RENORMALIZATION AND BLOW UP FOR CHARGE ONE EQUIVARIANT CRITICAL WAVE MAPS.

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Abstract. We prove the existence of equivariant finite time blow-up solutions for the wave map problem from $\mathbb{R}^{2+1} \to S^2$ of the form $u(t,r) = Q(\lambda(t)r) + \mathcal{R}(t,r)$ where u is the polar angle on the sphere, $Q(r) = 2\arctan r$ is the ground state harmonic map, $\lambda(t) = t^{-1-\nu}$, and $\mathcal{R}(t,r)$ is a radiative error with local energy going to zero as $t \to 0$. The number $\nu > \frac{1}{2}$ can be described arbitrarily. This is accomplished by first "renormalizing" the blow-up profile, followed by a perturbative analysis.

1. Introduction

We consider Wave Maps $U : \mathbb{R}^{2+1} \to S^2$ which are equivariant with co-rotation index 1. Specifically, they satisfy $U(t, \omega x) = \omega U(t,x)$ for $\omega \in SO(2,\mathbb{R})$, where the latter group acts in standard fashion on $\mathbb{R}^2$, and the action on S^2 is induced from that on $\mathbb{R}^2$ via stereographic projection. Wave maps are characterized by being critical with respect to the functional

$$U \to \int_{\mathbb{R}^{2+1}} \langle \partial_\alpha U, \partial^\alpha U \rangle \, d\sigma, \; \alpha = 0,1,2$$

with Einstein's summation convention being in force, $\partial^\alpha = m^{\alpha\beta}\partial_\beta$, $m_{\alpha\beta} = (m^{\alpha\beta})^{-1}$ the Minkowski metric on $\mathbb{R}^{2+1}$, and $d\sigma$ the associated volume element. Also, $\langle \cdot, \cdot \rangle$ refers to the standard inner product on $\mathbb{R}^3$ if we use ambient coordinates to describe u, $\partial_\alpha u$ etc. Recall that the energy is preserved:

$$\mathcal{E}(u) = \int_{\mathbb{R}^2} \langle DU(\cdot,t), DU(\cdot,t) \rangle \, dx = \text{const}$$

If one instead uses spherical coordinates, and lets u stand for the longitudinal angle, and similarly use polar coordinates r, θ on $\mathbb{R}^2$, we describe the Wave Map by $(t,r,\theta) \longrightarrow (u(t,r),\theta)$, where now $u(t,r)$, a scalar function, satisfies the equation

$$-u_{tt} + u_{rr} + \frac{u_r}{r} = \frac{\sin(2u)}{2r^2} \tag{1.1}$$

The problem at hand is *energy critical*, meaning that the conserved energy is invariant under the natural re-scaling $U \to U(\lambda t, \lambda x)$ (using the original coordinates and meaning of U). By contrast, the analogous wave map problem on $\mathbb{R}^{n+1}$, $n \geq 3$ is energy-supercritical in the sense that the natural scale-invariant Sobolev space is then $\dot{H}^{\frac{n}{2}}$, and the corresponding norm $\|u\|_{\dot{H}^{\frac{n}{2}}}$ is not expected to be controlled globally-in-time for general initial data, which leads to the general belief that in this case, there should not be a good well-posedness theory for general initial data, *irrespective of the target.* Indeed, singular wave maps stemming from C^∞-data have been constructed on background $\mathbb{R}^{3+1}$ with target S^3 in [Sha1], and with origin $\mathbb{R}^{n+1}$, $n \geq 4$ and for more general targets in [Sha2].

In the critical case, global well-posedness is expected for hyperbolic targets, while singularity development is expected for certain positively curved targets, such as S^2. More precisely, numerical evidence in [Bi], [Li]

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strongly suggests singularity development for equivariant wave maps of co-rotation index 1 from $\mathbb{R}^{2+1}$ to S^2 with smooth data, while wave maps from $\mathbb{R}^{2+1}$ to $\mathbf{H}^2$, and more generally $\mathbf{H}^k$, $k \geq 2$, are expected to preserve the regularity[1] of the initial data. Further evidence for possible singularity development up to this date specifically, in the co-rotation 1 equivariant case has been recently provided in [Co]. We note that a fairly satisfactory understanding has been achieved for small-energy wave maps from $\mathbb{R}^{2+1}$ to general targets [Tao], [Tat], [Kri], as well as for rotationally invariant wave maps and general initial data [Chr-Tah], [Str1]. In particular, *it is known that the latter never develop singularities* [Str1], and that for equivariant wave maps of co-rotation index 1, regularity breakdown can only occur in an *energy concentration scenario* [Str2]. For equivariant wave maps, it is known that regularity of the initial data is preserved (see previous footnote) provided the target satisfies a geodesic convexity condition [Sh-Tah].
Our objective in this paper is to rigorously demonstrate regularity breakdown for equivariant wave maps $u : \mathbb{R}^{2+1} \longrightarrow S^2$ of co-rotation index 1 with certain H^{1+} regular initial data. More precisely, the data $(u,\, u_t)$ will be of class $H^{1+\delta} \times H^{\delta}$ for some $\delta > 0$. It is well-known that such data result in unique local solutions of the same regularity until possible breakdown occurs via an energy-concentration scenario. We note that a result of Struwe shows that if the solution is indeed C^∞–smooth before breakdown[2], such a scenario can only happen by the bubbling off of a harmonic map [Str2]: specifically, let $Q(r) : \mathbb{R}^2 \longrightarrow S^2$ be an equivariant hamonic map, which can be constructed for every co-rotation index $k \in \mathbb{Z}$ (for example, for $k = 1$ stereographic projection will do). We shall identify $Q(r)$ with the longitudinal angle, as above. Then according to [Str2], if an equivariant wave map u of co-rotation index $k = 1$, again identified with the longitudinal angle, with smooth initial data at some time $t_0 > 0$ breaks down at time $T = 0$, then energy focuses at the origin, and there is a decomposition

$$u(t,r) = Q(\lambda(t)r) + \varepsilon(t,r),\ Q(r) \text{ a co-rotation } k = 1 \text{ index equivariant harmonic map}$$

where there is a sequence of times $t_i \to 0$, $t_i < 0$, $i = 1, 2, \ldots$, with $\lambda(t_i)|t_i| \to \infty$, such that the rescaled functions $u(t_i, \frac{r}{\lambda(t_i)})$ converge to $Q(r)$ in the strong energy topology.

This is borne out by our main theorem. We let $Q(r)$ represent the standard harmonic map of co-rotation $k = 1$, i.e., $Q(r) = 2 \arctan r$. Recall that in the equivariant formulation the energy is

$$\mathcal{E}(u) = \int_{\mathbb{R}^2} \Big[\frac{1}{2}(u_t^2 + u_r^2) + \frac{\sin^2(u)}{2r^2}\Big]\, r\, dr$$

The *local* energy relative to the origin is defined as

$$\mathcal{E}_{\rm loc}(u) = \int_{r<t} \Big[\frac{1}{2}(u_t^2 + u_r^2) + \frac{\sin^2(u)}{2r^2}\Big]\, r\, dr$$

It is well-known that for equivariant wave-maps singularities can only develop at the origin and this happens at time zero iff

$$\liminf_{t\to 0} \mathcal{E}_{\rm loc}(u)(t) > 0$$

The following theorem is the main result of this paper. Note that we need to "renormalize" the profile $Q(r\lambda(t))$ by means of a large perturbation (denoted u^e below). We find it convenient to solve backwards in time, with blow-up as $t \to 0+$.

Theorem 1.1. *Let $\nu > \frac{1}{2}$ be arbitrary and $t_0 > 0$ be sufficiently small. Define $\lambda(t) = t^{-1-\nu}$ and fix a large integer N. Then there exists a function*[3] *u^e satisfying*

$$u^e \in C^{\nu+1/2-}(\{t_0 > t > 0,\ |x| \leq t\}), \qquad \mathcal{E}_{\rm loc}(u^e)(t) \lesssim (t\lambda(t))^{-2}\, |\log t|^2 \quad \text{as} \quad t \to 0$$

and a blow-up solution u to (1.1) *in* $[0, t_0]$ *which has the form*

$$u(r,t) = Q(\lambda(t)r) + u^e(r,t) + \varepsilon(r,t), \qquad 0 \leq r \leq t$$

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[1] By this we mean that if initial data have regularity $H^{1+\delta}$, $\delta > 0$, the Wave Map can be uniquely globally extended in this class.

[2] This result most likely can be adapted to solutions of lesser smoothness

[3] We refer to this as an "elliptic profile modifier"; see Section 2 for a detailed explanation of this notion. Also, C^β for noninteger β means $C^{[\beta],\beta-[\beta]}$

where ε decays at $t = 0$; more precisely,

$$\varepsilon \in t^N H^{1+\nu-}_{\rm loc}(\mathbb{R}^2), \qquad \varepsilon_t \in t^{N-1} H^{\nu-}_{\rm loc}(\mathbb{R}^2), \qquad \mathcal{E}_{\rm loc}(\varepsilon)(t) \lesssim t^N \quad \text{as} \quad t \to 0$$

with spatial norms that are uniformly controlled as $t \to 0$. Also, $u(0,t) = 0$ for all $0 < t < t_0$. The solution $u(r,t)$ extends as an $H^{1+\nu-}$ solution to all of $\mathbb{R}^2$ and the energy of u concentrates in the cuspidal region $0 \le r \lesssim \frac{1}{\lambda(t)}$ leading to blow-up at $r = t = 0$.

We remark that a somewhat surprising feature of our theorem is that the blow-up rate is prescribed. This is in stark contrast to the usual modulation theoretic approach where the rate function is used to achieve orthogonality to unstable modes of the linearized problem. Heuristically speaking, there are two types of instabilities which typically arise in linearized problems: those due to symmetries of the nonlinear equation (typically leading to algebraic growth of the linear evolution) and those that produce exponential growth in the linear flow (due to some kind of discrete spectrum). For example, the latter arises in the recent work on "center-stable manifolds", see [Sch], [KrSch1], [KrSch2] whereas for the former see [KrSch3]. Both types can lead to blow up. Here we do not have any discrete spectrum in the linearized equation, but rather a zero-energy resonance which is due to the scaling symmetry. It is unclear at this point which role (other than a technical one) the resonance plays in the formation of the blow-up. Indeed, our approach is really non-perturbative as the crucial elliptic profile modifier produces a *large* perturbation of the basic profile Q. The perturbative component of our proof deals with the removal of errors produced by the elliptic profile modifier (it is crucial that these errors decay rapidly in time).

A recent preprint[4] by I. Rodnianski and J. Sterbenz [Ro-St] details the construction of generic sets of initial data (including smooth data) resulting in blow-up with a rate $\lambda(t) \sim \frac{\sqrt{|\log t|}}{t}$ for equivariant wave maps from $\mathbb{R}^{2+1}$ to S^2 with co-rotation index $k \ge 4$. These data can be chosen arbitrarily close to the corresponding co-rotation k harmonic map with respect to a suitable norm stronger than $\|.\|_{H^1}$. This latter behavior appears specific for sufficiently large co-rotation indices *but, due to numerical experiments (e.g.,* [Bi]*) is not expected for the case of co-rotation index one.* More precisely, numerical experiments suggest that perturbations of the "ground state" $Q(r) = 2\arctan(r)$ need to be of a certain size (depending on their profile) to result in blow-up. Indeed, our theorem, which is partly based on perturbative techniques, has a non-perturbative flavor in that the *elliptic profile modifier cannot be made small at fixed time $t = t_0 > 0$ and with a fixed profile $\lambda(t)$.* On a technical level, we remark that the corresponding linearlized operator has zero energy as an eigenvalue for $k > 1$ but for $k = 1$ zero energy becomes a *resonance* (indeed, $\partial_\lambda Q_k(\lambda r)|_{\lambda=1} \in L^2(0,\infty)$ iff $k > 1$ where $Q_k(r) = 2\arctan(r^k)$).

Our argument is correspondingly divided into two parts: first, we use a direct method, exploiting the algebraic fine structure of the system, to find an approximate solution $Q(\lambda(t)r) + u^e(t,r)$. Roughly speaking, one may think of $u_e(.,.)$ as being obtained by a finite sequence of approximations which alternately improve the accuracy near the light cone and near the origin. To model the solution near the light cone, one introduces the coordinates (a,t) where $a = \frac{r}{t}$ and reduces to solving an elliptic problem in a by neglecting time derivatives. More precisely, one treats time derivatives as error source terms, which get decimated by iterating the elliptic construction. Similarly, one improves accuracy near the origin $r = 0$ by working with the coordinates (R,t) where $R = \lambda(t)r$, again reducing to an elliptic problem by neglecting time derivatives. This process does not lead to an actual solution, as one "keeps losing time derivatives", which leads to worse and worse implicit constants. Thus in a second stage, we construct a parametrix for the wave equation which is obtained by passing to coordinates (R,τ) where $R = \lambda(t)r$, $\tau = \frac{1}{\nu}t^{-\nu}$. This in turn relies on a careful analysis of the spectral and scattering theory of the Schrödinger operator which arises by linearizing around $Q(r)$. The remaining error is then iterated away by continued application of the wave parametrix.

A surprising feature of Theorem 1.1 is the fact that *blow-up may be arbitrarily slow, as we can prescribe ν arbitrarily large.* However, the data leading to this blow-up are not generic, and indeed rather difficult to describe. We observe that in [Bi] solutions blowing up with $\lambda(t) \sim t^{-2.3}$ were observed numerically, corresponding to a transient regime dividing blow-up from global smoothness and scattering. In particular, this blow-up rate appears to correspond to a set of initial data of co-dimension one. Our initial data sets seem to lie on manifolds of very large co-dimension, which increases with ν (we plan to return to a rigorous

[4]The conclusions of our paper were reached before the appearance of this preprint

treatment of this conditional stability issue in a later paper). In particular, it appears unlikely that such solutions corresponding to $\nu >> 1$ would be detected numerically.

Finally, we note that our technique quite likely lends itself to constructing blow-up solutions for higher homotopy indices, too, as well as to problems of a similar nature, such as the critical Yang-Mills equation.

2. Approximate solutions

2.1. **The elliptic profile modifier.** In this section we show how to construct an arbitrarily good approximate solution to the wave map equation as a perturbation of a time-dependent harmonic map profile

$$u_0 = Q(R), \qquad R = r\lambda(t)$$

with the polynomial timescale

$$\lambda(t) = t^{-1-\nu}$$

To describe the approximate solution we use the time variable, the variable R which corresponds to the harmonic map scale, and the self-similar variable $a = r/t$ which is useful in analyzing the behavior near the cone. The only trade off in this construction is that we need to allow singularities of the form

$$(1-a^2)^\nu (\log(1-a^2))^k$$

as we approach the cone. Thus the larger the parameter ν, the better the regularity of the approximate solutions.

Theorem 2.1. *Let $k \in \mathbb{N}$. There exists an approximate solution u_{2k} for (1.1) of the form*

$$u_{2k-1}(r,t) = Q(\lambda(t)r) + \frac{c_k}{(t\lambda)^2} R\log(1+R^2) + O\left(\frac{R^{-1}(\log(1+R^2))^2}{(t\lambda)^2}\right)$$

so that the corresponding error has size

$$e_{2k-1} = O\left(\frac{R(\log(2+R))^{2k-1}}{t^2(t\lambda)^{2k}}\right)$$

Here the $O(\cdot)$ terms are uniform in $0 \le r \le t$ and $0 < t < t_0$ where t_0 is a fixed small constant.

Remark 2.2. In the proof we obtain u_{2k-1} and e_{2k-1} which are analytic inside the cone and $C^{\frac{1}{2}+\nu-}$, respectively $C^{-\frac{1}{2}+\nu-}$ on the cone, with a good asymptotic expansion both on the R scale and near the cone.

More precisely, using our notations defined below we have

$$u_{2k-1} \in Q(\lambda(t)r) + \frac{1}{(t\lambda)^2} IS^3(R\log R, \mathcal{Q}_k)$$

while the error satisfies

$$t^2 e_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^1(R(\log R)^{2k-1}, \mathcal{Q}'_{k-1})$$

Proof. We iteratively construct a sequence u_k of better approximate solutions by adding corrections v_k,

$$u_k = v_k + u_{k-1}$$

The error at step k is

$$e_k = (-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r)u_k - \frac{\sin(2u_k)}{2r^2}$$

To construct the increments v_k we first make a heuristic analysis. If u were an exact solution, then the difference

$$\varepsilon = u - u_{k-1}$$

would solve the equation

$$(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r)\varepsilon - \frac{\cos(2u_{k-1})}{2r^2}\sin(2\varepsilon) + \frac{\sin(2u_{k-1})}{2r^2}(1-\cos(2\varepsilon)) = e_{k-1} \tag{2.1}$$

In a first approximation we linearize this equation around $\varepsilon = 0$ and substitute u_{k-1} by u_0. Then we obtain the linear approximate equation

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{\cos(2u_0)}{r^2}\right)\varepsilon \approx e_{k-1} \tag{2.2}$$

For $r \ll t$ we expect the time derivative to play a lesser role so we neglect it and we are left with an elliptic equation with respect to the variable r,

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\cos(2u_0)}{r^2}\right)\varepsilon \approx e_{k-1}, \qquad r \ll t \tag{2.3}$$

For $r \approx t$ we can approximate $\cos(2u_0)$ by 1 and rewrite (2.2) in the form

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\varepsilon \approx e_{k-1}$$

Here the time and spatial derivatives have the same strength. However, we can identify another principal variable, namely $a = r/t$ and think of ε as a function of (t, a). As it turns out, neglecting a "higher order" part of e_{k-1} which can be directly included in e_k, we are able to use scaling and the exact structure of the principal part of e_{k-1} to reduce the above equation to a Sturm-Liouville problem in a which becomes singular at $a = 1$.

The above heuristics lead us to a two step iterative construction of the v_k's. The two steps successively improve the error in the two regions $r \ll t$, respectively $r \approx t$. To be precise, we define v_k by

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\cos(2u_0)}{r^2}\right)v_{2k+1} = e_{2k}^0 \tag{2.4}$$

respectively

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)v_{2k} = e_{2k-1}^0 \tag{2.5}$$

both equations having zero Cauchy data[5] at $r = 0$. Here at each stage the error term e_k is split into a principal part and a higher order term (to be made precise below),

$$e_k = e_k^0 + e_k^1$$

The successive errors are then computed as

$$e_{2k} = e_{2k-1}^1 + N_{2k}(v_{2k}), \qquad e_{2k+1} = e_{2k}^1 - \partial_t^2 v_{2k+1} + N_{2k+1}(v_{2k+1})$$

where

$$-N_{2k+1}(v) = \frac{\cos(2u_0) - \cos(2u_{2k})}{r^2}v + \frac{\sin(2u_{2k})}{2r^2}(1 - \cos(2v)) + \frac{\cos(2u_{2k})}{2r^2}(2v - \sin(2v)) \tag{2.6}$$

respectively

$$-N_{2k}(v) = \frac{1 - \cos(2u_{2k-1})}{r^2}v + \frac{\sin(2u_{2k-1})}{2r^2}(1 - \cos(2v)) + \frac{\cos(2u_{2k-1})}{2r^2}(2v - \sin(2v)) \tag{2.7}$$

To formalize this scheme we need to introduce suitable function spaces in the cone

$$\mathcal{C}_0 = \{(r,t) \, : \, 0 \le r < t, 0 < t < t_0\}$$

for the successive corrections and errors. We first consider the a dependence. For the corrections v_k we use

Definition 2.3. *For $i \in \mathbb{N}$ we let $j(i) = i$ if ν is irrational, respectively $j(i) = 2i^2$ if ν is rational.*
a) $\mathcal{Q}$ is the algebra of continuous functions $q : [0,1] \to \mathbb{R}$ with the following properties:
(i) q is analytic in $[0,1)$ with an even expansion at 0
(ii) Near $a = 1$ we have an absolutely convergent expansion of the form

$$q = q_0(a) + \sum_{i=1}^{\infty}\left((1-a)^{(2i-1)\nu+\frac{1}{2}}\sum_{j=0}^{j(2i-1)} q_{2i-1,j}(a)(\log(1-a))^j + (1-a)^{2i\nu+1}\sum_{j=0}^{j(2i)} q_{2i,j}(a)(\log(1-a))^j\right)$$

[5] The coefficients are singular at $r = 0$, therefore this has to be given a suitable interpretation

with analytic coefficients q_0, q_{ij}

b) $\mathcal{Q}_m$ is the algebra which is defined similarly, with the additional requirement that

$$q_{ij}(1) = 0 \quad \textit{if} \quad i \geq 2m+1, \textit{odd}$$

For the errors e_k we introduce

Definition 2.4. *a) With $j(i)$ as above, $\mathcal{Q}'$ is the space of continuous functions $q : [0,1] \to \mathbb{R}$ with the following properties:*

