y_n\| = dist(x_n, T(x_n)).$$

Suppose $x \in A(Fix(t), \{x_n\})$. Since $T(x)$ is compact, for each n, we choose $z_n \in T(x)$ such that

$$\|z_n - y_n\| = dist(y_n, T(x)).$$

By assumption there exist $\mu > 1$ such that

$$\|y_n - z_n\| = dist(y_n, Tx) \leq H(T(x_n), T(x)) \\ \leq \mu\, dist(x_n, T(x_n)) + \|x_n - x\|.$$

Since $T(x)$ is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k\to\infty} z_{n_k} = z \in T(x)$. Note that

$$\|x_{n_k} - z\| \leq \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - z_{n_k}\| + \|z_{n_k} - z\|$$

This entails

$$\limsup_{k\to\infty} \|x_{n_k} - z\| \leq \limsup_{k\to\infty} \|x_{n_k} - x\| \leq r.$$

Since $\{x_n\}$ is regular asymptotically uniform relative to $Fix(t)$, this shows that $z \in A(Fix(t), \{x_{n_k}\}) = A(Fix(t), \{x_n\})$ therefore

$$z \in Tx \cap A(Fix(t), \{x_n\}).$$

Now, we show that there exists a sequence $\{z_n\} \subset A(Fix(t), \{x_n\})$ such that $\lim dist_{n\to\infty}(z_n, Tz_n) = 0$. Indeed, take $z_0 \in A(Fix(t), \{x_n\})$, since $Tz_0 \cap A(Fix(t), \{x_n\}) \neq \emptyset$, there exists $y_0 \in Tz_0 \cap A(Fix(t), \{x_n\})$. We define

$$z_1 = (1-\lambda)z_0 + \lambda y_0.$$

Since $A(Fix(t), \{x_n\})$ is convex, we have $z_1 \in A(Fix(t), \{x_n\})$.
Similarly, since $Tz_1 \cap A(Fix(t), \{x_n\}) \neq \emptyset$, by the compactness of $Tz_1 \cap A(Fix(t), \{x_n\})$ we can choose $y_1 \in Tz_1 \cap A(Fix(t), \{x_n\})$ in such a way that

$$\|y_0 - y_1\| = dist(y_0, Tz_1 \cap A(Fix(t), \{x_n\})).$$

In this way we will find a sequence $\{z_n\} \in A(Fix(t), \{x_n\})$ such that

$$z_{n+1} = (1-\lambda)z_n + \lambda y_n$$

where $y_n \in Tz_n \cap A(Fix(t), \{x_n\})$ and

$$\|y_{n-1} - y_n\| = dist(y_{n-1}, Tz_n \cap A(Fix(t), \{x_n\})).$$

Therefore for every natural number $n \geq 1$ we have

$$\lambda \|z_n - y_n\| = \|z_n - z_{n+1}\|$$

from which it follows that

$$\lambda\, dist(z_n, T(z_n)) \leq \lambda \|z_n - y_n\| = \|z_n - z_{n+1}\|, \quad n \geq 1.$$

Our assumption now gives

$$H(T(z_n), T(z_{n+1})) \leq \|z_n - z_{n+1}\|, \quad n \geq 1,$$

hence

$$\|y_n - y_{n+1}\| = dist(y_n, T(z_{n+1})) \leq H(T(z_n), T(z_{n+1})) \leq \|z_n - z_{n+1}\|, \quad n \geq 1.$$

We now apply Lemma 2.11 to conclude that $\lim_{n\to\infty} \|z_n - y_n\| = 0$ where $y_n \in T(z_n)$.

Since $z_n \in A(Fix(t), \{x_n\})$, and $A(Fix(t), \{x_n\})$ is compact, by passing to subsequence we may assume that z_n is convergent to $z \in A(Fix(t), \{x_n\})$ as $n \to \infty$. Since Tz is compact, for each $n \geq 1$, we can choose $w_n \in Tz$ such that $\|w_n - z_n\| = dist(z_n, Tz)$. Moreover $w_n \in Fix(t)$ for all natural numbers $n \geq 1$. Indeed, since

$$\lambda \|z_n - tz_n\| = 0 \leq \|w_n - z_n\|, \qquad n \geq 1,$$

we have

$$\|z_n - tw_n\| = \|tw_n - tz_n\| \leq \|w_n - z_n\|.$$

Since $z \in Fix(t)$ and $w_n \in T(z)$, by the fact that the mappings t and T commute, we obtain $tw_n \in Ttz = Tz$. Now, by the uniqueness of w_n as the nearest point to z_n, we get $tw_n = w_n \in Fix(t)$.