(i) q is analytic in $[0,1)$ with an even expansion at 0

(ii) Near $a = 1$ we have a convergent expansion of the form

$$q = q_0(a) + \sum_{i=1}^{\infty} \left((1-a)^{(2i-1)\nu - \frac{1}{2}} \sum_{j=0}^{j(2i-1)} q_{2i-1,j}(a)(\log(1-a))^j + (1-a)^{2i\nu} \sum_{j=0}^{j(2i)} q_{2i,j}(a)(\log(1-a))^j \right)$$

with analytic coefficients q_0, q_{ij}

b) $\mathcal{Q}'_m$ is the space which is defined similarly, with the additional requirement that

$$q_{ij}(1) = 0 \quad \textit{if} \quad i \geq 2m+1, \;\; \textit{odd}$$

Next we define the class of functions of R:

Definition 2.5. *$S^m(R^k(\log R)^\ell)$ is the class of analytic functions $v : [0,\infty) \to \mathbb{R}$ with the following properties:*

(i) v vanishes of order m at $R = 0$

(ii) v has a convergent expansion near $R = \infty$,

$$v = \sum_{0 \leq j \leq \ell + i} c_{ij}\, R^{k-2i} (\log R)^j$$

We also introduce another auxiliary variable,

$$b = \frac{(\log(2+R^2))^2}{(t\lambda)^2} \tag{2.8}$$

Since we seek solutions inside the cone we can restrict b to a small interval $[0, b_0]$. We combine these three components in order to obtain the full function class which we need:

Definition 2.6. *a) $S^m(R^k(\log R)^\ell, \mathcal{Q}_n)$ is the class of analytic functions $v : [0,\infty) \times [0,1] \times [0,b_0] \to \mathbb{R}$ so that*

(i) v is analytic as a function of R, b,

$$v : [0,\infty) \times [0,b_0] \to \mathcal{Q}_n$$

(ii) v vanishes of order m at $R = 0$

(iii) v has a convergent expansion at $R = \infty$,

$$v(R,\cdot,b) = \sum_{0 \leq j \leq \ell + i} c_{ij}(\cdot, b) R^{k-2i} (\log R)^j$$

where the coefficients $c_{ij} : [0,b_0] \to \mathcal{Q}_m$ are analytic with respect to b

b) $IS^m(R^k(\log R)^\ell, \mathcal{Q}_n)$ is the class of analytic functions w on the cone $\mathcal{C}_0$ which can be represented as

$$w(r,t) = v(R,a,b), \qquad v \in S^m(R^k(\log R)^\ell, \mathcal{Q}_n)$$

We note that the representation of functions on the cone as in part (b) is in general not unique since R, a, b are dependent variables. Later we shall exploit this fact and switch from one representation to another as needed. We shall prove by induction that the successive corrections v_k and the corresponding error terms e_k can be chosen with the following properties: For each $k \geq 1$,

$$v_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^3(R(\log R)^{2k-1}, \mathcal{Q}_{k-1}) \tag{2.9}$$

$$t^2 e_{2k-1} \in \frac{1}{(t\lambda)^{2k}} IS^1(R(\log R)^{2k-1}, \mathcal{Q}'_{k-1}) \tag{2.10}$$

$$v_{2k} \in \frac{1}{(t\lambda)^{2k+2}} IS^3(R^3(\log R)^{2k-1}, \mathcal{Q}_k) \tag{2.11}$$

$$t^2 e_{2k} \in \frac{1}{(t\lambda)^{2k}} \left[IS^1(R^{-1}(\log R)^{2k}, \mathcal{Q}_k) + bIS^1(R(\log R)^{2k-1}, \mathcal{Q}'_k) \right] \tag{2.12}$$

Moreover, for $k = 0$,

$$t^2 e_0 \in S^1(R^{-1}) \tag{2.13}$$

Step 0: *We begin the analysis at $k = 0$, where we explicitly compute e_0.*
We have

$$\begin{aligned}
e_0 &= -u_{0tt} \\
&= -|\lambda'(t)|^2 r^2 Q''(R) - \lambda''(t) r Q'(R) \\
&= -\left(\frac{\lambda'}{\lambda}\right)^2 R^2 Q''(R) - \frac{\lambda''}{\lambda} R Q'(R) \\
&= \frac{1}{t^2}\left((\nu+1)^2 \frac{4R^3}{(1+R^2)^2} - (\nu+1)(\nu+2)\frac{2R}{1+R^2} \right) \\
&= \frac{1}{t^2}\left(-(\nu+1)^2 \frac{4R}{(1+R^2)^2} + \nu(\nu+1)\frac{2R}{1+R^2} \right)
\end{aligned}$$

With our notations

$$t^2 e_0 \in S^1(R^{-1})$$

as claimed. It remains to complete the induction step. Hence we assume we know the above relations hold up to $k-1$ with $k \geq 1$, and construct v_{2k-1}, respectively v_{2k}, so that they hold for the index k.

Step 1: *Begin with e_{2k-2} satisfying (2.12) or (2.13) and choose v_{2k-1} so that (2.9) holds.*
If $k = 1$, then define $e_0^0 := e_0$. If $k > 1$, we define the principal part e^0_{2k-2} of e_{2k-2} by setting $b = 0$, i.e.,

$$e^0_{2k-2}(R, a) := e_{2k-2}(R, a, 0)$$

For the difference we can pull out a factor of b and conclude that

$$\begin{aligned}
t^2 e^1_{2k-2} &\in \frac{b}{(t\lambda)^{2k-2}} \left[IS^1(R^{-1}(\log R)^{2k-2}, \mathcal{Q}_{k-1}) + IS^1(R(\log R)^{2k-3}, \mathcal{Q}'_{k-1}) \right] \\
&\subset \frac{1}{(t\lambda)^{2k}} IS^1(R(\log R)^{2k-1}, \mathcal{Q}'_{k-1})
\end{aligned}$$

which can be included in e_{2k-1}, cf. (2.10).

We define v_{2k-1} as in (2.4) neglecting the a dependence of e^0_{2k-2}. In other words, a is treated as a parameter. Changing variables to R in (2.4) we need to solve the equation

$$(t\lambda)^2 L v_{2k-1} = t^2 e^0_{2k-2} \in \frac{1}{(t\lambda)^{2k}} IS^1(R^{-1}(\log R)^{2k-2}, \mathcal{Q}_{k-1})$$

where the operator L is given by

$$L = \partial_R^2 + \frac{1}{R}\partial_R - \frac{\cos(2u_0)}{R^2} = \partial_R^2 + \frac{1}{R}\partial_R - \frac{1}{R^2}\frac{1 - 6R^2 + R^4}{(1+R^2)^2}$$

Then (2.9) is a consequence of the following ODE lemma.

Lemma 2.7. *Let $k \geq 1$. Then the solution v to the equation*

$$Lv = f \in S^1(R^{-1}(\log R)^{2k-2}), \qquad v(0) = v'(0) = 0$$

has the regularity

$$v \in S^3(R(\log R)^{2k-1})$$

Proof. Since f is analytic at 0 with a linear leading term, one can easily write down a Taylor series for v at 0 with a cubic leading term.

It remains to determine the asymptotic behavior of v at infinity. For this it is convenient to remove the first order derivative in L (to achieve constancy of the Wronskian). Thus, we seek a solution of

$$\tilde{L}\sqrt{R}\,v = \sqrt{R}f, \qquad \tilde{L} = \partial_R^2 - \frac{3}{4R^2} + \frac{8}{(1+R^2)^2}$$

We use this fundamental system of solutions for $\tilde{L}$:

$$\phi(R) = \frac{R^{\frac{3}{2}}}{1+R^2}, \qquad \theta(R) = \frac{-1 + 4R^2\log R + R^4}{\sqrt{R}\,(1+R^2)}$$

Their Wronskian is $W(\phi, \theta) = 2$. This allows us to obtain an integral representation for v using the variation of parameters formula, which gives

$$v = \frac{1}{2}R^{-\frac{1}{2}}\theta(R)\int_0^R \phi(R')\sqrt{R'}f(R')\;dR' - \frac{1}{2}R^{-\frac{1}{2}}\phi(R)\int_0^R \theta(R')\sqrt{R'}f(R')\;dR'$$

Carrying out the integration shows that the right-hand side grows like $R(\log R)^{2k-1}$ as claimed. □

As a special case of the above computation we also note the representation for v_1,

$$v_1 = \frac{1}{(t\lambda)^2}V(R), \qquad V \in S^3(R\log R) \tag{2.14}$$

Step 2: *Show that if v_{2k-1} is chosen as above then* (2.10) *holds.*

Thinking of v_{2k-1} as a function of t, R and a we can write e_{2k-1} in the form

$$e_{2k-1} = N_{2k-1}(v_{2k-1}) + E^t v_{2k-1} + E^a v_{2k-1}$$

Here $N_{2k-1}(v_{2k-1})$ accounts for the contribution from the nonlinearity and is given by (2.6). $E^t v_{2k-1}$ contains the terms in

$$\partial_t^2 v_{2k-1}(t, R, a)$$

where no derivative applies to the variable a, while $E^a v_{2k-1}$ contains the terms in

$$(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r)v_{2k-1}(t, R, a)$$

where at least one derivative applies to the variable a. We begin with the terms in N_{2k-1}. We first note that, by summing the v_j over $1 \leq j \leq 2k-2$,

$$u_{2k-2} - u_0 \in \frac{1}{(t\lambda)^2}IS^3(R\log R, \mathcal{Q}_{k-1}) \tag{2.15}$$

To switch to trigonometric functions we need

Lemma 2.8. *Let*

$$v \in \frac{1}{(t\lambda)^2}IS^3(R\log R, \mathcal{Q}_{k-1})$$

Then

$$\sin v \in \frac{1}{(t\lambda)^2}IS^3(R\log R, \mathcal{Q}_{k-1}), \qquad \cos v \in IS^0(1, \mathcal{Q}_{k-1})$$

Proof. We write

$$\sin v = vg(v^2)$$

with g an entire function. Then it suffices to show that $g(v^2) \in IS^0(1, \mathcal{Q}_{k-1})$. We begin with

$$v^2 \in \frac{1}{(t\lambda)^4} IS^6(R^2(\log R)^2, \mathcal{Q}_{k-1}) \subset \frac{R^2}{(t\lambda)^2}\frac{1}{(t\lambda)^2} IS^4((\log R)^2, \mathcal{Q}_{k-1})$$

But $a^2 = R^2(t\lambda)^{-2}$ and the remaining $(t\lambda)^{-2}$ together with up to two $\log R$ factors combines to give one b factor. We conclude that

$$v^2 \in a^2 b IS^2(1, \mathcal{Q}_{k-1}) \subset IS^2(1, \mathcal{Q}_{k-1})$$

For $w \in S^2(1, \mathcal{Q}_{k-1})$ we evaluate $g(w)$. Since g is analytic we conclude that $g(w)$ is analytic in R, b when interpreted as

$$g(w) \ : \ [0,\infty) \times [0, b_0] \to \mathcal{Q}_{k-1}$$

We consider the asymptotic expansion at $R = \infty$. Since we have an absolutely convergent asymptotic expansion for w and a convergent Taylor series for g at 0, we obtain an absolutely convergent asymptotic expansion for $g(w)$. This gives

$$g(w) \in S^0(1, \mathcal{Q}_{k-1})$$

and concludes the proof of the lemma. □

Using Lemma 2.8 and (2.15) we compute

$$\begin{aligned}
\cos(2u_0) - \cos(2u_{2k-2}) &= 2\cos(2u_0)\sin^2(u_{2k-2} - u_0) + 2\sin(2u_0)\sin(u_{2k-2}-u_0)\cos(u_{2k-2}-u_0) \\
&\subset \frac{1}{(t\lambda)^4} IS^6(R^2(\log R)^2, \mathcal{Q}_{k-1}) + \frac{1}{(t\lambda)^2} IS^4(\log R, \mathcal{Q}_{k-1})
\end{aligned}$$

Hence

$$\begin{aligned}
& t^2 \frac{\cos(2u_0) - \cos(2u_{2k-2})}{r^2} v_{2k-1} \\
\in\ & \frac{(t\lambda)^2}{R^2}\left(\frac{1}{(t\lambda)^2} IS^4(\log R, \mathcal{Q}_{k-1}) + \frac{1}{(t\lambda)^4} IS^6(R^2(\log R)^2, \mathcal{Q}_{k-1})\right) \frac{1}{(t\lambda)^{2k}} IS^3(R(\log R)^{2k-1}, \mathcal{Q}_{k-1}) \\
\subset\ & \frac{1}{(t\lambda)^{2k}}\left(IS^5(R^{-1}(\log R)^{2k}, \mathcal{Q}_{k-1}) + \frac{1}{(t\lambda)^2} IS^7(R(\log R)^{2k+1}, \mathcal{Q}_{k-1})\right) \\
\subset\ & \frac{1}{(t\lambda)^{2k}} IS^5(R(\log R)^{2k-1}, \mathcal{Q}_{k-1})
\end{aligned}$$

where at the last step we have pulled a b factor out of the second term. Similarly we have

$$\begin{aligned}
& t^2 \frac{\sin(2u_{2k-2})}{2r^2}(1 - \cos(2v_{2k-1})) \\
\in\ & \frac{(t\lambda)^2}{R^2}\left(IS^1(R^{-1}, \mathcal{Q}_{k-1}) + \frac{1}{(t\lambda)^2} IS^3(R\log R, \mathcal{Q}_{k-1})\right)\left(\frac{1}{(t\lambda)^{2k}} IS^3(R(\log R)^{2k-1}, \mathcal{Q}_{k-1})\right)^2 \\
=\ & \frac{1}{(t\lambda)^{2k}}\left(\frac{1}{(t\lambda)^{2k-2}} IS^5(R^{-1}(\log R)^{4k-2}, \mathcal{Q}_{k-1}) + \frac{1}{(t\lambda)^{2k}} IS^7(R(\log R)^{4k-1}, \mathcal{Q}_{k-1})\right) \\
\subset\ & \frac{1}{(t\lambda)^{2k}} IS^5(R(\log R)^{2k-1}, \mathcal{Q}_{k-1})
\end{aligned}$$

where we have used a power of b^k to pass to the final inclusion. Finally,

$$\begin{aligned}
t^2 \frac{\cos(2u_{2k-2})}{r^2}(2v_{2k-1} - \sin(2v_{2k-1})) &\in \frac{(t\lambda)^2}{R^2} IS^0(1, \mathcal{Q}_{k-1})\left(\frac{1}{(t\lambda)^{2k}} IS^3(R(\log R)^{2k-1}, \mathcal{Q}_{k-1})\right)^3 \\
&\subset \frac{1}{(t\lambda)^{6k-2}} IS^7(R(\log R)^{6k-3}, \mathcal{Q}_{k-1}) \\
&\subset \frac{1}{(t\lambda)^{2k}} IS^7(R(\log R)^{2k-1}, \mathcal{Q}_{k-1})
\end{aligned}$$

This concludes the analysis of $N_{2k-1}(v_{2k-1})$. We continue with the terms in $E^t v_{2k-1}$, where we can neglect the a dependence. Therefore, it suffices to compute

$$t^2\partial_t^2\left(\frac{1}{(t\lambda)^{2k}}IS^3(R(\log R)^{2k-1})\right)\subset\frac{1}{(t\lambda)^{2k}}IS^1(R(\log R)^{2k-1})$$

Finally, we consider the terms in $E^a v_{2k-1}$. For

$$v_{2k-1}(r,t)=\frac{1}{(t\lambda)^{2k}}w(R,a),\qquad w\in S^3(R(\log R)^{2k-1},\mathcal{Q}_{k-1})$$

we have

$$\begin{aligned} t^2E^a v_{2k-1} &= \frac{1}{(t\lambda)^{2k}}\big[2k\nu a w_a(R,a)-(\nu+1)Raw_{Ra}(R,a)+2R^{-1}a^{-1}w_{Ra}(R,a)+a^{-1}w_a(R,a)\\ &+ (1-a^2)w_{aa}(R,a)-aw_a(R,a)\big]\end{aligned}$$

Since $\mathcal{Q}_{k-1}$ are even in a we conclude that

$$a\partial_a,\,a^{-1}\partial_a,\,(1-a^2)\partial_a^2:\mathcal{Q}_{k-1}\to\mathcal{Q}'_{k-1}$$

Also the R^{-1} factor simply removes one order of vanishing at $R=0$. Hence we easily obtain

$$t^2E^a v_{2k-1}\in\frac{1}{(t\lambda)^{2k}}IS^1(R(\log R)^{2k-1},\mathcal{Q}'_{k-1})$$

This concludes the proof of (2.10). We remark that for the special case of $k=1$, i.e., with v_1 as in (2.14), these arguments yield

$$t^2e_1\in(t\lambda)^{-2}S^3(R\log R) \tag{2.16}$$

Step 3: *Define v_{2k} so that* (2.11) *holds.*

We begin the analysis with e_{2k-1} replaced by its main asymptotic component f_{2k-1} at $R=\infty$ for $b=0$. This has the form

$$t^2f_{2k-1}=\frac{R}{(t\lambda)^{2k}}\sum_{j=0}^{2k-1}q_j(a)(\log R)^j,\qquad q_j\in\mathcal{Q}'_{k-1}$$

which we rewrite as

$$t^2f_{2k-1}=\frac{1}{(t\lambda)^{2k-1}}\sum_{j=0}^{2k-1}aq_j(a)(\log R)^j$$

We remark that (2.16) implies that $t^2f_1(a)=(t\lambda)^{-1}a\log R$. Consider the equation (2.5) with f_{2k-1} on the right-hand side,

$$t^2\left(-\partial_t^2+\partial_r^2+\frac{1}{r}\partial_r-\frac{1}{r^2}\right)w_{2k}=t^2f_{2k-1}$$

Homogeneity considerations suggest that we should look for a solution w_{2k} which has the form

$$w_{2k}=\frac{1}{(t\lambda)^{2k-1}}\sum_{j=0}^{2k-1}W^j_{2k}(a)(\log R)^j$$

The one-dimensional equations for W^j_{2k} are obtained by matching the powers of $\log R$. This gives the system of equations

$$t^2\left(-\partial_t^2+\partial_r^2+\frac{1}{r}\partial_r-\frac{1}{r^2}\right)\left(\frac{1}{(t\lambda)^{2k-1}}W^j_{2k}(a)\right)=\frac{1}{(t\lambda)^{2k-1}}(aq_j(a)-F_j(a))$$

where

$$\begin{aligned}F_j(a)=(j+1)&\left[((\nu+1)\nu(2k-1)+a^{-2})W^{j+1}_{2k}+(a^{-1}-(1+\nu)a)\partial_aW^{j+1}_{2k}\right]\\ &+(j+2)(j+1)((\nu+1)^2+a^{-2})W^{j+2}_{2k}\end{aligned} \tag{2.17}$$

Here we make the convention that $W_{2k}^j = 0$ for $j \geq 2k$. Then we solve the equations in this system successively for decreasing values of j from $2k-1$ to 0.

Conjugating out the power of t we get

$$t^2\Big(-\Big(\partial_t + \frac{(2k-1)\nu}{t}\Big)^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big) W_{2k}^j(a) = aq_j(a) - F_j(a)$$

which we rewrite as an equation in the a variable,

$$L_{(2k-1)\nu} W_{2k}^j = aq_j(a) - F_j(a) \tag{2.18}$$

where the one-parameter family of operators L_β is defined by

$$L_\beta = (1-a^2)\partial_a^2 + (a^{-1} + 2a\beta - 2a)\partial_a + (-\beta^2 + \beta - a^{-2}) \tag{2.19}$$

We claim that solving this system with 0 Cauchy data at $a = 0$ yields solutions which satisfy

$$W_{2k}^j \in a^3 \mathcal{Q}_k, \qquad j = \overline{0, 2k-1} \tag{2.20}$$

To prove this we need the following

Lemma 2.9. *Let $0 \leq m(j) \lesssim j^2$. Let f be an analytic function in $[0,1)$ with an odd expansion at 0 and an absolutely convergent expansion near $a = 1$ of the form*

$$\begin{aligned} f(a) = f_0(a) + \sum_{j=1}^{\infty} \Bigg[&(1-a)^{(2j-1)\nu - \frac{1}{2}} \sum_{m=0}^{m(2j-1)} f_{2j-1,m}(a) \left[\log(1-a)\right]^m + \\ &+ (1-a)^{2j\nu} \sum_{m=0}^{m(2j)} f_{2j,m}(a) \left[\log(1-a)\right]^m \Bigg] \end{aligned} \tag{2.21}$$

with $f_{i,j}$ analytic near $a = 1$. Then there is a unique solution w to the equation

$$L_{(2k-1)\nu}\, w = f, \qquad w(0) = 0,\ \partial_a w(0) = 0 \tag{2.22}$$

with the following properties:

(i) w is an analytic function in $[0,1)$

(ii) w is cubic at 0 and has an odd expansion at 0

(iii) w has an absolutely convergent expansion near $a = 1$ of the form

$$\begin{aligned} w(a) = w_0(a) + \sum_{j=1}^{\infty} \Bigg[&(1-a)^{(2j-1)\nu + \frac{1}{2}} \sum_{\ell=0}^{\ell(2j-1)} w_{2j-1,\ell}(a) [\log(1-a)]^\ell + \\ &+ (1-a)^{2j\nu+1} \sum_{\ell=0}^{\ell(2j)} w_{2j,\ell}(a) [\log(1-a)]^\ell \Bigg] \end{aligned} \tag{2.23}$$

with $w_{i,j}$ analytic near $a = 1$ and $\ell(i) = m(i)$ with one exception, namely $\ell(2k-1) = m(2k-1)+1$. If however $f_{2k-1,m(2k-1)}(1) = 0$, then $\ell(2k-1) = m(2k-1)$. In that case also $w_{2k-1,\ell}(1) = 0$ if $\ell > 0$, but not necessarily when $\ell = 0$. Finally, if $f_{2i-1,j}(1) = 0$ for all $i > k$ and all j, then also $w_{2i-1,\ell}(1) = 0$ for all $i > k$ and all ℓ.

Proof. Denote $\beta = (2k-1)\nu$. Since $k \geq 1$ and $\nu > \frac{1}{2}$, also $\beta > \frac{1}{2}$. Cearly, L_β is the sum of two pieces: that part which is homogeneous of degree -2 in a, viz.,

$$L_\beta^0 = \partial_a^2 + a^{-1}\partial_a - a^{-2} = a^{-1}\partial_a(a\partial_a) - a^{-2}$$

and the remainder which is homogeneous of degree 0. The equation $L_\beta^0\, y = 0$ has fundamental solutions a and a^{-1}. A standard power-series ansatz then leads to fundamental solutions of $L_\beta\, y = 0$ of the form

$$\phi_1(a) = a(1 + O(a^2)), \qquad \phi_2(a) = a^{-1}(1 + O(a^2))$$

where the $O(\cdot)$ terms are analytic functions of a^2. Since our right-hand side f has size $O(a)$ at 0, this implies that we can use the equation to write a complete Taylor expansion for w near 0. Matching coefficients in $L_\beta w = f$ with

$$f(a) = \sum_{j=1}^{\infty} f_j \, a^{2j-1}, \qquad w(a) = \sum_{j=2}^{\infty} w_j \, a^{2j-1}$$

yields

$$2j(2j-2)w_j = (2j(2j-1) - (4j-1)\beta + \beta^2)w_{j-1} + f_{j-1}$$

where we take $w_1 = 0$. The coefficient of w_j is always nonzero; this allows us to successively compute the coefficients w_j. The convergence of the series for w easily follows from the convergence of the series for f.
It remains to study the solution w near $a = 1$. The behavior of L_β at 1 is well approximated by

$$L_\beta^1 = 2(1-a)\partial_a^2 + (2\beta - 1)\partial_a + (\beta - \beta^2 - 1) = 2(1-a)^{\beta+\frac{1}{2}}\partial_a\big[(1-a)^{-\beta+\frac{1}{2}}\partial_a\big] + (\beta - \beta^2 - 1)$$

in the sense that

$$L_\beta = L_\beta^1 + (a-1)\big[(1-a)\partial_a^2 + (2(\beta-1) - a^{-1})\partial_a + (a+1)a^{-2}\big] =: L_\beta^1 + (a-1)L_\beta^2 \tag{2.24}$$

The differential operator

$$2(1-a)^{\beta+\frac{1}{2}}\partial_a\big[(1-a)^{-\beta+\frac{1}{2}}\partial_a\big] \tag{2.25}$$

annihilates 1 and $(1-a)^{\beta+\frac{1}{2}}$. Therefore, we seek a fundamental system for $L_\beta^1 y = 0$ of the form

$$\phi_1(a) = 1 + \sum_{\ell=1}^{\infty} \mu_\ell (1-a)^\ell, \qquad \phi_2(a) = (1-a)^{\beta+\frac{1}{2}}\Big[1 + \sum_{\ell=1}^{\infty} \tilde{\mu}_\ell (1-a)^\ell\Big] \tag{2.26}$$

This leads to the conditions, for $\ell \geq 1$,

$$\mu_1(1-2\beta) + \beta - \beta^2 - 1 = 0, \quad \mu_{\ell+1}(\ell+1)(2\ell+1-2\beta) + (\beta - \beta^2 - 1)\mu_\ell = 0 \tag{2.27}$$

$$\tilde{\mu}_1(2\beta+3) + \beta - \beta^2 - 1 = 0, \quad \tilde{\mu}_{\ell+1}(\ell+1)(2\ell+3+2\beta) + (\beta - \beta^2 - 1)\tilde{\mu}_\ell = 0 \tag{2.28}$$

Clearly, (2.28) always has a solution whereas (2.27) requires $\beta - \frac{1}{2} \notin \mathbb{Z}^+$; in the latter case, the series in (2.26) define entire functions. If, on the other hand, $\ell_0 := \beta - \frac{1}{2} \in \mathbb{Z}^+$, then ϕ_1 is modified to

$$\phi_1(a) = 1 + \sum_{\ell=1}^{\infty} \mu_\ell (1-a)^\ell + c_1\, \phi_2(a)\log(1-a) \tag{2.29}$$

with the unique choice $c_1 = -(2\beta+1)^{-1}(\beta - \beta^2 - 1)\mu_{\ell_0}$. Here (2.27) is unchanged and can be solved for μ_ℓ up to $\ell \leq \ell_0$; for $\ell > \ell_0$ this equation is then modified by the terms from the ϕ_2 series (in particular, for $\ell = \ell_0 + 1$ the choice of c_1 assures the validity of the equation, whereas for all $\ell > \ell_0 + 1$ we can again solve for μ_ℓ). Finally, observe that the same process also leads to a fundamental system for L_β; indeed, the remainder $(a-1)L_\beta^2$ in (2.24) does not change the coefficients of $\mu_{\ell+1}$ or $\tilde{\mu}_{\ell+1}$ in (2.27) and (2.28). In conclusion, the preceding power series argument leads to a fundamental system of $L_\beta y = 0$, which we again denote by $\phi_1(a)$ and $\phi_2(a)$.

Modulo a linear combination of ϕ_1, ϕ_2 it suffices to find one solution to the inhomogeneous equation $L_\beta w = f$ near $a = 1$. At this point, it will be convenient to write L_β as a Sturm-Liouville operator. Thus, we write

$$L_\beta = q_1^{-1}(a)\partial_a(q_2(a)\partial_a) + q_3(a)$$

with, cf. (2.19),

$$q_1^{-1}q_2(a) = 1 - a^2, \qquad q_1^{-1}q_2'(a) = a^{-1} + 2a(\beta - 1), \qquad q_3(a) = -\beta^2 + \beta - a^{-2}$$

One checks that for a close to 1 the first two equations admit solutions

$$q_2(a) = (1-a)^{-\beta+\frac{1}{2}}[1 + (1-a)\tilde{q}_1(a)], \qquad q_1(a) = \frac{1}{2}(1-a)^{-\beta-\frac{1}{2}}[1 + (1-a)\tilde{q}_2(a)]$$

with $\tilde{q}_1, \tilde{q}_2$ analytic near $a = 1$. The Wronskian can now be computed as

$$q_2(a)[\phi_1(a)\phi_2'(a) - \phi_1'(a)\phi_2(a)] = \beta + 1/2$$

Thus, a particular solution of the inhomogeneous problem is given by

$$w(a) = (\beta+1/2)^{-1}\phi_1(a)\int_a^1 \phi_2(a')q_1(a')f(a')\,da' + (\beta+1/2)^{-1}\phi_2(a)\int_{a_0}^a \phi_1(a')q_1(a')f(a')\,da' \tag{2.30}$$

where $a_0 < 1$ is some number close to 1. For the first integral, note that $\phi_2(a')q_1(a')$ is an analytic function in the neighborhood of $a' = 1$. Let $\gamma \neq -1$ and m be a positive integer. Iterating the relation

$$\int_a^1 (1-a')^\gamma [\log(1-a')]^m\,da' = \frac{1}{\gamma+1}[\log(1-a)]^m(1-a)^{\gamma+1} - \frac{m}{\gamma+1}\int_a^1 (1-a')^\gamma[\log(1-a')]^{m-1}\,da' \tag{2.31}$$

shows that each summand on the right-hand side of (2.21), inserted into the first integral in (2.30), makes an admissible contribution to w in the sense of (2.23) (for this it does not matter whether ϕ takes the form (2.26) or (2.29)). The analysis of the second integral in (2.30) is again based on (2.31) provided $j \neq k$, since then $\gamma \neq -1$. If $j = k$, then we encounter

$$\int_{a_0}^a (1-a')^{-1}[\log(1-a')]^m\,da' = -(m+1)^{-1}[\log(1-a)]^{m+1} + C$$

which explains why we obtain an extra log-factor when $j = k$. Clearly, if $f_{2k-1,m(2k-1)}(1) = 0$ then there is no extra log-factor and the lemma is proved. In that case also we write, with

$$f = (1-a)^{\beta-\frac12}\sum_{m=0}^{m(2k-1)} f_{2k-1,m}(a)\,[\log(1-a)]^m$$

the second integral in (2.30) as

$$\phi_2(a)\int_{a_0}^a \phi_1(a')q_1(a')f(a')\,da' = \phi_2(a)\int_{a_0}^1 \phi_1(a')q_1(a')f(a')\,da' - \phi_2(a)\int_a^1 \phi_1(a')q_1(a')f(a')\,da'$$

The first term on the right-hand side here is just a multiple of $\phi_2(a)$, whereas the second one possesses the extra vanishing at $a = 1$, as claimed. The final claim of the lemma follows similarly. ☐

Before turning to the proof of (2.20) in full generality, we first discuss the special case $k = 1$. This will also serve to explain how the algebra $\mathcal{Q}_k$ arises at all in the iteration. If $k = 1$, then (2.18) reduces to the system

$$L_\nu W_2^1(a) = a, \qquad L_\nu W_2^0(a) = -(\nu(\nu+1)+a^{-2})W_2^1(a) - (a^{-1}-(\nu+1)a)\partial_a W_2^1(a)$$

due to $t^2 f_1(a) = (t\lambda)^{-1} a\log R$. In view of the solution formula (2.30) with $\beta = \nu$, provided $\nu - \frac12 \notin \mathbb{Z}^+$,

$$\begin{aligned}
W_2^1(a) &= g_0(a) + g_1(a)(1-a)^{\nu+\frac12}\\
W_2^0(a) &= h_0(a) + h_1(a)(1-a)^{\nu+\frac12} + h_2(a)(1-a)^{\nu+\frac12}\log(1-a)
\end{aligned}$$

where $g_j(a), h_j(a)$ are analytic around $a = 1$. Note that the term $(1-a)^{\nu+\frac12}\log(1-a)$ appears in W_2^0 due to $\partial_a W_2^1$. Similarly, if $\nu - \frac12 \in \mathbb{Z}^+$, then

$$\begin{aligned}
W_2^1(a) &= g_0(a) + g_1(a)(1-a)^{\nu+\frac12} + g_2(a)(1-a)^{\nu+\frac12}\log(1-a)\\
W_2^0(a) &= h_0(a) + (1-a)^{\nu+\frac12}\sum_{\ell=0}^{2} h_{\ell+1}(a)[\log(1-a)]^\ell + (1-a)^{2\nu+1}\sum_{\ell=0}^{2} h_{\ell+4}(a)[\log(1-a)]^\ell,
\end{aligned}$$

with analytic g_j, h_j. The terms involving the $(1-a)^{2\nu+1}$ factor in W_2^0 are due to the modified ϕ, see (2.29). Thus, we see that in all cases $W_2^j \in \mathcal{Q}_1$ for $j = 0, 1$ and a near 1.

We now continue with the proof of (2.20) for general k. At first we consider the easier case when ν is irrational. We apply the lemma in (2.18) using for the right-hand side the fact that $q_{2k-1} \in Q'_{k-1}$. This implies that the coefficient of $(1-a)^{(2k-1)\nu-\frac12}$ in q_{2k-1} vanishes at $a = 1$. The lemma gives a similar expansion for W_{2k}^{2k-1} with the required vanishing conditions. Hence $W_{2k}^{2k-1} \in \mathcal{Q}_k$, with one extra $(1-a)^{(2k-1)\nu+\frac12}$ term (this is the $w_{2k-1,0}(1) \neq 0$ statement of the lemma) – we refer to this as the "free term" in what follows.

Next we reiterate the argument for the remaining W_{2k}^j which solve (2.18). At each step we have to compute F_j, see (2.17). Since W_{2k}^{j+1} and W_{2k}^{j+2} have an odd Taylor expansion at 0 beginning with a cubic term, it follows that F_j has an odd Taylor expansion at 0 beginning with a linear term. The expansion of F_j around $a=1$ is similar to the one for W^{j+1}_{2k} except that one $(1-a)$ factor is lost in the "free term". For $j=2k-2$ this produces the term $(1-a)^{(2k-1)\nu+\frac12}\log(1-a)$ in W^j_{2k} etc. At the conclusion of the iteration we have gained at most $2k-1$ logarithms in the free term for the W^j_{2k}'s. Then (2.20) follows.

Next we consider the case when ν is rational. This is more difficult since now the term $(1-a)^{(2k-1)\nu-\frac12}$ can also arise in expressions of the form

$$f_{2j-1,m}(a)(1-a)^{(2j-1)\nu-\frac12}[\log(1-a)]^m \quad or \quad f_{2j,m}(a)(1-a)^{2j\nu}[\log(1-a)]^m$$

using the notations of the lemma. This leads to more logarithms than in the irrational case. The first term above will be of interest if $2(k-j)\nu$ is an integer, while the second needs to be considered if $(2k-2j-1)\nu-\frac12$ is an integer. The worst case is $j=k-1$. Then we can have $m(2k-2)$ logarithms in the second term above, while $2k$ more logarithms are produced by the $2k$ applications of the lemma. Hence we need the relation

$$m(j)\ge m(j-1)+j+1$$

which is verified e.g. by $m(j)=j^2$ (we pick $n_j=2j^2$ because of $j=1$, see above).
We cannot use w_{2k} for v_{2k} due to the singularity of $\log R$ at $R=0$. However, we define instead

$$v_{2k}:=\frac{1}{(t\lambda)^{2k-1}}\sum_{j=0}^{2k-1}W^j_{2k}(a)\Big(\frac12\log(1+R^2)\Big)^j$$

In doing this we add an additional component to the error. This is large near $R=0$, but this is not so important since the aim of this correction is to improve the error for large R. Since $a^3=R^3/(t\lambda)^3$, pulling a cubic factor a^3 out of the W's it is easy to see that (2.11) holds.

Step 4: For v_{2k} defined as above we show that the corresponding error e_{2k} satisfies (2.12). We can write e_{2k} in the form

$$t^2e_{2k}=t^2\left(e_{2k-1}-e^0_{2k-1}\right)+t^2\left(e^0_{2k-1}-\left(-\partial_t^2+\partial_r^2+\frac1r\partial_r-\frac{1}{r^2}\right)v_{2k}\right)+t^2N_{2k}(v_{2k})$$

where N_{2k} is defined by (2.7) and

$$e^0_{2k-1}=\frac{R}{(t\lambda)^{2k}}\sum_{j=0}^{2k-1}q_j(a)\Big(\frac12\log(1+R^2)\Big)^j$$

We begin with the first term in e_{2k}, which has the form

$$t^2(e_{2k-1}-e^0_{2k-1})\in\frac{1}{(t\lambda)^{2k}}\left[IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}'_{k-1})+bIS^1(R(\log R)^{2k-1},\mathcal{Q}'_{k-1})\right]$$

The second term is contained in the second term of (2.12). It remains to show that

$$IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}'_{k-1})\subset IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}_{k-1})+bIS^1(R(\log R)^{2k-1},\mathcal{Q}'_{k-1}) \tag{2.32}$$

For $w\in IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}'_{k-1})$ we write

$$w=(1-a^2)w+\frac{1}{(t\lambda)^2}R^2w$$

Then

$$(1-a^2)w\in IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}_{k-1}),\qquad \frac{1}{(t\lambda)^2}R^2w\in bIS^1(R(\log R)^{2k-1},\mathcal{Q}'_{k-1})$$

as desired. The second term in e_{2k} would equal 0 if we were to replace $\frac12\log(1+R^2)$ by $\log R$ in both e^0_{2k} and v_{2k}. Hence the difference is obtained when we replace the derivatives of $\frac12\log(1+R^2)$ by derivatives of $\log R$ in the expression

$$t^2\left(-\partial_t^2+\partial_r^2+\frac1r\partial_r\right)v_{2k}=t^2\left(-\partial_t^2+\partial_r^2+\frac1r\partial_r\right)\left(\frac{1}{(t\lambda)^{2k-1}}\sum_{j=0}^{2k-1}W^j_{2k}(a)\Big(\frac12\log(1+R^2)\Big)^j\right)$$

Computing these differences one finds that the second term in e_{2k} is a sum of expressions of the form

$$\frac{1}{(t\lambda)^{2k-1}}\sum_{j=0}^{2k-1}\frac{W_{2k}^j(a)}{a^2}\Big[S(R^{-2})(\log(1+R^2))^{j-1}+S(R^{-2})(\log(1+R^2))^{j-2}\Big]+\frac{\partial_a W_{2k}^j(a)}{a}S(R^{-2})(\log(1+R^2))^{j-1}$$

Since W_{2k}^j are cubic at 0 it follows that we can pull out an a factor and see that this part of the error is in

$$\frac{1}{(t\lambda)^{2k}}IS^1(R^{-1}(\log R)^{2k-2},\mathcal{Q}_k')$$

which is admissible by (2.32).

Finally, we consider the nonlinear terms in N_{2k}. Again the a,b dependence is uninteresting since $\mathcal{Q}_k$ is an algebra. We shall use that

$$u_{2k-1}-u_0\in\frac{1}{(t\lambda)^2}IS^3(R\log R,\mathcal{Q}_k)$$

By Lemma 2.8, for the linear term we therefore have

$$\begin{aligned}
&t^2\frac{1-\cos(2u_{2k-1})}{r^2}v_{2k}\\
\in\ &\frac{(t\lambda)^2}{R^2}\left(IS^1(R^{-1},\mathcal{Q}_k)+\frac{1}{(t\lambda)^2}IS^3(R\log R,\mathcal{Q}_k)\right)^2\frac{1}{(t\lambda)^{2k+2}}IS^3(R^3(\log R)^{2k-1},\mathcal{Q}_k)\\
\subset\ &\frac{1}{(t\lambda)^{2k}}\left(IS^3(R^{-1}(\log R)^{2k-1},\mathcal{Q}_k)+\frac{1}{(t\lambda)^2}IS^5(R(\log R)^{2k},\mathcal{Q}_k)+\frac{1}{(t\lambda)^4}IS^7(R^3(\log R)^{2k+1},\mathcal{Q}_k)\right)\\
\subset\ &\frac{1}{(t\lambda)^{2k}}\left(IS^3(R^{-1}(\log R)^{2k-1},\mathcal{Q}_k)+\frac{b}{(t\lambda)^2}IS^5(R(\log R)^{2k-1},\mathcal{Q}_k)\right)
\end{aligned}$$

For the quadratic term we obtain

$$\begin{aligned}
&t^2\frac{\sin(2u_{2k-1})}{2r^2}(1-\cos(2v_{2k}))\\
\in\ &\frac{(t\lambda)^2}{R^2}\left(IS^1(R^{-1},\mathcal{Q}_k)+\frac{1}{(t\lambda)^2}IS^3(R\log R,\mathcal{Q}_k)\right)\left(\frac{1}{(t\lambda)^{2k+2}}IS^3(R^3(\log R)^{2k-1}),\mathcal{Q}_k)\right)^2\\
\subset\ &\frac{1}{(t\lambda)^{2k}}\left(\frac{1}{(t\lambda)^{2k+2}}IS^5(R^3(\log R)^{4k-2},\mathcal{Q}_k)+\frac{1}{(t\lambda)^{2k+4}}IS^7(R^5(\log R)^{4k-1},\mathcal{Q}_k)\right)\\
\subset\ &\frac{1}{(t\lambda)^{2k}}\left(IS^1(R^{-1}(\log R)^{2k},\mathcal{Q}_k)+bIS^3(R(\log R)^{2k-1},\mathcal{Q}_k)\right)
\end{aligned}$$

Finally, the cubic term is

$$\begin{aligned}
t^2\frac{\cos(2u_{2k-1})}{r^2}(2v_{2k}-\sin(2v_{2k}))\ &\in\ \frac{(t\lambda)^2}{R^2}\left(\frac{1}{(t\lambda)^{2k+2}}IS^3(R^3(\log R)^{2k-1}),\mathcal{Q}_k)\right)^3\\
&\subset\ \frac{1}{(t\lambda)^{2k}}\frac{1}{(t\lambda)^{4k+4}}IS^7(R^7(\log R)^{6k-3},\mathcal{Q}_k))\\
&\subset\ \frac{a^6b^{4k-2}}{(t\lambda)^{2k}}IS^1(R(\log R)^{2k-1},\mathcal{Q}_k))\\
&\subset\ \frac{b}{(t\lambda)^{2k}}IS^1(R(\log R)^{2k-1},\mathcal{Q}_k'))
\end{aligned}$$

This concludes the proof of Theorem 2.1. □

3. The perturbed equation

We now need to complement the approximate solution found in the first section to an actual solution. The mechanism for achieving this will rely on the construction of an approximate parametrix for a suitable wave-type equation. We now set about deriving this equation: we seek an exact solution of the form

$$u(t,r)=u_{2k-1}(t,r)+\varepsilon(t,r)$$

where u_{2k-1} is as in the previous section and ε will be obtained by means of an iteration procedure. To motivate this procedure, note that we need to solve the following equation, see (2.1),

$$-\varepsilon_{tt}+\varepsilon_{rr}+\frac{1}{r}\varepsilon_r-\frac{\cos(2Q(\lambda r))}{r^2}\varepsilon=N_{2k-1}(\varepsilon)+e_{2k-1} \tag{3.1}$$

where N_{2k-1} is defined in (2.6) but with u_{2k-2} replaced by u_{2k-1}.