Since Tz is compact the sequence $\{w_n\}$ has a convergent subsequence $\{w_{n_k}\}$ with $\lim_{k\to\infty} w_{n_k} = w \in Tz$. Because $w_{n_k} \in Fix(t)$ for all n, and $Fix(t)$ is closed, we obtain $w \in Fix(t)$. By assumption there exists $\mu > 1$ such that

$$dist(z_{n_k}, Tz) \leq \mu dist(z_{n_k}, T(z_{n_K})) + \|z_{n_k} - z\|.$$

Note that

$$\begin{aligned}\|z_{n_k} - w\| &\leq \|z_{n_k} - w_{n_k}\| + \|w_{n_k} - w\| \\ &\leq \mu dist(z_{n_k}, T(z_{n_k})) + \|w_{n_k} - w\| + \|z_{n_k} - z\|.\end{aligned}$$

This entails

$$\limsup_{k\to\infty} \|z_{n_k} - w\| \leq \limsup_{k\to\infty} \|z_{n_k} - z\|.$$

We conclude that $z = w$, hence $z = tz \in Tz$.

□

Theorem 3.6. *Let D be a nonempty closed convex bounded subset of a uniformly convex Banach space X, $t : D \to D$, and $T : D \to KC(D)$ a single valued and a multivalued mapping, both satisfying the conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. Assume that t, T commute. Then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Proof. By Theorem 3.3 the mapping t has a nonempty fixed point set $Fix(t)$ which is a closed convex subset of X (by Lemma 2.5). Since X is uniformly convex, we have asymptotic center relative $Fix(t)$ of each sequence in $Fix(t)$ is nonempty and singleton (hence compact). Therefore, by theorem 3.5, T has a fixed point. □

Theorem 3.7. *Let D be a nonempty weakly compact convex subset of a UCED Banach space X. Let $t : D \to D$, and $T : D \to KC(D)$ be a single* *valued and a multivalued mapping respectively, both satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. Assume that t, T commute. Then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Proof. By Theorem 3.3 the mapping t has a nonempty fixed point set $Fix(t)$ which is a closed convex subset of X (by Lemma 2.5). Since D is weakly compact we conclude that $Fix(t)$ is weakly compact. Since X is UCED, we conclude that the asymptotic center relative $Fix(t)$ of each sequence in $Fix(t)$ is nonempty and singleton (hence compact). Therefore, by Theorem 3.5, T has a fixed point. □

Theorem 3.8. *Let D be a nonempty compact convex subset of a strictly convex Banach space X, $t : D \to D$, and $T : D \to KC(D)$ a single valued and a multivalued mapping, both satisfying the conditions (E) and (C_λ) for some $\lambda \in (0,1)$. Assume that t, T commute. Then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Proof. By Theorem 3.3 the mapping t has a nonempty fixed point set $Fix(t)$ which is a closed convex subset of X (by Lemma 2.5). Since D is compact we conclude that $Fix(t)$ is compact. Since X is strictly convex, we infer that the asymptotic center relative $Fix(t)$ of each sequence in $Fix(t)$ is nonempty and compact. Therefore, by Theorem 3.5, T has a fixed point. □

Theorem 3.9. *Let D be a nonempty closed convex bounded subset of a strictly convex Banach space X. Let $t : D \to D$, and $T : D \to KC(D)$ be two nonexpansive mappings. Assume that t, T commute. Suppose the asymptotic center relative $Fix(t)$ of each sequence in $Fix(t)$ is nonempty and compact. Then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Proof. By Lemma 2.10 the mapping T satisfies the condition E_1. we also note that T satisfies the condition (C_λ) for all $\lambda \in (0,1)$. So the result follows from Theorem 3.5 □

Corollary 3.10. *Let D be a nonempty closed convex bounded subset of a uniformly convex Banach space X. Let $t : D \to D$, and $T : D \to KC(D)$ be two nonexpansive mappings. Assume that t, T commute. Then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Corollary 3.11. *Let E be a nonempty weakly compact convex subset of a UCED Banach space X. Let $t : D \to D$, and $T : D \to KC(D)$ be two nonexpansive mappings. Assume that t, T commute. Then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

Corollary 3.12. *Let D be a nonempty compact convex subset of a strictly convex Banach space X. Let $t : D \to D$, and $T : D \to KC(D)$ be two nonexpansive mappings. Assume that t, T commute. Then there exists a point $z \in D$ such that $z = t(z) \in T(z)$.*

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