In order to remove the time dependence of the potential in (3.1), we now introduce new coordinates: first, the new time is to satisfy the relation

$$\frac{\partial}{\partial\tau}=\frac{1}{\lambda(t)}\frac{\partial}{\partial t}$$

Specifically, we may put $\tau=-\int_t^1\lambda(s)ds+\frac{1}{\nu}=\frac{1}{\nu}t^{-\nu}$. Thus, the singularity now corresponds to $\tau=\infty$. Next, introduce the new dependent variable $v(\tau,R):=\varepsilon(t(\tau),\lambda^{-1}R)$, where we now understand λ as a function of τ. Then we have

$$\frac{\partial}{\partial\tau}v=t'(\tau)\varepsilon_t(t(\tau),\lambda^{-1}R)-\frac{\lambda_\tau}{\lambda^2}R\varepsilon_r(t(\tau),\lambda^{-1}R),\ \frac{\partial}{\partial R}v=\lambda^{-1}\varepsilon_r(t(\tau),\lambda^{-1}R)$$

This entails that

$$(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)v=\lambda^{-1}\varepsilon_t(t(\tau),\lambda^{-1}R)$$

From here we get

$$(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)^2v=(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)[\lambda^{-1}\varepsilon_t]=\lambda^{-2}\varepsilon_{tt}-\frac{\lambda_\tau}{\lambda^2}\varepsilon_t=\lambda^{-2}\varepsilon_{tt}-\frac{\lambda_\tau}{\lambda}\partial_\tau v-[\frac{\lambda_\tau}{\lambda}]^2R\partial_R v$$

We conclude that we may recast the wave equation (3.1) in the following way:

$$-[(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)^2+\frac{\lambda_\tau}{\lambda}(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)]v+\Big(\partial_R^2+\frac{1}{R}\partial_R-\frac{\cos[2Q(R)]}{R^2}\Big)v=\frac{1}{\lambda^2}[N_{2k-1}(\varepsilon)+e_{2k-1}](t(\tau),\lambda^{-1}R)$$

In order to turn the above second order elliptic operator in R into a selfadjoint operator we introduce the new variable $\tilde\varepsilon(\tau,R):=R^{\frac12}v(\tau,R)$. This leads to

$$(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)\tilde\varepsilon=R^{\frac12}(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)v+\frac12R^{\frac12}\frac{\lambda_\tau}{\lambda}v(\tau,R)$$

$$(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)^2\tilde\varepsilon=R^{\frac12}(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)^2v+R^{\frac12}\frac{\lambda_\tau}{\lambda}(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)v+\frac12R^{\frac12}\partial_\tau(\frac{\lambda_\tau}{\lambda})v+\frac14R^{\frac12}(\frac{\lambda_\tau}{\lambda})^2v$$

One checks that

$$R^{\frac12}(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)^2v+R^{\frac12}\frac{\lambda_\tau}{\lambda}(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)v=(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)^2\tilde\varepsilon-\frac14(\frac{\lambda_\tau}{\lambda})^2\tilde\varepsilon-\frac12\partial_\tau(\frac{\lambda_\tau}{\lambda})\tilde\varepsilon$$

as well as

$$R^{\frac12}\Big(\partial_R^2+\frac{1}{R}\partial_R-\frac{\cos[2Q(R)]}{R^2}\Big)v=(\partial_R^2-\frac{3}{4R^2}+\frac{8}{(1+R^2)^2})\tilde\varepsilon$$

Combining these observations with (3.1), we now obtain the wave equation

$$\Big(-(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)^2+\frac14(\frac{\lambda_\tau}{\lambda})^2+\frac12\partial_\tau(\frac{\lambda_\tau}{\lambda})\Big)\tilde\varepsilon-\mathcal{L}\tilde\varepsilon=\lambda^{-2}R^{\frac12}\Big(N_{2k-1}(R^{-\frac12}\tilde\varepsilon)+e_{2k-1}\Big) \tag{3.2}$$

where

$$\mathcal{L}:=-\partial_R^2+\frac{3}{4R^2}-\frac{8}{(1+R^2)^2} \tag{3.3}$$

Equation (3.2) is the main equation which we need to solve in this paper. As a first step, in the following section we will carefully analyze the spectral properties of the underlying linear operator $\mathcal{L}$.

4. The analysis of the underlying strongly singular Sturm-Liouville operator

The goal of this section is to develop the scattering theory of $\mathcal{L}$ from (3.3). We start with the basic[6]

Definition 4.1. *Let*

$$\mathcal{L}_0 := -\frac{d^2}{dr^2} + \frac{3}{4r^2}, \qquad \mathcal{L} := \mathcal{L}_0 - \frac{8}{(1+r^2)^2} =: \mathcal{L}_0 + V$$

be half-line operators on $L^2(0,\infty)$. They are self-adjoint with the same domain, namely

$$\mathrm{Dom}(\mathcal{L}) = \mathrm{Dom}(\mathcal{L}_0) = \{f \in L^2((0,\infty)) \ : \ f, f' \in AC_{\mathrm{loc}}((0,\infty)),\ \mathcal{L}_0 f \in L^2((0,\infty))\}$$

It is important to realize that because of the strong singularity of the potential at $r = 0$ no boundary condition is needed there to insure self-adjointness. Technically speaking, this means that $\mathcal{L}_0$ and $\mathcal{L}$ are in the *limit point case* at $r = 0$, see Gesztesy, Zinchenko [GesZin]. It is worth noting that the potential $\frac{3}{4r^2}$ is critical with respect to this property — any number smaller than $\frac{3}{4}$ leads to an operator which is in the limit circle case at $r = 0$. We remark that $\mathcal{L}_0$ and $\mathcal{L}$ are in the limit point case at $r = \infty$ by a standard criterion (sub-quadratic growth of the potential).

Lemma 4.2. *The spectrum of $\mathcal{L}$ is purely absolutely continuous and equals* $\mathrm{spec}(\mathcal{L}) = [0,\infty)$.

Proof. That $\mathcal{L}$ has no negative spectrum follows from

$$\mathcal{L}\phi_0 = 0, \qquad \phi_0(r) = \frac{r^{3/2}}{1+r^2} \tag{4.1}$$

with ϕ_0 positive (by the Sturm oscillation theorem, see [DS]). And since $\phi_0 \notin L^2((0,\infty))$, zero is not an eigenvalue. The pure absolute continuity of the spectrum of $\mathcal{L}$ is an immediate consequence of the fact that the potential of $\mathcal{L}$ is integrable at infinity. □

Since $\phi_0 \notin L^2((0,\infty))$, one refers to zero energy as a *resonance*. Heuristically speaking, this notion can be thought of as follows: by inspection, $\mathcal{L}_0 r^{-\frac{1}{2}} = 0$ and $\mathcal{L}_0 r^{\frac{3}{2}} = 0$. A "generic" perturbation $\tilde{\mathcal{L}} = \mathcal{L}_0 + \tilde{V}$ with $\tilde{V}$ bounded, smooth, and nicely decaying, will have zero energy solutions that behave just like $r^{-\frac{1}{2}}$ and $r^{\frac{3}{2}}$, respectively. However, in some cases $\tilde{V}$ is such that these two $\mathcal{L}_0$ solutions will be "in resonance" and produce a globally bounded zero energy solution of $\tilde{\mathcal{L}}$ which behaves like $r^{\frac{3}{2}}$ around zero and $r^{-\frac{1}{2}}$ around infinity — just like ϕ_0.

For the parametrix construction in the following sections the relevance of the zero energy resonance lies with the singularity of the spectral measure of $\mathcal{L}$ at zero energy. Indeed, for $\mathcal{L}_0$ the density of the spectral measure behaves like ξ as $\xi \to 0$, whereas for $\mathcal{L}$ we will show that it behaves like $(\xi \log^2 \xi)^{-1}$ as $\xi \to 0$. We now briefly summarize the results from [GesZin] relevant for our purposes, see Section 3 in their paper, in particular Example 3.10.

Theorem 4.3. *a) For each $z \in \mathbb{C}$ there exists a fundamental system $\phi(r,z)$, $\theta(r,z)$ for $\mathcal{L} - z$ which is analytic in z for each $r > 0$ and has the asymptotic behavior*

$$\phi(r,z) \sim r^{\frac{3}{2}}, \qquad \theta(r,z) \sim \frac{1}{2} r^{-\frac{1}{2}} \quad \text{as} \quad r \to 0 \tag{4.2}$$

In particular, their Wronskian is $W(\theta(\cdot,z),\phi(\cdot,z)) = 1$ for all $z \in \mathbb{C}$. We remark that $\phi(\cdot,z)$ is the Weyl-Titchmarsh solution[7] of $\mathcal{L} - z$ at $r = 0$. By convention, $\phi(\cdot,z), \theta(\cdot,z)$ are real-valued for $z \in \mathbb{R}$.

b) For each $z \in \mathbb{C}$, $\mathrm{Im}\, z > 0$, let $\psi^+(r,z)$ denote the Weyl-Titchmarsh solution of $\mathcal{L} - z$ at $r = \infty$ normalized so that

$$\psi^+(r,z) \sim z^{-\frac{1}{4}}\, e^{iz^{\frac{1}{2}} r} \quad \text{as} \quad r \to \infty,\ \mathrm{Im}\, z^{\frac{1}{2}} > 0$$

---

[6] In this section we use the variable $r > 0$ for the independent variable. The reader should note that this now plays the role of R in the previous section.

[7] Our $\phi(\cdot,z)$ is the $\tilde{\phi}(z,\cdot)$ function from [GesZin] where the analyticity is only required in a strip around $\mathbb{R}$ – but here there is no need for this restriction.

If $\xi > 0$, then the limit $\psi^+(r, \xi + i0)$ exists point-wise for all $r > 0$ and we denote it by $\psi^+(r, \xi)$. Moreover, define $\psi^-(\cdot, \xi) := \overline{\psi^+(\cdot, \xi)}$. Then $\psi^+(r, \xi)$, $\psi^-(r, \xi)$ form a fundamental system of $\mathcal{L} - \xi$ with asymptotic behavior $\psi^\pm(r, \xi) \sim \xi^{-\frac14}\, e^{\pm i\xi^{\frac12} r}$ as $r \to \infty$.

c) The spectral measure of $\mathcal{L}$ is absolutely continuous and its density is given by

$$\rho(\xi) = \frac{1}{\pi} \operatorname{Im}\, m(\xi + i0)\, \chi_{[\xi > 0]} \tag{4.3}$$

with the "generalized Weyl-Titchmarsh" function

$$m(\xi) = \frac{W(\theta(.,\xi),\, \psi^+(.,\xi))}{W(\psi^+(.,\xi), \phi(.,\xi))} \tag{4.4}$$

d) The distorted Fourier transform defined as

$$\mathcal{F} : f \longrightarrow \widehat{f}(\xi) = \lim_{b\to\infty} \int_0^b \phi(r, \xi) f(r)\, dr$$

is a unitary operator from $L^2(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+, \rho)$ and its inverse is given by

$$\mathcal{F}^{-1} : \widehat{f} \longrightarrow f(r) = \lim_{\mu\to\infty} \int_0^\mu \phi(r, \xi) \widehat{f}(\xi)\, \rho(\xi)\, d\xi$$

Here lim *refers to the $L^2(\mathbb{R}^+, \rho)$, respectively the $L^2(\mathbb{R}^+)$, limit.*

Needless to say, part b) above has nothing to do with [GesZin] and is standard. Most relevant for our computations are (4.4) (which is formula (3.22) in [GesZin]), as well as the Fourier inversion theorem in this context (see Theorem 3.5 in [GesZin]).

Theorem 4.3 of course also holds for $\mathcal{L}_0$ instead of $\mathcal{L}$. In that case we have a Bessel equation with solutions

$$\begin{aligned} \phi(r; z) &= 2z^{-1/2} r^{1/2} J_1(z^{1/2} r) \\ \theta(r; z) &= \frac{\pi}{4} z^{1/2} r^{1/2} [-Y_1(z^{1/2} r) + \pi^{-1} \log(z) J_1(z^{1/2} r)] \\ \psi(r; z) &= z^{1/2} r^{1/2} [-Y_1(z^{1/2} r) + i J_1(z^{1/2} r)] = z^{1/2} r^{1/2} i H_1^{(1)}(z^{1/2} r) \\ &= \theta(r; z) + m(z)\phi(r; z) \\ m(z) &= \frac{\pi}{4} z[i - \pi^{-1} \log(z)], \quad z \in \mathbb{C} \setminus \mathbb{R}^+ \end{aligned} \tag{4.5}$$

The last formula shows that for strongly singular potentials the Weyl-Titchmarsh function ceases to be Herglotz, see [GesZin] for further discussion. Although we shall make no use of these formulas for $\mathcal{L}_0$, the reader should note the similarities between the asymptotic expansions on ϕ, θ and ψ^+ we derive below and the classical ones for the Bessel functions, cf. [Wat].

4.1. Asymptotic behavior of ϕ and θ.

Beginning with two explicit solutions for $\mathcal{L} f = 0$, namely

$$\phi_0(r) = \frac{r^{\frac32}}{1 + r^2}, \qquad \theta_0(r) = \frac{1 - 4r^2 \log r - r^4}{2r^{\frac12}(1 + r^2)} = r^{-\frac12}(1 - r^2)/2 - 2\phi_0(r) \log r$$

we shall construct power series expansions for ϕ and θ from (4.2) in $z \in \mathbb{C}$ when $r > 0$ is fixed.

Proposition 4.4. *For any $z \in \mathbb{C}$ the fundamental system $\phi(r, z)$, $\theta(r, z)$ from Theorem 4.3 admits absolutely convergent asymptotic expansions*

$$\begin{aligned} \phi(r, z) &= \phi_0(r) + r^{-\frac12} \sum_{j=1}^\infty (r^2 z)^j \phi_j(r^2) \\ \theta(r, z) &= r^{-\frac12} \Big(1 - r^2 - \sum_{j=1}^\infty (r^2 z)^j \theta_j(r^2)\Big)/2 - (2 + z/4)\phi(r, z) \log r \end{aligned}$$

where the functions ϕ_j, θ_j are holomorphic in $U = \{\,\mathrm{Re}\, u > -\frac{1}{2}\}$ and satisfy the bounds

$$|\phi_j(u)| \leq \frac{3C^j}{(j-1)!}\log(1+|u|), \quad |\phi_1(u)| > \frac{1}{2}\log u \quad \text{if} \quad u \gg 1$$

$$|\theta_1(u)| \leq C|u|, \qquad |\theta_j(u)| \leq \frac{C^j}{(j-1)!}(1+|u|), \qquad u \in U$$

Furthermore,

$$\phi_1(u) = \begin{cases} -\frac{1}{4}\log u + \frac{1}{2} + O(u^{-1}\log^2 u) & \text{as} \quad u \to \infty \\ -\frac{u}{8} + \frac{u^2}{12} + O(u^3) & \text{as} \quad u \to 0 \end{cases} \tag{4.6}$$

Proof. We begin with ϕ. We formally write

$$\phi(r,z) = r^{-\frac{1}{2}} \sum_{j=0}^{\infty} z^j f_j(r)$$

This becomes rigorous once we verify the convergence of the series in any reasonable sense. The functions f_j should solve

$$\mathcal{L}(r^{-\frac{1}{2}} f_j) = r^{-\frac{1}{2}} f_{j-1}, \qquad f_{-1} = 0$$

The forward fundamental solution for $\mathcal{L}$ is

$$H(r,s) = \frac{1}{2}(\phi_0(r)\theta_0(s) - \phi_0(s)\theta_0(r))1_{[r>s]}$$

Hence we have the iterative relation

$$f_j(r) = \frac{1}{2}\int_0^s r^{\frac{1}{2}} s^{-\frac{1}{2}} (\phi_0(r)\theta_0(s) - \phi_0(s)\theta_0(r)) f_{j-1}(s)\, ds, \qquad f_0(r) = \frac{r^2}{1+r^2}$$

Using the expressions for ϕ_0, θ_0 we rewrite this as

$$f_j(r) = \int_0^r \frac{r^2(-1+4s^2\log s + s^4) - s^2(-1+4r^2\log r + r^4)}{2s(1+r^2)(1+s^2)} f_{j-1}(s) ds$$

We claim that all functions f_j extend to even holomorphic functions in any even simply connected domain not containing $\pm i$, vanishing at 0. Indeed, we now suppose that f_{j-1} has these properties and we shall prove them for f_j. Clearly, f_j extends to a holomorphic function in any even simply connected domain not containing $\pm i$ and 0. We first show that at 0 there is at most an isolated singularity. For this we consider a branch of the logarithm which is holomorphic in $\mathbb{C} \setminus \mathbb{R}^+$ and show that $f_j(r+i0) = f_j(r-i0)$ for $r < 0$. Disregarding the terms not involving logarithms, we need to show that for any holomorphic function g we have

$$\int_0^{r+i0} (\log s - \log(r+i0)) g(s) ds = \int_0^{r-i0} (\log s - \log(r-i0)) g(s) ds$$

This is obvious since for $s < 0$ we have

$$\log(s+i0) - \log(r+i0) = \log(s-i0) - \log(r-i0)$$

The singularity at 0 is a removable singularity. Indeed, for s close to 0 we have $|f_{j-1}(s)| \lesssim |s|$ which by a crude bound on the denominator in the above integral leads to $|f_j(r)| \lesssim |r|$ (again with r close to 0). This also shows that f_j vanishes at 0.

The fact that f_j is even is obvious if we substitute $2\log s$ and $2\log r$ by $\log s^2$ respectively $\log s^2$ in the integral. This is allowed since due to the above discussion we can use any branch of the logarithm. Indeed, denoting $\tilde{f}_{j-1}(s^2) = f_{j-1}(s)$ the change of variable $s^2 = v$ yields the iterative relation

$$\tilde{f}_j(u) = \int_0^u \frac{u(-1+2v\log v + v^2) - v(-1+2u\log u + u^2)}{4v(1+u)(1+v)} \tilde{f}_{j-1}(v) dv, \qquad \tilde{f}_0(u) = \frac{u}{1+u} \tag{4.7}$$

Next, we obtain bounds on the functions $\tilde f_j$. To avoid the singularity at -1 we restrict ourselves to the region $U = \{\,\mathrm{Re}\, u > -\frac12\}$. We claim that the $\tilde f_j$ satisfy the bound

$$|\tilde f_j(u)| \le \frac{3C^j}{(j-1)!}|u|^j \log(1+|u|)$$

The kernel above can be estimated by

$$\left| \frac{u(-1+2v\log v + v^2) - v(-1+2u\log u + u^2)}{2v(1+u)(1+v)} \right| \le C\frac{|u|}{|v|}$$

We have

$$|\tilde f_0(u)| \le 3\frac{|u|}{1+|u|}$$

which yields

$$|\tilde f_1(u)| \le 3C|u| \int_0^{|u|} \frac{1}{1+x}dx = 3C|u|\log(1+|u|)$$

From here on we use induction, noting that

$$\int_0^{|u|} x^{j-1}\log(1+x)dx \le \frac1j |u|^j \log(1+|u|)$$

To conclude the proof, we note that the functions ϕ_j are given by $\phi_j(u) = u^{-j}\tilde f_j(u)$ and satisfy the desired pointwise bound. Finally, (4.6) follows by an asymptotic evaluation of the explicit integral (4.7) with $j=1$, which we leave to the reader.

The argument for the function θ is similar. The ansatz

$$\begin{aligned}\theta(r,z) &= r^{-\frac12}\Big(1 - r^2 - \sum_{j=1}^\infty z^j g_j(r)\Big)/2 - (2+z/4)\phi(r,z)\log r \\ &= r^{-\frac12}\Big(1 - r^2 - \sum_{j=1}^\infty z^j g_j(r)\Big)/2 + (2+z/4)\big(\phi_0(r) - \sum_{j=1}^\infty z^j r^{-\frac12} f_j(r)\big)\log r\end{aligned}$$

yields a recurrence relation for the g_j via $(\mathcal{L}-z)\theta = 0$. Indeed, for $j=1$,

$$\begin{aligned}\mathcal{L}(r^{-\frac12} g_1(r)) &= \theta_0(r) - \mathcal{L}\big(\frac12\phi_0(r)\log r + 4r^{-\frac12} f_1(r)\log r\big) \\ &= r^{-\frac12}\Big[r^2 - \frac{r^2(3+r^2)}{(1+r^2)^2} - \frac{8}{r^2}f_1(r) + \frac{8}{r}f_1'(r)\Big]\end{aligned}$$

where the important fact is that the quantity in brackets is even analytic around 0 and vanishes at 0. A similar computation yields for $j\ge 2$

$$\mathcal{L}(r^{-\frac12} g_j(r)) = r^{-\frac12}\Big[g_{j-1}(r) - r^{-2}f_{j-1}(r) + r^{-1}f_{j-1}'(r) - 8r^{-2}f_j(r) + 8r^{-1}f_j'(r)\Big]$$

The same considerations as in the case of f_j show that each g_j is an even holomorphic function in any even simply connected domain not containing $\pm i$. Also, the same bound for the fundamental solution for $\mathcal{L}$ leads to $|g_1(r)| \le Cr^4$ and more generally, for $j \ge 2$,

$$|g_j(r)| \le \frac{C^j}{(j-1)!} r^{2j}(1+r^2)$$

The proof of the proposition is concluded. □

Remark 4.5. The logarithmic behavior of $\phi_1(u)$ for large u is inherited by $\phi(r,\xi)$; indeed, suppose that $1 \gg \xi > 0$ and $r = \delta\xi^{-\frac12}$ where $\delta > 0$ is small. Then the proposition shows that

$$\phi(r,\xi) \gtrsim r^{-\frac12}\log r$$

The size of δ here only depends on various constants in the expansion of ϕ and is thus itself an absolute constant. We remark that the appearance of the $\log r$ term is a specific feature of $\mathcal{L}$ — it does not occur for

$\mathcal{L}_0$, see (4.5) — indicative of the fact that $\mathcal{L}$ is a *long range* perturbation of $\mathcal{L}_0$. We shall see later that the logarithm in ϕ produces crucial logarithmic factors in the small ξ asymptotics of the spectral density of $\mathcal{L}$, see Proposition 4.7 below.

We note that although the above series for ϕ converges for all r, z, we can only use it to obtain various estimates for ϕ in the region $|z|r^2 \lesssim 1$. On the other hand, in the region $\xi r^2 \gtrsim 1$ where $z = \xi > 0$, we will represent ϕ in terms of ψ^+ and use the ψ^+ asymptotic expansion, described in what follows.

4.2. **The asymptotic behavior of ψ^+.** The following result provides good asymptotics for ψ^+ in the region $r^2\xi \gtrsim 1$.

Proposition 4.6. *For any $\xi > 0$, the solution $\psi^+(\cdot, \xi)$ from Theorem 4.3 is of the form*

$$\psi^+(r, \xi) = \xi^{-\frac{1}{4}} e^{ir\xi^{\frac{1}{2}}} \sigma(r\xi^{\frac{1}{2}}, r), \qquad r^2\xi \gtrsim 1$$

where σ admits the asymptotic series approximation

$$\sigma(q, r) \approx \sum_{j=0}^{\infty} q^{-j} \psi_j^+(r), \qquad \psi_0^+ = 1, \qquad \psi_1^+ = \frac{3i}{8} + O(\frac{1}{1+r^2})$$

with zero order symbols $\psi_j^+(r)$ that are analytic at infinity,

$$\sup_{r>0} |(r\partial_r)^k \psi_j^+(r)| < \infty$$

in the sense that for all large integers j_0, and all indices α, β, we have

$$\sup_{r>0} \Big| (r\partial_r)^\alpha (q\partial_q)^\beta \Big[\sigma(q, r) - \sum_{j=0}^{j_0} q^{-j} \psi_j^+(r) \Big] \Big| \leq c_{\alpha,\beta,j_0} q^{-j_0-1}$$

for all $q > 1$.

Proof. With the notation

$$\sigma(q, r) = \xi^{\frac{1}{4}} \psi^+(r, \xi) e^{-ir\xi^{\frac{1}{2}}}$$

we need to solve the conjugated equation

$$\left(-\partial_r^2 - 2i\xi^{\frac{1}{2}}\partial_r + \frac{3}{4r^2} - \frac{8}{(1+r^2)^2} \right) \sigma(r\xi^{\frac{1}{2}}, r) = 0 \tag{4.8}$$

We look for a formal power series solving this equation,

$$\sum_{j=0}^{\infty} \xi^{-\frac{j}{2}} f_j(r) \tag{4.9}$$

This yields a recurrence relation for the f_j's,

$$2i\partial_r f_j = \left(-\partial_r^2 + \frac{3}{4r^2} - \frac{8}{(1+r^2)^2} \right) f_{j-1}, \qquad f_0 = 1$$

which is solved by

$$f_j = \frac{i}{2}\partial_r f_{j-1} + \frac{i}{2} \int_r^\infty \left(\frac{3}{4s^2} - \frac{8}{(1+s^2)^2} \right) f_{j-1}(s)\, ds$$

Extending this into the complex domain, it is easy to see that the functions f_j are holomorphic in $\mathbb{C} \setminus [-i, i]$. They are also holomorphic at ∞, and the leading term in the Taylor series at ∞ is r^{-j}. At 0, on the other hand, f_j are singular. The worst singularity is of power type, namely r^{-j}; however, weaker terms contain logarithms and powers of logarithms so it is not easy to obtain a complete expansion. Instead we contend ourselves with a weaker estimate, namely

$$|(r\partial_r)^k f_j| \leq c_{jk}\, r^{-j} \qquad \forall r > 0$$

which is easy to establish inductively. The functions

$$\psi_j^+(r) := r^j f_j(r)$$

now satisfy the desired bounds due to the bounds above on f_j.

Unlike in the expansion for small r, here we make no effort to obtain a uniform estimate on the size of the derivatives of ψ_j^+. This is because we do not expect the formal series (4.9) to converge, on account of the fact that derivatives are lost in the iterative construction of the f_j's. Instead we can construct an approximate sum, i.e., a function $\sigma_{ap}(q,r)$ with the property that for each $j_0 \geq 0$ we have

$$\Big|(r\partial_r)^\alpha (q\partial_q)^\beta \big[\sigma_{ap}(q,r) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r)\big]\Big| \leq c_{\alpha,\beta,j_0}\, q^{-j_0-1} \tag{4.10}$$

The construction of $\sigma_{ap}(q,r)$ is standard in symbol calculus; indeed, we can set

$$\sigma_{ap}(q,r) := \sum_{j=0}^{\infty} q^{-j}\psi_j^+(r)\chi(q\delta_j)$$

where $\delta_j \to 0$ sufficiently fast and χ is a cut-off function which vanishes around zero and is equal to one for large arguments. The bound (4.10) implies that $\sigma_{ap}(r\xi^{\frac12}, r)$ is a good approximate solution for (4.8) at infinity, namely the error

$$e(r\xi^{\frac12}, r) = \left(-\partial_r^2 - 2i\xi^{\frac12}\partial_r + \frac{3}{4r^2} - \frac{8}{(1+r^2)^2}\right)\sigma_{ap}(r,\xi)$$

satisfies for all indices α, β, j

$$|(r\partial_r)^\alpha (q\partial_q)^\beta e(q,r)| \leq c_{\alpha,\beta,j}\, r^{-2} q^{-j}$$

To conclude the proof it remains to solve the equation for the difference $\sigma_1 = -\sigma + \sigma_{ap}$,

$$\left(-\partial_r^2 - 2i\xi^{\frac12}\partial_r + \frac{3}{4r^2} - \frac{8}{(1+r^2)^2}\right)\sigma_1(r\xi^{\frac12}, r) = e(r\xi^{\frac12}, r)$$

with zero Cauchy data at infinity. We claim that the solution σ_1 satisfies

$$|(r\partial_r)^\alpha (q\partial_q)^\beta \sigma_1(q,r)| \leq c_{\alpha,\beta,j}\, q^{-j}, \qquad j \geq 2$$

Note that this finishes the proof by defining $\sigma = \sigma_{ap} - \sigma_1$. A change of variable allows us to switch from the pair of operators $(r\partial_r, q\partial_q)$ to $(r\partial_r, \xi\partial_\xi)$ with comparable bounds. We rewrite the above equation as a first order system for $(v_1, v_2) = (\sigma_1, r\partial_r\sigma_1)$:

$$\partial_r \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 0 & r^{-1} \\ \frac{3}{4r} - \frac{8r}{(1+r^2)^2} & r^{-1} - 2i\xi^{\frac12} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ re \end{pmatrix}$$

Then we have

$$\frac{d}{dr}|v|^2 \gtrsim -r^{-1}|v|^2 - r|v||e|$$

which gives

$$\frac{d}{dr}|v| \geq -C(r^{-1}|v| + r|e|)$$

and by Gronwall

$$|v(r)| \leq \int_r^\infty \left(\frac{s}{r}\right)^C s|e(s)|ds$$

Then for large j we have

$$|e| \lesssim \xi^{-\frac{j}{2}} r^{-j-2} \implies |v| \lesssim \xi^{-\frac{j}{2}} r^{-j} = q^{-j} \tag{4.11}$$

To estimate derivatives of v we commute them with the operator. For derivatives with respect to r we have

$$\partial_r(r\partial_r)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 0 & \frac1r \\ \frac{3}{4r} - \frac{8r}{(1+r^2)^2} & \frac1r - 2i\xi^{\frac12} \end{pmatrix} (r\partial_r)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac1r \\ \frac{3}{4r} - \frac{8r(3r^2-1)}{(1+r^2)^3} & \frac1r \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ r\partial_r(re) \end{pmatrix}$$

But the right-hand side is bounded by r^{-j-1} from the previous step and the hypothesis on e, therefore as above $r\partial_r v$ is bounded by r^{-j}.

We argue similarly for the ξ derivatives. We have

$$\partial_r(\xi\partial_\xi)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{r} \\ \frac{3}{4r} - \frac{8r}{(1+r^2)^2} & \frac{1}{r} - 2i\xi^{\frac{1}{2}} \end{pmatrix}(\xi\partial_\xi)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & i\xi^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \xi\partial_\xi(re) \end{pmatrix}$$

The only difference is in the first term on the right, for which we write $\xi^{\frac{1}{2}} = r^{-1}q$ and we use the decay property of v with j replaced by $j+1$:

$$|\xi^{\frac{1}{2}} v_2| \lesssim \xi^{\frac{1}{2}} q^{-j-1} \lesssim r^{-1}q^{-j}, \qquad |\xi\partial_\xi(re)| \lesssim r^{-1}q^{-j}$$

as desired. Finally, higher order derivatives are estimated by induction using the above arguments at each step. □

4.3. **Structure of the spectral measure of $\mathcal{L}$.** We begin by relating the functions ϕ, θ and $\psi^{\pm}$. By examining the asymptotics at $r = 0$ we see that

$$W(\theta, \phi) = 1 \tag{4.12}$$

Also by examining the asymptotics as $r \to \infty$ we obtain

$$W(\psi^+, \psi^-) = -2i \tag{4.13}$$

Hence we can express the $\mathcal{L} - \xi$ solutions in either the ϕ, θ basis or the $\psi^{\pm}$ basis. On the other hand, ϕ, θ are real-valued while the real and imaginary parts of $\psi^{\pm}$ are equally strong. Hence the two bases are quite separated. These are the main ingredients of the next result.

Proposition 4.7. *a) We have*

$$\phi(r, \xi) = a(\xi)\psi^+(r, \xi) + \overline{a(\xi)\psi^+(r, \xi)} \tag{4.14}$$

where a is smooth, always nonzero, and has size[8]

$$|a(\xi)| \asymp \begin{cases} -\xi^{\frac{1}{2}}\log\xi & \xi \ll 1 \\ \xi^{-\frac{1}{2}} & \xi \gtrsim 1 \end{cases}$$

Moreover, it satisfies the symbol type bounds

$$|(\xi\partial_\xi)^k a(\xi)| \le c_k |a(\xi)| \quad \forall\, \xi > 0$$

b) The spectral measure $\rho(\xi)d\xi$ has density

$$\rho(\xi) = \frac{1}{\pi}|a(\xi)|^{-2}$$

and therefore satisfies

$$\rho(\xi) \asymp \begin{cases} \frac{1}{\xi(\log\xi)^2} & \xi \ll 1 \\ \xi & \xi \gtrsim 1 \end{cases}$$

Proof. a) Since ϕ is real-valued, due to (4.13), the relation (4.14) above holds with

$$a(\xi) = -\frac{i}{2}W(\phi(\cdot, \xi), \psi^-(\cdot, \xi))$$

We evaluate the Wronskian in the region where both the $\psi^+(r,\xi)$ and $\phi(r,\xi)$ asymptotics are useful, i.e., where $r^2\xi \approx 1$. By Proposition 4.4 we obtain that both $\phi(\xi^{-\frac{1}{2}}, \xi)$ and $(r\partial_r\phi)(\xi^{-\frac{1}{2}}, \xi)$ can be expressed in the form $\xi^{\frac{1}{4}}f(\xi^{-1})$ with f holomorphic and satisfying

$$|f(u)| \lesssim \log(1 + |u|), \qquad \mathrm{Re}\, u > \frac{1}{4}$$

[8] $a \asymp b$ means that for some constant C one has $C^{-1}a < b < Ca$

On the other hand, it follows from Proposition 4.6 that both $\psi^+(\xi^{-\frac12},\xi)$ and $(r\partial_r\psi^+)(\xi^{-\frac12},\xi)$ can be expressed in the form $\xi^{-\frac14}h(\xi^{-\frac12})$ with h satisfying symbol type bounds

$$|(r\partial_r)^k h(r)| \leq c_k$$

Combining the two expressions above, it follows that a is a sum of terms of the form $\xi^{\frac12}f(\xi^{-1})h(\xi^{-\frac12})$ with f,h as above. The bounds from above on a and its derivatives follow.

It remains to prove the bound from below on a, which is more delicate. By (4.13) we have

$$\mathrm{Im}\,(\psi^+(r,\theta)\partial_r\psi^-(r,\theta)) = -1$$

Since ϕ is real-valued, this gives

$$\mathrm{Im}\left[\partial_r\psi^+(r,\xi)W(\phi(\cdot,\xi),\psi^-(\cdot,\xi))\right] = -\partial_r\phi(r,\xi)$$

which implies that for all r we have

$$|a(\xi)| \geq \frac{|\partial_r\phi(r,\xi)|}{2|\partial_r\psi^+(r,\xi)|}$$

We use this relation for $r=\delta\xi^{-\frac12}$ with a small constant δ. Then by Proposition 4.4 we have

$$|\partial_r\phi(r,\xi)| \gtrsim r^{-\frac32}\log(1+r^2)$$

while by Proposition 4.6

$$|\partial_r\psi^+(r,\xi)| \lesssim \xi^{\frac14}(r^2\xi)^{-j_0}$$

This give the desired bound from below on a.

b) By (4.12) we can express ψ^+ in terms of θ and ϕ by

$$\psi^+ = -\phi W(\psi^+,\theta) + \theta W(\psi^+,\phi)$$

Since both ϕ and θ are real-valued, by inserting this into (4.13) we obtain the relation

$$\mathrm{Im}\,(W(\psi^+,\theta)W(\psi^-,\phi)) = -1$$

Inserting this in the expression for the spectral measure (4.3) and taking (4.4) into account we obtain

$$\rho(\xi) = \frac{1}{\pi}\frac{\mathrm{Im}\,(W(\psi^+,\theta)W(\psi^-,\phi))}{|W(\psi^+,\phi)|^2} = \frac{1}{\pi}|W(\psi^+,\phi)|^{-2} = \frac{1}{\pi|a(\xi)|^2}$$

as desired. □

5. The transference identity

Returning to the radiation part $\tilde{\varepsilon}$ in (3.2), the idea is to expand it in terms of the generalized Fourier basis[9] $\phi(R,\xi)$ associated with the operator $\mathcal{L} = -\partial_R^2 + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2}$, i.e., write

$$\tilde{\varepsilon}(\tau,R) = \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\,d\xi$$

and deduce a transport equation for the Fourier coefficients $x(\tau,\xi)$. The main difficulty in doing this is caused by the operator $R\partial_R$ which is not diagonal in the Fourier basis. Our strategy for dealing with this is to replace it with $2\xi\partial_\xi$ modulo an error which we treat perturbatively. The operator $R\partial_R - 2\xi\partial_\xi$ is natural since it annihilates the expression $e^{i\xi^{\frac12}R}$ arising in the asymptotic expansion of $\phi(R,\xi)$ for large R. Consequently, we define the error operator $\mathcal{K}$ by

$$\widehat{R\partial_R u} = -2\xi\partial_\xi\hat{u} + \mathcal{K}\hat{u} \tag{5.1}$$

where $\hat{f} = \mathcal{F}f$ is the "distorted Fourier transform" from Theorem 4.3. Using the expressions for the direct and inverse Fourier transform in that theorem we obtain

$$\mathcal{K}f(\eta) = \left\langle \int_0^\infty f(\xi)R\partial_R\phi(R,\xi)\rho(\xi)\,d\xi\,,\,\phi(R,\eta)\right\rangle_{L^2_R} + \left\langle \int_0^\infty 2\xi\partial_\xi f(\xi)\phi(R,\xi)\rho(\xi)\,d\xi\,,\,\phi(R,\eta)\right\rangle_{L^2_R}$$

[9]We now return to the variable R as the independent spatial variable instead of r as in the previous section.

Integrating by parts with respect to ξ in the second expression we obtain

$$\mathcal{K}f(\eta) = \Big\langle \int_0^\infty f(\xi)[R\partial_R - 2\xi\partial_\xi]\phi(R,\xi)\rho(\xi)\,d\xi\,,\,\phi(R,\eta)\Big\rangle_{L^2_R} - 2\left(1+\frac{\eta\rho'(\eta)}{\rho(\eta)}\right)f(\eta) \tag{5.2}$$

where the scalar product is to be interpreted in the principal value sense with $f \in C_0^\infty((0,\infty))$. Apriori we have

$$\mathcal{K} : C_0^\infty((0,\infty)) \to C^\infty((0,\infty))$$

therefore we can write

$$\mathcal{K}f(\eta) = \int_0^\infty K(\eta,\xi)f(\xi)\,d\xi$$

for a distribution valued function $\eta \to K(\eta,\xi)$. We refer to (5.1) as the *transference identity* to indicate that we are transferring derivatives from R to ξ. To asses its usefulness we need to understand the boundedness properties of the operator $\mathcal{K}$. We begin with a description of the kernel $K(\eta,\xi)$.

Theorem 5.1. *The operator $\mathcal{K}$ can be written as*

$$\mathcal{K} = -\Big(\frac{3}{2}+\frac{\eta\rho'(\eta)}{\rho(\eta)}\Big)\delta(\xi-\eta) + \mathcal{K}_0 \tag{5.3}$$

where the operator $\mathcal{K}_0$ has a kernel $K_0(\eta,\xi)$ of the form[10]

$$K_0(\eta,\xi) = \frac{\rho(\xi)}{\xi-\eta}F(\xi,\eta) \tag{5.4}$$

with a symmetric function $F(\eta,\xi)$ of class C^2 in $(0,\infty)\times(0,\infty)$ satisfying the bounds

$$|F(\xi,\eta)| \lesssim \begin{cases} \xi+\eta & \xi+\eta\le 1 \\ (\xi+\eta)^{-\frac{3}{2}}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N} & \xi+\eta\ge 1\end{cases}$$

$$|\partial_\xi F(\xi,\eta)| + |\partial_\eta F(\xi,\eta)| \lesssim \begin{cases} 1 & \xi+\eta\le 1 \\ (\xi+\eta)^{-2}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N} & \xi+\eta\ge 1\end{cases}$$

$$\sup_{j+k=2}|\partial_\xi^j\partial_\eta^k F(\xi,\eta)| \lesssim \begin{cases} |\log(\xi+\eta)|^3 & \xi+\eta\le 1 \\ (\xi+\eta)^{-\frac{5}{2}}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N} & \xi+\eta\ge 1\end{cases}$$

where N an arbitrary large integer.

Proof. We first establish the off-diagonal behavior of K, and later return to the issue of identifying the δ-measure that sits on the diagonal. We begin with (5.2) with $f \in C_0^\infty((0,\infty))$. The integral

$$u(R) = \int_0^\infty f(\xi)[R\partial_R - 2\xi\partial_\xi]\phi(R,\xi)\rho(\xi)\,d\xi$$

behaves like $R^{\frac32}$ at 0 and is a Schwartz function at infinity. The second factor $\phi(R,\eta)$ in (5.2) also decays like $R^{\frac32}$ at 0 but at infinity it is only bounded with bounded derivatives. Then the following integration by parts is justified:

$$\eta\mathcal{K}f(\eta) = \Big\langle u, \mathcal{L}\phi(R,\eta)\Big\rangle_{L^2_R} = \Big\langle \mathcal{L}u, \phi(R,\eta)\Big\rangle_{L^2_R}$$

Moreover,

$$\begin{aligned}\mathcal{L}u &= \int_0^\infty f(\xi)[\mathcal{L},R\partial_R]\phi(R,\xi)\rho(\xi)\,d\xi + \int_0^\infty f(\xi)(R\partial_R - 2\xi\partial_\xi)\xi\phi(R,\xi)\rho(\xi)\,d\xi \\ &= \int_0^\infty f(\xi)[\mathcal{L},R\partial_R]\phi(R,\xi)\rho(\xi)\,d\xi + \int_0^\infty \xi f(\xi)(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi)\rho(\xi)\,d\xi - 2\int_0^\infty \xi f(\xi)\phi(R,\xi)\rho(\xi)\,d\xi\end{aligned}$$

with the commutator

$$[\mathcal{L},R\partial_R] = 2\mathcal{L} + \frac{16}{(1+R^2)^2} - \frac{32R^2}{(1+R^2)^3} =: 2\mathcal{L} + W(R)$$

[10]The kernel below is interpreted in the principal value sense

Thus,

$$\mathcal{L}u = \int_0^\infty f(\xi)W(R)\phi(R,\xi)\rho(\xi)\,d\xi + \int_0^\infty \xi f(\xi)(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi)\rho(\xi)\,d\xi$$

Hence we obtain

$$\eta\mathcal{K}f(\eta) - \mathcal{K}(\xi f)(\eta) = \Big\langle \int_0^\infty f(\xi)W(R)\phi(R,\xi)\rho(\xi)\,d\xi, \phi(R,\eta)\Big\rangle_{L^2_R}$$

The double integral on the right-hand side is absolutely convergent, therefore we can change the order of integration to obtain

$$(\eta-\xi)K(\eta,\xi) = \rho(\xi)\Big\langle W(R)\phi(R,\xi),\phi(R,\eta)\Big\rangle_{L^2_R}$$

This leads to the representation in (5.4) when $\xi \neq \eta$ with

$$F(\xi,\eta) = \Big\langle W(R)\phi(R,\xi),\phi(R,\eta)\Big\rangle_{L^2_R}$$

It remains to study its size and regularity. First, due to our pointwise bound from the previous section,

$$\begin{aligned}
&\sup_{R\geq 0}|\phi(R,\xi)| \lesssim \langle\xi\rangle^{-\frac34},\\
&|R\partial_R\phi(R,\xi)| \lesssim \min(R\xi^{-\frac14}, R^{\frac32}) \qquad \forall\, \xi>1\\
&|\partial_\xi\phi(R,\xi)| \lesssim \min(R\xi^{-\frac54}, R^{\frac72}) \qquad \forall\, \xi>1/2\\
&|\partial_\xi\phi(R,\xi)| \lesssim \min(R^{\frac32}\log(1+R^2), \xi^{-\frac14}|\log\xi|R) \qquad \forall\, 0<\xi<1/2\\
&|\partial_\xi^2\phi(R,\xi)| \lesssim \min(R^2\xi^{-\frac74}, R^{\frac{11}{2}}) \qquad \forall\, \xi>1/2\\
&|\partial_\xi^2\phi(R,\xi)| \lesssim \min(R^{\frac72}\log(1+R^2), \xi^{-\frac34}|\log\xi|\,R^2) \qquad \forall\, 0<\xi<1/2
\end{aligned} \tag{5.5}$$

we always have the estimates

$$\begin{aligned}
&|F(\xi,\eta)| \lesssim \langle\xi\rangle^{-\frac34}\langle\eta\rangle^{-\frac34},\\
&|\partial_\xi F(\xi,\eta)| \lesssim \langle\xi\rangle^{-\frac54}\langle\eta\rangle^{-\frac34}, \quad |\partial_\eta F(\xi,\eta)| \lesssim \langle\xi\rangle^{-\frac34}\langle\eta\rangle^{-\frac54},\\
&|\partial^2_{\xi\eta}F(\xi,\eta)| \lesssim \langle\xi\rangle^{-\frac54}\langle\eta\rangle^{-\frac54} \qquad \forall\, \xi+\eta\gtrsim 1\\
&|\partial^2_\xi F(\xi,\eta)| \lesssim \xi^{-\frac74}\eta^{-\frac34} \qquad \forall\, \xi>1,\ \eta>1\\
&|\partial^2_\eta F(\xi,\eta)| \lesssim \xi^{-\frac34}\eta^{-\frac74} \qquad \forall\, \xi>1,\ \eta>1
\end{aligned} \tag{5.6}$$

They are only useful when ξ and η are very close. To improve on them, we consider two cases:

Case 1: $1\lesssim \xi+\eta$**.** To capture the cancelations when ξ and η are separated we resort to another integration by parts,

$$\eta F(\xi,\eta) = \Big\langle W(R)\phi(R,\xi), \mathcal{L}\phi(R,\eta)\Big\rangle = \Big\langle [\mathcal{L}, W(R)]\phi(R,\xi),\phi(R,\eta)\Big\rangle + \xi F(\xi,\eta)$$

Hence, evaluating the commutator,

$$(\eta-\xi)F(\xi,\eta) = -\Big\langle (2W_R\partial_R + W_{RR})\phi(R,\xi),\phi(R,\eta)\Big\rangle \tag{5.7}$$

Since $W_R(0)=0$ it follows that $(2W_R\partial_R + W_{RR})\phi(R,\xi)$ has the same behavior as $\phi(R,\xi)$ in the first region. Then we can repeat the argument above to obtain

$$(\eta-\xi)^2F(\xi,\eta) = -\Big\langle [\mathcal{L}, 2W_R\partial_R + W_{RR}]\phi(R,\xi),\phi(R,\eta)\Big\rangle$$

The second commutator has the form, with $V(R) := -8(1+R^2)^{-2}$,

$$[\mathcal{L}, 2W_R\partial_R + W_{RR}] = 4W_{RR}\mathcal{L} - 4W_{RRR}\partial_R - W_{RRRR} + 3R^{-2}(R^{-1}W_R - W_{RR}) - 2W_RV_R - 4W_{RR}V$$

Since $R^{-1}W_R(R) - W_{RR}(R) = O(R^2)$ this leads to

$$(\eta-\xi)^2F(\xi,\eta) = \Big\langle (W^{odd}(R)\partial_R + W^{even}(R) + \xi W^{even}(R))\phi(R,\xi),\phi(R,\eta)\Big\rangle$$

where by W^{odd}, respectively W^{even}, we have generically denoted odd, respectively even, nonsingular rational functions with good decay at infinity. Inductively, one now verifies the identity

$$
\begin{aligned}
&(\eta-\xi)^{2k}F(\xi,\eta)=\Big\langle\Big(\sum_{j=0}^{k-1}\xi^j\,W^{odd}_{kj}(R)\,\partial_R+\sum_{\ell=0}^{k}\xi^\ell W^{even}_{k\ell}(R)\Big)\phi(R,\xi),\phi(R,\eta)\Big\rangle \qquad (5.8)\\
&\langle R\rangle|W^{odd}_{kj}(R)|+|W^{even}_{k\ell}(R)|\lesssim\langle R\rangle^{-4-2k}\qquad\forall\, j,\ell
\end{aligned}
$$

By means of the pointwise bounds on ϕ and $\partial_R\phi$ from (5.5) we infer from this that

$$|F(\xi,\eta)|\lesssim\frac{\xi^{k-\frac34}\langle\eta\rangle^{-\frac34}}{(\eta-\xi)^{2k}}\qquad\forall\,\xi\gtrsim 1,\ \eta>0$$

Combining this estimate with (5.6) yields, for arbitrary N,

$$|F(\xi,\eta)|\lesssim(\xi+\eta)^{-\frac32}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N}\ \ \text{provided}\ \ \xi+\eta\gtrsim 1,$$

as claimed. For the derivatives of F we follow a similar procedure. If ξ and η are comparable, then from (5.6), $|\partial_\eta F(\xi,\eta)|\lesssim\langle\xi\rangle^{-2}$. Otherwise we differentiate with respect to η in (5.8). This yields

$$(\eta-\xi)^{2k}\partial_\eta F(\xi,\eta)=\Big\langle\Big(\sum_{j=0}^{k-1}\xi^j\,W^{odd}_{kj}(R)\,\partial_R+\sum_{\ell=0}^{k}\xi^\ell W^{even}_{k\ell}(R)\Big)\phi(R,\xi),\partial_\eta\phi(R,\eta)\Big\rangle-2k(\eta-\xi)^{2k-1}F(\xi,\eta)$$

Using also the bound on F from above we obtain

$$|\partial_\eta F(\xi,\eta)|\lesssim\frac{\xi^{k-\frac34}\eta^{-\frac54}}{(\eta-\xi)^{2k}},\qquad 1\lesssim\xi,\eta$$

respectively

$$|\partial_\eta F(\xi,\eta)|\lesssim\frac{\eta^{-\frac54}}{(\eta-\xi)^{2k}}\qquad\xi\ll 1\lesssim\eta$$

and

$$|\partial_\eta F(\xi,\eta)|\lesssim\frac{\xi^{k-\frac34}}{(\eta-\xi)^{2k}}\qquad\eta\ll 1\lesssim\xi$$

which again yield the desired bounds. Finally, we consider the second order derivatives with respect to ξ and η. For ξ and η close we again use the bound from (5.6). Otherwise we differentiate twice in (5.8) and continue as before. We note that it is important here that the decay of W^{odd}_{kj} and $W^{even}_{k\ell}$ improves with k. This is because the second order derivative bound at 0 has a sizeable growth at infinity which has to be canceled,

$$|\partial_\xi^2\phi(R,0)|\approx R^{\frac72}\log R$$

Case 2: $\xi,\eta\ll 1$. Our first observation is that $F(0,0)=0$. This is easy to verify by direct integration, and is heuristically justified by the fact that $W=[L,R\partial_R]$. The pointwise bound

$$|\partial_\xi F(\xi,\eta)|\lesssim 1$$

follows by a direct computation. The second order derivative bound is, however, more delicate. We have at our disposal the pointwise bounds

$$|\partial_\xi^j\phi(R,\xi)|\lesssim\left\{\begin{array}{ll}R^{-\frac12+2j}\log(1+R^2) & R<\xi^{-\frac12}\\ \xi^{\frac14-j/2}|\log\xi|\,R^j & R\geq\xi^{-\frac12}\end{array}\right.,\qquad j=0,1,2$$

If $\eta<\xi<1/2$, then these bounds imply that

$$
\begin{aligned}
|\partial^2_{\xi\eta}F(\xi,\eta)|\lesssim&\int_0^{\xi^{-\frac12}}\langle R\rangle^{-4}R^3(\log(1+R^2))^2\,dR+\int_{\xi^{-\frac12}}^{\eta^{-\frac12}}\langle R\rangle^{-4}R^{\frac52}\xi^{-\frac14}|\log\xi|\log(1+R^2)\,dR\\
&+\int_{\eta^{-\frac12}}^{\infty}\langle R\rangle^{-2}\xi^{-\frac14}\eta^{-\frac14}|\log\xi||\log\eta|\,dR
\end{aligned}
$$

The main contribution comes from the first term. Integrating we obtain

$$|\partial^2_{\xi\eta}F(\xi,\eta)|\lesssim|\log\xi|^3$$

A similar computation yields, again when $\eta < \xi < 1/2$,

$$|\partial_\xi^2 F(\xi,\eta)| \lesssim \int_0^{\xi^{-\frac12}} \langle R\rangle^{-4} R^3 (\log(1+R^2))^2\, dR + \int_{\xi^{-\frac12}}^{\eta^{-\frac12}} \langle R\rangle^{-4} R^{\frac32} \xi^{-\frac34} |\log \xi| \log(1+R^2)\, dR$$
$$+ \int_{\eta^{-\frac12}}^\infty \langle R\rangle^{-2} \xi^{-\frac34} \eta^{\frac14} |\log\xi| |\log\eta|\, dR \lesssim |\log\xi|^3$$

It remains to consider the expression $\partial_\xi^2 F(\xi,\eta)$ for $\xi \ll \eta < 1/2$. Differentiating in (5.7) we obtain

$$(\eta-\xi)\partial_\xi^2 F(\xi,\eta) = 2\partial_\xi F(\xi,\eta) - \Big\langle \partial_\xi^2 \phi(R,\xi), (2W_R\partial_R + W_{RR})\phi(R,\eta)\Big\rangle$$

We differentiate and integrate with respect to η to obtain

$$(\eta-\xi)\partial_\xi^2 F(\xi,\eta) = \int_\xi^\eta \Big[2\partial^2_{\xi\zeta} F(\xi,\zeta) - \Big\langle \partial_\xi^2\phi(R,\xi), (2W_R\partial_R + W_{RR})\partial_\zeta \phi(R,\zeta)\Big\rangle\Big]\, d\zeta \tag{5.9}$$

Using also the bound

$$|\partial_R\partial_\zeta \phi(R,\zeta)| \lesssim \begin{cases} R^{\frac12}\log(1+R^2) & R < \zeta^{-\frac12} \\ \zeta^{-\frac14}|\log\zeta| & R \ge \zeta^{-\frac12}\end{cases}$$

we can evaluate the inner product in (5.9) as follows:

$$\Big|\Big\langle \partial_\xi^2\phi(R,\xi), (2W_R\partial_R + W_{RR})\phi(R,\eta)\Big\rangle\Big| \lesssim \int_0^{\zeta^{-\frac12}} \langle R\rangle^{-6} R^{\frac72}\log(1+R^2) R^{\frac32}\log(1+R^2)\, dR$$
$$+ \int_{\zeta^{-\frac12}}^{\xi^{-\frac12}} \langle R\rangle^{-6} R^{\frac72}\log(1+R^2)\zeta^{-\frac14}|\log\zeta|\, R\, dR + \int_{\xi^{-\frac12}}^\infty \langle R\rangle^{-6}\xi^{-\frac34}|\log\xi| R^2 \zeta^{-\frac14}|\log\zeta| R\, dR \lesssim |\log\zeta|^3$$

Thus, (5.9) is controlled by

$$|(\eta-\xi)\partial_\eta^2 F(\xi,\eta)| \lesssim \Big| \int_\xi^\eta (\log\zeta)^3\, d\zeta\Big| \lesssim \eta|\log\eta|^3$$

Since $\eta \ll \xi$ this yields

$$|\partial_\eta^2 F(\xi,\eta)| \lesssim |\log\eta|^3$$

which concludes the analysis of the off-diagonal part of the kernel.

Next, we extract the δ measure that sits on the diagonal of the kernel K from the representation formula (5.2), see also (5.3). To do so, we can restrict ξ,η to a compact subset of $(0,\infty)$. This is convenient, as we then have the following asymptotics of $\phi(R,\xi)$ for $R\xi^{\frac12} \gg 1$:

$$\phi(R,\xi) = \operatorname{Re}\Big[a(\xi)\xi^{-\frac14} e^{iR\xi^{\frac12}}\Big(1+\frac{3i}{8R\xi^{\frac12}}\Big)\Big] + O(R^{-2})$$
$$(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi) = -2\operatorname{Re}\Big[\xi\partial_\xi(a(\xi)\xi^{-\frac14}) e^{iR\xi^{\frac12}}\Big(1+\frac{3i}{8R\xi^{\frac12}}\Big)\Big] + O(R^{-2})$$

where the $O(\cdot)$ terms depend on the choice of the compact subset. The R^{-2} terms are integrable so they contribute a bounded kernel to the inner product in (5.2). The same applies to the contribution of a bounded R region. Using the above expansions, we conclude that the δ-measure contribution of the inner product in (5.2) can only come from one of the following integrals:

$$-\int_0^\infty\int_0^\infty f(\xi)\chi(R)\operatorname{Re}\Big[\xi\partial_\xi(a(\xi)\xi^{-\frac14})a(\eta)\eta^{-\frac14}e^{iR(\xi^{\frac12}+\eta^{\frac12})}\Big(1+\frac{3i}{8R\xi^{\frac12}}\Big)\Big(1+\frac{3i}{8R\eta^{\frac12}}\Big)\Big]\rho(\xi)\, d\xi dR \tag{5.10}$$

$$-\frac12\int_0^\infty\int_0^\infty f(\xi)\chi(R)\,\xi\partial_\xi(a(\xi)\xi^{-\frac14})\bar a(\eta)\eta^{-\frac14}e^{iR(\xi^{\frac12}-\eta^{\frac12})}\Big(1+\frac{3i}{8R\xi^{\frac12}}\Big)\Big(1-\frac{3i}{8R\eta^{\frac12}}\Big)\rho(\xi)\, d\xi dR \tag{5.11}$$

$$-\frac12\int_0^\infty\int_0^\infty f(\xi)\chi(R)\,\xi\partial_\xi(\bar a(\xi)\xi^{-\frac14})a(\eta)\eta^{-\frac14}e^{-iR(\xi^{\frac12}-\eta^{\frac12})}\Big(1-\frac{3i}{8R\xi^{\frac12}}\Big)\Big(1+\frac{3i}{8R\eta^{\frac12}}\Big)\rho(\xi)\, d\xi dR \tag{5.12}$$

where χ is a smooth cutoff function which equals 0 near $R = 0$ and 1 near $R = \infty$. In all of the above integrals we can argue as in the proof of the classical Fourier inversion formula to change the order of integration. Integrating by parts in the first integral (5.10) reveals that it cannot contribute a δ-measure. Discarding the R^{-2} terms from (5.11) and (5.12) reduces us further to the expressions

$$-\int_0^\infty \int_0^\infty f(\xi)\chi(R)\,\mathrm{Re}\left[\xi\partial_\xi(a(\xi)\xi^{-\frac14})\bar{a}(\eta)\eta^{-\frac14}e^{iR(\xi^{\frac12}-\eta^{\frac12})}\right]\rho(\xi)\,d\xi dR \tag{5.13}$$

$$+\frac38\int_0^\infty \int_0^\infty f(\xi)\chi(R)\,\mathrm{Im}\left[\xi\partial_\xi(a(\xi)\xi^{-\frac14})\bar{a}(\eta)\eta^{-\frac14}e^{iR(\xi^{\frac12}-\eta^{\frac12})}\right]R^{-1}(\xi^{-\frac12}-\eta^{-\frac12})\rho(\xi)\,d\xi dR \tag{5.14}$$

The second integral (5.14) has both an R^{-1} and a $(\xi^{-\frac12}-\eta^{-\frac12})$ factor so its contribution to K is bounded. The first integral (5.13) contributes both a Hilbert transform type kernel as well as a δ-measure to K. By inspection, the δ contribution is

$$\begin{aligned}
&-\frac12\int_{-\infty}^\infty \mathrm{Re}\left[\xi\partial_\xi(a(\xi)\xi^{-\frac14})\bar{a}(\eta)\eta^{-\frac14}e^{iR(\xi^{\frac12}-\eta^{\frac12})}\right]\rho(\xi)\,dR\\
&= -\pi\,\mathrm{Re}\left[\xi\partial_\xi(a(\xi)\xi^{-\frac14})\bar{a}(\eta)\eta^{-\frac14}\right]\rho(\xi)\delta(\xi^{\frac12}-\eta^{\frac12})\\
&= -2\pi\xi^{\frac12}\rho(\xi)\,\mathrm{Re}\left[\xi\partial_\xi(a(\xi)\xi^{-\frac14})\bar{a}(\xi)\xi^{-\frac14}\right]\delta(\xi-\eta)\\
&= -2\pi\xi^{\frac12}\rho(\xi)\,\mathrm{Re}\left[-\frac14\xi^{-\frac12}|a(\xi)|^2+\xi^{\frac12}a(\xi)\bar{a}'(\xi)\right]\delta(\xi-\eta)\\
&= \left[\frac12+\frac{\xi\rho'(\xi)}{\rho(\xi)}\right]\delta(\xi-\eta)
\end{aligned}$$

where we used that $\rho(\xi)^{-1} = \pi|a|^2$ in the final step. Combining this with the δ-measure in (5.2) yields (5.3). □

Next we consider the L^2 mapping properties for $\mathcal{K}$. We introduce the weighted L^2 spaces $L^{2,\alpha}_\rho$ with norm

$$\|f\|_{L^{2,\alpha}_\rho} := \Big(\int_0^\infty |f(\xi)|^2\langle\xi\rangle^{2\alpha}\rho(\xi)\,d\xi\Big)^{\frac12} \tag{5.15}$$

Then we have

Proposition 5.2. *a) The operator $\mathcal{K}_0$ from (5.3) maps*

$$\mathcal{K}_0\;:\;L^{2,\alpha}_\rho \to L^{2,\alpha+1/2}_\rho$$

b) In addition, we have the commutator bound

$$[\mathcal{K}_0,\xi\partial_\xi]\;:\;L^{2,\alpha}_\rho \to L^{2,\alpha}_\rho$$

Both statements hold for all $\alpha \in \mathbb{R}$.

Proof. a) This is equivalent to showing that the kernel

$$\rho^{\frac12}(\eta)\langle\eta\rangle^{\alpha+1/2}K_0(\eta,\xi)\langle\xi\rangle^{-\alpha}\rho^{-\frac12}(\xi)\;:\;L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)$$

With the notation of the previous theorem, the kernel on the left-hand side is

$$\tilde{K}_0(\eta,\xi) := \langle\eta\rangle^{\alpha+1/2}\langle\xi\rangle^{-\alpha}\frac{\sqrt{\rho(\xi)\rho(\eta)}}{\xi-\eta}F(\xi,\eta)$$

We first separate the diagonal and off-diagonal behavior of $\tilde{K}_0$, considering several cases.

Case 1: $(\xi,\eta)\in Q := [0,4]\times[0,4]$.

We cover the unit interval with dyadic subintervals $I_j = [2^{j-1}, 2^{j+1}]$. We cover the diagonal with the union of squares

$$A = \bigcup_{j=-\infty}^{2} I_j \times I_j$$

and divide the kernel $\tilde{K}_0$ into

$$1_Q\tilde{K}_0 = 1_{A\cap Q}\tilde{K}_0 + 1_{Q\setminus A}\tilde{K}_0$$

Case 1(a): Here we show that the diagonal part $1_{A\cap Q}\tilde{K}_0$ of $\tilde{K}_0$ maps L^2 to L^2. By orthogonality it suffices to restrict ourselves to a single square $I_j \times I_j$. We recall the $T1$ theorem for Calderon-Zygmund operators, see page 293 in [Ste]: suppose the kernel $K(\eta,\xi)$ on $\mathbb{R}^2$ defines an operator $T : \mathcal{S} \to \mathcal{S}'$ and has the following pointwise properties with some $\gamma \in (0,1]$ and a constant C_0:

(i) $|K(\eta,\xi)| \le C_0|\xi-\eta|^{-1}$
(ii) $|K(\eta,\xi) - K(\eta',\xi)| \le C_0|\eta-\eta'|^\gamma|\xi-\eta|^{-1-\gamma}$ for all $|\eta-\eta'| < |\xi-\eta|/2$
(iii) $|K(\eta,\xi) - K(\eta,\xi')| \le C_0|\xi-\xi'|^\gamma|\xi-\eta|^{-1-\gamma}$ for all $|\xi-\xi'| < |\xi-\eta|/2$

If in addition T has the restricted L^2 boundedness property, i.e., for all $r > 0$ and $\xi_0, \eta_0 \in \mathbb{R}$, $\|T(\omega^{r,\xi_0})\|_2 \le C_0 r^{\frac12}$ and $\|T^*(\omega^{r,\eta_0})\|_2 \le C_0 r^{\frac12}$ where $\omega^{r,\xi_0}(\xi) = \omega((\xi-\xi_0)/r)$ with a fixed bump-function ω, then T and T^* are $L^2(\mathbb{R})$ bounded with an operator norm that only depends on C_0.

Within the square $I_j \times I_j$, Theorem 5.1 shows that the kernel of $\tilde{K}_0$ satisfies these properties with $\gamma = 1$, and is thus bounded on L^2.

Case 1(b): Consider now the off-diagonal part $1_{Q\setminus A}\tilde{K}_0$. In this region, by Theorem 5.1,

$$|\tilde{K}_0(\eta,\xi)| \lesssim \frac{1}{\sqrt{\xi\eta}|\log\xi||\log\eta|}$$

which is a Hilbert-Schmidt kernel on Q and thus L^2 bounded.

Case 2: $(\xi,\eta) \in Q^c$. We cover the diagonal with the union of squares

$$B = \bigcup_{j=1}^\infty I_j \times I_j$$

and divide the kernel $\tilde{K}_0$ into

$$1_{Q^c}\tilde{K}_0 = 1_{B\cap Q^c}\tilde{K}_0 + 1_{Q^c\setminus B}\tilde{K}_0$$

Case 2a: Here we consider the estimate on B. As in case 1a) above, we use Calderon-Zygmund theory. Evidently, $|\tilde{K}_0(\eta,\xi)| \lesssim |\xi-\eta|^{-1}$ on B by Theorem 5.1. To check (ii) and (iii), we differentiate $\tilde{K}_0$. It will suffice to consider the case where the ∂_ξ derivative falls on $F(\xi,\eta)$. We distinguish two cases: if $|\xi^{\frac12} - \eta^{\frac12}| \le 1$, then $|\xi-\eta| \lesssim \xi^{\frac12}$ which implies that

$$\frac{\xi^{-\frac12}|\xi-\xi'|}{|\xi-\eta|} \lesssim \frac{|\xi-\xi'|^{\frac12}}{|\xi-\eta|^{\frac32}} \qquad \forall\, |\xi-\xi'| < |\xi-\eta|/2$$

if, on the other hand, $|\xi^{\frac12} - \eta^{\frac12}| > 1$, then

$$\frac{\xi^{-\frac12}|\xi-\xi'|}{|\xi-\eta||\xi^{\frac12}-\eta^{\frac12}|} \lesssim \frac{|\xi-\xi'|}{|\xi-\eta|^2} \qquad \forall\, |\xi-\xi'| < |\xi-\eta|/2$$

which proves property (iii) on B with $\gamma = \frac12$, and by symmetry also (ii). The restricted L^2 property follows form the cancelation in the kernel and the previous bounds on the kernel. Hence, $\tilde{K}_0$ is L^2 bounded on B.

Case 2b: Finally, in the exterior region $Q_c \setminus B$ we have the bound, with arbitrarily large N,

$$|\tilde{K}_0(\eta,\xi)| \lesssim (1+\xi)^{-N}(1+\eta)^{-N}$$

which is L^2 bounded by Schur's lemma.

b) A direct computation shows that the kernel K_0^{com} of the commutator $[\xi\partial_\xi, K_0]$ is given by

$$K_0^{com}(\eta,\xi) = (\eta\partial_\eta + \xi\partial_\xi)K_0(\eta,\xi) + K_0(\eta,\xi) = \frac{\rho(\xi)}{\xi-\eta}F^{com}(\xi,\eta)$$

interpreted in the principal value sense and with F^{com} given by

$$F^{com}(\xi,\eta) = \frac{\xi\rho'(\xi)}{\rho(\xi)}F(\xi,\eta) + (\eta\partial_\eta + \xi\partial_\xi)F(\xi,\eta)$$

By Theorem 5.1 this satisfies the same pointwise off-diagonal bounds as F. Near the diagonal the bounds for F^{com} and its derivatives are worse[11] than those for F by a factor of $(1+\xi)^{\frac{1}{2}}$. Then the proof of the L^2 commutator bound is similar to the argument in part (a). □

6. The final equation

To rewrite the equation (3.2) in a final form, we begin by expressing the operator $R\partial_R$ in terms of the kernel $\mathcal{K}$ in the transference identity (5.1). We have, with $\mathcal{F}$ as in Theorem 4.3,

$$\mathcal{F}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big) = \Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}(-2\xi\partial_\xi + \mathcal{K})\Big)\mathcal{F}$$

which gives

$$\begin{aligned}\mathcal{F}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big)^2 =& \Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}(-2\xi\partial_\xi + \mathcal{K})\Big)^2\mathcal{F}\\ =& \Big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\Big)^2\mathcal{F} + 2\frac{\lambda_\tau}{\lambda}\mathcal{K}\Big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\Big)\mathcal{F} + \frac{\lambda_\tau^2}{\lambda^2}(\mathcal{K}^2 + 2[\xi\partial_\xi, \mathcal{K}])\mathcal{F}\end{aligned}$$

This leads to a transport type equation for the Fourier transform $x(\tau,\xi)$ of $\tilde{\varepsilon}$ by applying $\mathcal{F}$ to (3.2). Indeed, in view of the preceding

$$\begin{aligned}-\Big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\Big)^2 x - \xi x =& 2\frac{\lambda_\tau}{\lambda}\mathcal{K}\Big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\Big)x + \frac{\lambda_\tau^2}{\lambda^2}(\mathcal{K}^2 + 2[\xi\partial_\xi, \mathcal{K}])x\\ &- \Big(\frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\Big)x + \lambda^{-2}\mathcal{F}R^{\frac{1}{2}}(N_{2k-1}(R^{-\frac{1}{2}}\mathcal{F}^{-1}x) + e_{2k-1})\end{aligned} \tag{6.1}$$

We want to obtain solutions to (6.1) which decay as $\tau \to \infty$, which means we need to solve the equation backward in time, i.e., with zero Cauchy data at $\tau = \infty$. We treat this problem iteratively, as a small perturbation of the linear equation governed by the operator on the left–hand side. For this we need to solve the following **transport equation**

$$-\Big[\Big(\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi\Big)^2 + \xi\Big]x(\tau,\xi) = b(\tau,\xi), \tag{6.2}$$

We denote by H the backward fundamental solution for the operator

$$\Big(\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi\Big)^2 + \xi$$

and by $H(\tau,\sigma)$ its kernel,

$$x(\tau) = \int_\tau^\infty H(\tau,\sigma)f(\sigma)\,d\sigma$$

The mapping properties of H are described in the following result, which will be proven in the next section.

Proposition 6.1. *For any $\alpha \geq 0$ there exists some (large) constant $C = C(\alpha)$ so that the operator $H(\tau,\sigma)$ satisfies the bounds*

$$\|H(\tau,\sigma)\|_{L^{2,\alpha}_\rho \to L^{2,\alpha+1/2}_\rho} \lesssim \tau\Big(\frac{\sigma}{\tau}\Big)^C \tag{6.3}$$

$$\Big\|\Big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\Big)H(\tau,\sigma)\Big\|_{L^{2,\alpha}_\rho \to L^{2,\alpha}_\rho} \lesssim \Big(\frac{\sigma}{\tau}\Big)^C \tag{6.4}$$

uniformly in $\sigma \geq \tau$.

This leads us to introduce the spaces $L^{\infty,N}L^{2,\alpha}_\rho$ with norm

$$\|f\|_{L^{\infty,N}L^{2,\alpha}_\rho} := \sup_{\tau\geq 1}\tau^N\|f(\tau)\|_{L^{2,\alpha}_\rho}$$

Then the above proposition immediately allows us to draw the following conclusions:

[11] The one derivative loss can be avoided by a more careful analysis, but this does not seem necessary here.

Corollary 6.2. *Given $\alpha \geq 0$, let N be large enough. Then*

$$\|Hb\|_{L^{\infty,N-2}L^{2,\alpha+1/2}_\rho} + \left\|\left(\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi\right)Hb\right\|_{L^{\infty,N-1}L^{2,\alpha}_\rho} \leq C_0\,\frac{1}{N}\|b\|_{L^{\infty,N}L^{2,\alpha}_\rho}$$

with a constant C_0 that depends on α but does not depend on N.

The small factor N^{-1} is crucial here for our argument to work. On the other hand, the nonlinear operator N_{2k-1} from (6.1) has the following mapping properties:

Proposition 6.3. *Assume that N is large enough and $\frac{\nu}{2} + \frac{3}{4} > \alpha > \frac{1}{4}$. Then the map*

$$x \to \lambda^{-2}\mathcal{F}R^{\frac{1}{2}}(N_{2k-1}(R^{-\frac{1}{2}}\mathcal{F}^{-1}x))$$

is locally Lipschitz from $L^{\infty,N-2}L^{2,\alpha+1/2}_\rho$ to $L^{\infty,N}L^{2,\alpha}_\rho$.

The above two results, combined with Proposition 5.2, allow us to use a contraction argument to solve equation (6.1). The next two sections are devoted to proving Propositions 6.1, 6.3. Finally, in the last section we close the argument.

7. The transport equation

Here we consider the backward fundamental solution H for (6.2) and prove Proposition 6.1. Observe that (6.2) implies

$$[\partial_\tau^2 + \lambda^{-2}(\tau)\xi]x(\tau, \lambda^{-2}(\tau)\xi) = b(\tau, \lambda^{-2}(\tau)\xi)$$

We introduce the operator

$$L_{\xi,\tau} := \partial_\tau^2 + \lambda^{-2}(\tau)\xi$$

and the fundamental solutions $S(\tau,\sigma,\xi)$, $U(\tau,\sigma,\xi)$, which satisfy

$$L_{\xi,\tau}S(\tau,\sigma,\xi) = 0,\ S(\tau,\tau,\xi) = 0,\ \partial_\tau S(\tau,\sigma,\xi)|_{\tau=\sigma} = -1$$

$$L_{\xi,\tau}U(\tau,\sigma,\xi) = 0,\ U(\tau,\tau,\xi) = 1,\ \partial_\tau U(\tau,\sigma,\xi)|_{\tau=\sigma} = 0$$

Then (6.2) may be solved via

$$x(\tau, \lambda^{-2}(\tau)\xi) = -\int_\tau^\infty S(\tau,\sigma,\xi)b(\sigma,\lambda^{-2}(\sigma)\xi)\,d\sigma$$

Given this representation, we note that the index α plays no role in (6.3) and (6.4) since

$$\frac{(1+\lambda^{-2}(\tau)\xi)^\alpha}{(1+\lambda^{-2}(\sigma)\xi)^\alpha} \lesssim \left(\frac{\sigma}{\tau}\right)^C$$

Hence without loss of generality we set $\alpha = 0$. Similarly, we can neglect the measure of integration $\rho(\xi)d\xi$ which also has a polynomial behavior both at 0 and at infinity,

$$\frac{\rho(\lambda^{-2}(\tau)\xi)}{\rho(\lambda^{-2}(\sigma)\xi)} \lesssim \left(\frac{\sigma}{\tau}\right)^C$$

Then the bounds (6.3) and (6.4) reduce to proving that

$$|S(\tau,\sigma,\xi)| \lesssim \sigma\left(\frac{\sigma}{\tau}\right)^C(1+\lambda^{-2}(\tau)\xi)^{-\frac{1}{2}}, \qquad |\partial_\tau S(\tau,\sigma,\xi)| \lesssim \left(\frac{\sigma}{\tau}\right)^C, \qquad 1 \lesssim \tau < \sigma$$

Recalling that $\lambda(\tau) = \tau^{1+\nu^{-1}}$ (we are ignoring a multiplicative constant here), we strengthen the first bound and prove instead that

$$|S(\tau,\sigma,\xi)| \lesssim \sigma\left(\frac{\sigma}{\tau}\right)^C(1+\tau^{-\frac{2}{\nu}}\xi)^{-\frac{1}{2}}, \qquad |\partial_\tau S(\tau,\sigma,\xi)| \lesssim \left(\frac{\sigma}{\tau}\right)^C \qquad 0 < \tau < \sigma \tag{7.1}$$

The advantage of doing this is that the last bound is scale invariant. Precisely, one verifies directly the scaling relation

$$S(\tau,\sigma,\xi) = \xi^{\frac{\nu}{2}}S(\tau\xi^{-\frac{\nu}{2}}, \sigma\xi^{-\frac{\nu}{2}}, 1)$$

which leaves (7.1) unchanged. Hence in what follows it suffices to prove (7.1) in the case $\xi = 1$. We begin by constructing two special solutions for the operator $L_{1,\tau}$. For small[12] τ we use a standard WKB ansatz.

Lemma 7.1. *a) (Large τ solutions) If ν is not an even integer then there exist two analytic solutions ϕ_0 and ϕ_1 of $L_{1,\tau}\phi_j = 0$ with a series representation*

$$\phi_j(\tau) = \sum_{k=0}^{\infty} c_{jk}\, \tau^{j-\frac{2k}{\nu}}, \qquad c_{j0} = 1$$

which is convergent for all $\tau > 0$. If ν is an even integer then the result still holds with a modification in the expression for ϕ_1, namely

$$\phi_1(\tau) = c_1\phi_0(\tau)\log\tau + \sum_{k=0}^{\infty} c_{1k}\, \tau^{1-\frac{2k}{\nu}}, \qquad c_{10} = 1$$

b) (Small τ solutions) There is a solution ϕ_2 for $L_{1,\tau}$ of the form

$$\phi_2(\tau) = \tau^{\frac{1}{2}+\frac{1}{2\nu}} e^{i\nu\tau^{-\frac{1}{\nu}}}[1 + a(\tau^{\frac{1}{\nu}})]$$

with a smooth and satisfying $a(0) = 0$.

Proof. a) We substitute the formal series in the equation

$$(\partial_\tau^2 + \tau^{-2-\frac{2}{\nu}})\phi_j = 0$$

in the equation and identify the coefficients of the similar terms. This yields

$$c_{j,k}(j - \frac{2k}{\nu})(j - 1 - \frac{2k}{\nu}) + c_{j,k-1} = 0 \qquad k \geq 1$$

Hence the coefficients c_{jk} can be iteratively computed and satisfy a bound of the type

$$|c_{j,k}| \leq \frac{C^k}{(k!)^2}$$

which implies that the series converges for all τ.

If $j = 0$ then the argument works for all $\nu > 0$. If $j = 1$ then there is an obstruction if ν is an even integer; indeed, this happens precisely when $2k = \nu$. As usual, this is compensated for by adding in the logarithmic term, since

$$L_{1,\tau}(\phi_0(\tau)\log\tau) = -\tau^{-2}\phi_0 + \tau^{-1}2\partial_\tau\phi_0$$

has a nonzero coefficient on the τ^{-2} term.

b) In this case, we use the usual WKB-ansatz which we now recall in a more general setting: we wish to solve the equation $(\partial_\tau^2 + Q)\psi = 0$ where $Q(\tau)$ is a smooth potential for $\tau > 0$. Fix some (small) $\tau_0 > 0$. WKB means that we seek a solution of the form $\psi(\tau) = \psi_0(\tau)[1 + a(\tau)]$ with

$$\psi_0(\tau) = Q^{-\frac{1}{4}}(\tau)e^{iS(\tau)}, \qquad S(\tau) = \int_{\tau_0}^{\tau} Q^{\frac{1}{2}}(\sigma)\, d\sigma$$

Since

$$\partial_\tau^2\psi_0 + Q\psi_0 = V\psi_0, \qquad V = -\frac{1}{4}\frac{Q''}{Q} + \frac{5}{16}\Big(\frac{Q'}{Q}\Big)^2$$

we obtain the following equation for $a(\tau)$:

$$(a'\psi_0^2)'(\tau) = -V\psi_0^2(\tau)[1 + a(\tau)]$$

which we solve in the form

$$\begin{aligned} a(\tau) &= -\int_0^{\tau}\int_0^{\tau'} \psi_0^{-2}(\tau')\psi_0^2(\sigma)V(\sigma)[1 + a(\sigma)]\, d\sigma d\tau' \\ &= \frac{i}{2}\int_0^{\tau} Q^{-\frac{1}{2}}(\sigma)\big[1 - e^{2i(S(\sigma)-S(\tau))}\big]\, V(\sigma)[1 + a(\sigma)]\, d\sigma \end{aligned}$$

[12]The reader should bear in mind that by this τ we mean the rescaled one, i.e, $\xi^{-\frac{\nu}{2}}\tau$, which can be arbitrarily close to zero.

provided these integrals converge at zero. They do in our case: in fact, $Q(\tau) = \lambda^{-2}(\tau)$ which implies that

$$\psi_0(\tau) = \tau^{\frac{1}{2}+\frac{1}{2\nu}} e^{i\nu\tau^{-\frac{1}{\nu}}}$$
$$a(\tau) = c\, i \int_0^\tau \sigma^{-1+\frac{1}{\nu}} \left[1 - e^{2i\nu(\sigma^{-\frac{1}{\nu}} - \tau^{-\frac{1}{\nu}})}\right] [1 + a(\sigma)]\, d\sigma$$

or, after changing variables to $a(\tau^\nu) = \tilde{a}(\tau)$,

$$\tilde{a}(\tau) = ic\nu \int_0^\tau \left[1 - e^{2i\nu(\sigma^{-1} - \tau^{-1})}\right] [1 + \tilde{a}(\sigma)]\, d\sigma \tag{7.2}$$

By the boundedness of the kernel, this Volterra equation has a solution $\tilde{a} \in C([0, \infty))$ which is clearly then also smooth for all $\tau > 0$. We now claim that in fact $\tilde{a} \in C^\infty([0, \infty))$. Indeed, the zero order iterate here is a smooth function at $\tau = 0$:

$$\int_0^\tau \left[1 - e^{2i\nu(\sigma^{-1} - \tau^{-1})}\right] d\sigma = \int_{\tau^{-1}}^\infty \left[1 - e^{2i\nu(u - \tau^{-1})}\right] \frac{du}{u^2}$$
$$= \tau - \int_{\tau^{-1}}^\infty e^{2i\nu(u-\tau^{-1})}\, \frac{du}{u^2} = \sum_{j=1}^m c_j\, \tau^j + O(\tau^{m+1})$$

for any positive integer m by repeated integration by parts. One now proceeds to show the same for the higher Volterra iterates; alternatively, we insert the ansatz

$$\tilde{a}(\tau) = \sum_{j=1}^m d_j\, \tau^j + O(\tau^{m+1})$$

into (7.2) and solve for the coefficients d_j. In either case, the conclusion is that (7.2) has a smooth solution, as claimed. □

We now use this lemma to prove (7.1), which will then conclude the proof of Proposition 6.1. Considering the limits at infinity, respectively at 0, one finds that

$$W(\phi_0, \phi_1) = 1, \qquad W(\phi_2, \overline{\phi}_2) = -2i$$

This allows us to express the backward fundamental solution $S(\tau, \sigma)$ in terms of these bases. Note that we suppress the ξ variable as $\xi = 1$ is fixed. We consider two cases.

Case 1: $\sigma > 1$. Then we have

$$S(\tau, \sigma) = \phi_1(\sigma)\phi_0(\tau) - \phi_0(\sigma)\phi_1(\tau)$$

If $1 \leq \tau \leq \sigma$, then (7.1) follows directly from the properties of ϕ_0 and ϕ_1. If $\tau < 1$ then we express $\phi_0(\tau)$ and $\phi_1(\tau)$ in terms of the $\{\phi_2, \overline{\phi}_2\}$ basis to obtain

$$S(\tau, \sigma) = \operatorname{Re}(c(\sigma)\phi_2(\tau)), \qquad |c(\sigma)| \lesssim \sigma$$

This gives

$$|S(\tau, \sigma)| \lesssim \sigma\tau^{\frac{1}{2}+\frac{1}{2\nu}}, \qquad |\partial_\tau S(\tau, \sigma)| \lesssim \sigma\tau^{-\frac{1}{2}-\frac{1}{2\nu}}$$

Again (7.1) follows.

Case 2: $\sigma < 1$. Then we express $S(\tau, \sigma)$ in the $\{\phi_2, \overline{\phi}_2\}$ basis to obtain

$$S(\tau, \sigma) = \operatorname{Im}(\phi_2(\sigma)\overline{\phi}_2(\tau))$$

This gives the bounds

$$|S(\tau, \sigma)| \lesssim \sigma^{\frac{1}{2}+\frac{1}{2\nu}}\tau^{\frac{1}{2}+\frac{1}{2\nu}}, \qquad |\partial_\tau S(\tau, \sigma)| \lesssim \sigma^{\frac{1}{2}+\frac{1}{2\nu}}\tau^{-\frac{1}{2}-\frac{1}{2\nu}}$$

which imply (7.1).

8. The nonlinear terms

In this section we consider the *nonlinear source terms*, i.e., those given by the right-hand side of (3.2), and prove Proposition 6.3. Recalling that $R = r\lambda$, we write

$$\begin{aligned}\lambda^{-2}R^{\frac{1}{2}}N_{2k-1}(R^{-\frac{1}{2}}\tilde{\varepsilon}) = \frac{\cos(2u_{2k-1})-\cos(2Q)}{R^2}2\tilde{\varepsilon} + \frac{\sin(2u_{2k-1})}{2R}\frac{\cos(2\tilde{\varepsilon}R^{-\frac{1}{2}})-1}{R^{\frac{1}{2}}} \\ + \cos(2u_{2k-1})\frac{\sin(2\tilde{\varepsilon}R^{-\frac{1}{2}})-2\tilde{\varepsilon}R^{-\frac{1}{2}}}{2R^{\frac{3}{2}}}\end{aligned} \tag{8.1}$$

where the regularity of the coefficients above is computed as in Step 2 of the proof of Theorem 2.1,

$$\frac{\cos(2u_{2k-1})-\cos(2Q)}{R^2} \in \tau^{-2}IS^2(R^{-2}(\log R)^2, \mathcal{Q}_{k-1}) \tag{8.2}$$

$$\frac{\sin(2u_{2k-1})}{2R} \in IS^0(R^{-2}\log R, \mathcal{Q}_{k-1}) \tag{8.3}$$

$$\cos(2u_{2k-1}) \in IS^0(1, \mathcal{Q}_{k-1}) \tag{8.4}$$

where we used here that $t\lambda(t) \asymp \tau$ and also that $R \lesssim \tau$ (recall the algebras $\mathcal{Q}$ and $\mathcal{Q}_k$ from Definition 2.3). Proposition 6.3 amounts to proving multiplicative estimates in the context of the classical Sobolev spaces. Here we use Sobolev spaces adapted to the operator $\mathcal{L}$, namely

$$\|u\|_{H^\alpha_\rho} := \|\widehat{u}\|_{L^{2,\alpha}_\rho}$$

Restating Proposition 6.3 with this notation shows that we need to prove that the map

$$\tilde{\varepsilon} \mapsto \lambda^{-2}R^{\frac{1}{2}}N_{2k-1}(R^{-\frac{1}{2}}\tilde{\varepsilon})$$

is locally Lipschitz from $L^{\infty,N-2}H^{\alpha+1/2}_\rho$ to $L^{\infty,N}H^\alpha_\rho$. Note that (8.2) has an explicit gain of τ^{-2} which explains why we can improve the time-decay of the first (linear) term on the right-hand side of (8.1) from $N-2$ to N. On the other hand, there is no such gain in (8.3) and (8.4). What saves us here is that both the second and third terms on the right-hand side of (8.1) are truly *nonlinear terms* in $\tilde{\varepsilon}$.
As a technical tool we introduce an inhomogeneous Littlewood-Paley decomposition

$$f = \sum_{\lambda=1}^{\infty} P_\lambda f = \sum_\lambda \int_0^\infty p_\lambda(\xi)\phi(R,\xi)\widehat{f}(\xi)\rho(\xi)\,d\xi$$

corresponding to a smooth partition of unity $\{p_\lambda\}$ in the Fourier space. Here $\lambda \in \{2^j\}_{j=0}^\infty$ and p_λ is adapted to frequencies of size λ. Our first result is

Lemma 8.1. *Let $q \in S(1,\mathcal{Q})$ and $|\alpha| < \frac{\nu}{2} + \frac{3}{4}$. Then*

$$\|qf\|_{H^\alpha_\rho} \lesssim \|f\|_{H^\alpha_\rho}$$

Proof. We decompose the multiplication operator into its Littlewood-Paley pieces:

$$q = \sum_{\lambda,\mu} P_\lambda\, q P_\mu$$

The diagonal sum corresponding to $\lambda \asymp \mu$ is estimated using only the L^∞ bound on q. For the off-diagonal component it suffices to show rapid decay. In fact, we claim that

$$\|P_\lambda\, qP_\mu\|_{L^2\to L^2} \lesssim (\mu+\lambda)^{-\frac{1}{4}-\frac{\nu}{2}}[\log(\mu+\lambda)]^m$$

where m is some large integer. The Fourier kernel of $P_\lambda\, qP_\mu$ is

$$K_{\lambda,\mu}(\eta,\xi) = \sqrt{\rho(\xi)\rho(\eta)}\; p_\lambda(\xi)p_\mu(\eta)\int q(R)\phi(\xi,R)\phi(\eta,R)\,dR$$

in the sense that

$$\sqrt{\rho(\eta)}\;\mathcal{F}(P_\lambda\, qP_\mu f)(\eta) = \int K_{\lambda,\mu}(\eta,\xi)\,\widehat{f}(\xi)\sqrt{\rho(\xi)}\,d\xi$$

Therefore, the above L^2 bound would follow from the pointwise estimate (recall $\rho(\xi) \asymp \xi$ for $\xi > 1$)

$$\left| \int q(R)\phi(\xi,R)\phi(\eta,R)dR \right| \lesssim \langle \xi \rangle^{-1} \langle \eta \rangle^{-1} \langle \xi+\eta \rangle^{-\frac14-\frac{\nu}{2}} [\log(2+\xi+\eta)]^m$$

The function q has a symbol type behavior with respect to R except near $R=\tau$, where it has a power type singularity $(1-a)^{\nu+\frac12}$, $a=R/\tau$, possibly involving also logarithms[13]. To separate this singularity from the behavior at 0 we use a smooth cutoff to split q into (recall that τ is a large parameter)

$$q = q_{<\tau/2} + q_{>\tau/2}$$

The first term is a symbol of order 0 with respect to R. To proceed, we recall the calculations leading up to (5.8). The main tool there is the following double commutator identity: if $\xi \neq \eta$ and U is a zero order symbol, then

$$\begin{aligned} &(\xi-\eta)^2 \langle U(R)\phi(R,\xi), \phi(R,\eta)\rangle = \Big\langle \big[[U,\mathcal{L}],\mathcal{L}\big]\phi(R,\xi), \phi(R,\eta)\Big\rangle \\ &= \Big\langle \big(-4U_{RR}\xi + 3R^{-2}(U_{RR} - R^{-1}U_R) + 4U_{RR}V + U_{RRRR} + 2U_RV_R + 4U_{RRR}\partial_R \big)\phi(R,\xi), \phi(R,\eta)\Big\rangle \end{aligned} \tag{8.5}$$

where the inner products exist in the principal value sense (recall that $V(R) = -8(1+R^2)^{-2}$). Iterating this identity k times yields

$$(\xi-\eta)^{2k} \Big\langle q_{<\tau/2}(R)\phi(R,\xi), \phi(R,\eta)\Big\rangle = \Big\langle \Big[\sum_{j=0}^{k-1} \xi^j q_j^{odd}(R)\partial_R + \sum_{\ell=0}^{k} \xi^\ell q_\ell^{even}(R) \Big] \phi(R,\xi), \phi(R,\eta)\Big\rangle$$

where q_j^{odd} and q_ℓ^{even} are symbols of order at most $-2k$ with odd, respectively even, expansions around $R=0$. For $1+\xi \not\asymp 1+\eta$ this gives

$$|\langle q_{<\tau/2}(R)\phi(R,\xi), \phi(R,\eta)\rangle| \lesssim \langle \xi+\eta\rangle^{-k}$$

for all k which is more than we need.

The second term $q_{>\tau/2}$ can be thought of as a function of a,

$$q_{>\tau/2}(R) = q_1(a), \qquad a = \frac{R}{\tau}$$

where q_1 is supported in $[\frac12, 2]$ and has a $\mathcal{Q}$ type singularity[14] at $a=1$. We divide it into a singular and a nonsingular component,

$$q_{>\tau/2} = q^{\rm s}_{>\tau/2} + q^{\rm ns}_{>\tau/2}, \qquad q^{\rm s}_{>\tau/2} := q_{>\tau/2}\chi_{[|R-\tau|<\langle\xi+\eta\rangle^{-\frac12}]}, \qquad q^{\rm ns}_{>\tau/2} := q_{>\tau/2}\chi_{[|R-\tau|>\langle\xi+\eta\rangle^{-\frac12}]},$$

where the χ's define a smooth partition of unity relative to the indicated sets. For the singular component we bound the integral directly using the pointwise bounds on $\phi(R,\xi)$ to obtain

$$\begin{aligned} \left| \int q^{\rm s}_{>\tau/2}(R)\phi(R,\xi)\phi(R,\eta)\, dR \right| &\lesssim \int_{\frac{\tau}{2}}^{\tau} (1-R/\tau)^{\nu+\frac12} |\log(1-R/\tau)|^m \, 1_{[|R-\tau|<\langle\xi+\eta\rangle^{-\frac12}]} \langle\xi\rangle^{-\frac34}\langle\eta\rangle^{-\frac34}\, dR \\ &\lesssim \langle\xi\rangle^{-\frac34}\langle\eta\rangle^{-\frac34} \tau^{-\nu-\frac12} \langle\xi+\eta\rangle^{-\frac{\nu}{2}-\frac34} [\log(2+\xi+\eta)]^m \\ &\lesssim \langle\xi\rangle^{-1}\langle\eta\rangle^{-1}\langle\xi+\eta\rangle^{-\frac{\nu}{2}-\frac14}[\log(2+\xi+\eta)]^m \end{aligned}$$

For the nonsingular component, a k-fold iteration of (8.5) yields

$$(\xi-\eta)^{2k} \Big\langle q^{\rm ns}_{>\tau/2}(R)\phi(R,\xi), \phi(R,\eta)\Big\rangle = \Big\langle \Big[\sum_{j=0}^{k-1} \xi^j q^{\rm odd}_{k,j}(R)\partial_R + \sum_{\ell=0}^{k} \xi^\ell q^{\rm even}_{k,\ell}(R) \Big] \phi(R,\xi), \phi(R,\eta)\Big\rangle \tag{8.6}$$

with

$$q^{\rm odd}_{k,j}(R) = \sum_{i=0}^{2k-j-1} r^{\rm odd}_{k,j,i}(R)\, \partial_R^{2i+1} q^{\rm ns}_{>\tau/2}(R), \qquad q^{\rm even}_{k,\ell}(R) = \sum_{i=1}^{2k-\ell} r^{\rm even}_{k,\ell,i}(R)\, \partial_R^{2i} q^{\rm ns}_{>\tau/2}(R)$$

[13]Strictly speaking, there is a multiplicative constant in $a = cR/\tau$, but we ignore it

[14]q_1 also has a nonsingular part, which by a slight abuse of notation we include in $q_{<\tau/2}$

where the coefficients are rational functions, smooth for all $R \geq 0$, decaying at rates

$$|r^{\rm odd}_{k,j,i}(R)| \lesssim R^{-2-(4k-2j-2i)}, \qquad |r^{\rm even}_{k,\ell,i}(R)| \lesssim R^{-4-(4k-2\ell-2i)}$$

The logic behind the numerology here is simple: a factor ξ^j consumes $2j$ derivatives, so the remaining $4k$ derivatives need to hit either the symbol $q^{\rm ns}_{>\tau/2}(R)$ or the weight V (the latter leading to the rational functions).

We show how to apply these formulas for the case of the even weights, the odd ones being analogous. As for the derivatives

$$\partial_R^{2i} q^{\rm ns}_{>\tau/2}(R) = \partial_R^{2i}\Big(q_{>\tau/2}\chi_{[|R-\tau|>\langle\xi+\eta\rangle^{-\frac12}]}\Big)$$

it will suffice to consider two extreme cases: when all derivatives fall on the symbol, or all fall on the cut-off function. The contribution by the latter to $|\langle q^{\rm ns}_{>\tau/2}(R)\phi(R,\xi),\phi(R,\eta)\rangle|$ is bounded by (ignoring logs)

$$\begin{aligned}
&(\xi+\eta)^{-2k}\int_{[|R-\tau|\asymp\langle\xi+\eta\rangle^{-\frac12}]} R^{-4-(4k-2\ell-2i)}(1-a)^{\nu+\frac12}\langle\xi+\eta\rangle^i\xi^\ell\langle\xi\rangle^{-\frac34}\langle\eta\rangle^{-\frac34}\,dR\\
&\lesssim (\xi+\eta)^{-2k}\tau^{-3-(4k-2\ell-2i)}\tau^{-\nu-\frac32}\langle\xi+\eta\rangle^{-\frac{\nu}{2}-\frac34+i}\langle\xi\rangle^{\ell-\frac34}\langle\eta\rangle^{-\frac34}\\
&\lesssim \langle\xi\rangle^{-1}\langle\eta\rangle^{-1}\langle\xi+\eta\rangle^{-\frac{\nu}{2}-\frac14}
\end{aligned}$$

as desired. The other cases are checked similarly and we skip them. □

This allows us to deal with the coefficients in front of the $\tilde\varepsilon$ terms. As remarked above, the τ decay for the first term in N_{2k-1} comes from the τ^{-2} factor in the coefficient and from the quadratic (respectively, cubic) expressions in $\tilde\varepsilon$ for the remaining terms. It remains to prove the following:

Proposition 8.2. *Let $\alpha > \frac14$. Then the maps*

$$\tilde\varepsilon \mapsto R^{-\frac12}(\cos(2\tilde\varepsilon R^{-\frac12})-1) \tag{8.7}$$

$$\tilde\varepsilon \mapsto R^{-\frac32}(\sin(2\tilde\varepsilon R^{-\frac12})-2\tilde\varepsilon R^{-\frac12}) \tag{8.8}$$

are locally Lipschitz from $H^{\alpha+1/2}_\rho$ to H^α_ρ.

The proof will be split up into the following four lemmas. We first obtain a pointwise bound for frequency localized L^2 functions:

Lemma 8.3. *For dyadic $\lambda \geq 1$ we have*

$$|P_\lambda f(R)| \lesssim \lambda\min\{R^{\frac32},\lambda^{-\frac34}\}\|f\|_{L^2}$$

for all $f\in L^2(\mathbb{R}^+)$.

Proof. Using the inversion formula we write

$$P_\lambda f(R) = \int_0^\infty p_\lambda(\xi)\widehat{f}(\xi)\phi(R,\xi)\rho(\xi)\,d\xi$$

The pointwise bounds for ϕ,

$$|\phi(R,\xi)| \lesssim \min\{R^{\frac32},\xi^{-\frac34}\}$$

and the Cauchy-Schwarz inequality finish the proof. □

We also have estimates for the derivative:

Lemma 8.4. *For dyadic $\lambda\geq 1$ we have*

$$|\partial_R P_\lambda f(R)| \lesssim \lambda\min\{R^{\frac12},\lambda^{-\frac14}\}\|f\|_{L^2}$$

and

$$\|\partial_R P_\lambda f\|_{L^2} \lesssim \lambda^{\frac12}\|f\|_{L^2}$$

for all $f\in L^2(\mathbb{R}^+)$.

Proof. The first estimate follows from the pointwise bounds on $\partial_R \phi$. For the second bound we can integrate by parts (justified by the first bound) to obtain

$$\lambda \|P_\lambda f(R)\|_{L^2}^2 \gtrsim \Big\langle \mathcal{L} f, f \Big\rangle \geq \|\partial_R f\|_{L^2}^2 + \frac{3}{4}\|R^{-1} f\|_{L^2}^2 - C\|f\|_{L^2}^2$$

which leads to the desired conclusion. □

Next we consider bilinear estimates but with a weight that is singular at 0. This suffices in order to estimate the quadratic and the cubic terms in the proposition. The logic behind the Lemma 8.5 is the following: dividing by $R^{\frac{3}{2}}$ should amount to a loss of $\xi^{\frac{3}{4}}$ on the Fourier side (since the scaling relation is $R\xi^{\frac{1}{2}} = 1$). Inspection of the following estimates shows that we do indeed lose a combined $\frac{3}{4}$ weight in ξ on the right-hand side.

Lemma 8.5. *Let $\alpha > \frac{1}{4}$. Then*

$$\|R^{-\frac{3}{2}} fg\|_{H_\rho^{\alpha+\frac{1}{4}}} \lesssim \|f\|_{H_\rho^{\alpha+\frac{1}{2}}} \|g\|_{H_\rho^{\alpha+\frac{1}{2}}}$$

respectively

$$\|R^{-\frac{3}{2}} fg\|_{H_\rho^{\alpha}} \lesssim \|f\|_{H_\rho^{\alpha+\frac{1}{4}}} \|g\|_{H_\rho^{\alpha+\frac{1}{2}}}$$

for all f, g so that the right-hand sides are finite.

Proof. We first use the above pointwise bound to obtain an L^2 estimate,

$$\|R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g\|_{L^2} \lesssim \min\{\lambda_1, \lambda_2\} \|P_{\lambda_1} f\|_{L^2} \|P_{\lambda_2} g\|_{L^2}$$

This suffices for both of the above estimates provided that the output is measured at frequency $\sigma \lesssim \max\{\lambda_1, \lambda_2\}$. Indeed, in that case

$$\begin{aligned}
&\sum_{\lambda_1,\lambda_2} \sum_{\sigma < \max(\lambda_1,\lambda_2)} \sigma^{\alpha+\frac{1}{4}} \big\|P_\sigma[R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g]\big\|_2 \\
&\lesssim \sum_{\lambda_1 > \lambda_2} \lambda_1^{\alpha+\frac{1}{4}} \lambda_2 \|P_{\lambda_1} f\|_2 \|P_{\lambda_2} g\|_2 + \sum_{\lambda_1 \leq \lambda_2} \lambda_2^{\alpha+\frac{1}{4}} \lambda_1 \|P_{\lambda_1} f\|_2 \|P_{\lambda_2} g\|_2 \\
&\lesssim \sum_{\lambda_1 > \lambda_2} \lambda_1^{-\frac{1}{4}} \lambda_2^{\frac{1}{2}-\alpha} \|f\|_{H_\rho^{\alpha+\frac{1}{2}}} \|g\|_{H_\rho^{\alpha+\frac{1}{2}}}
\end{aligned}$$

which gives the desired bound since $\alpha > \frac{1}{4}$.
For larger σ, however, we need some additional decay. For this we compute using integration by parts

$$\begin{aligned}
\Big\langle R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g, P_\sigma h \Big\rangle &= \Big\langle R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g, \mathcal{L}^k \mathcal{L}^{-k} P_\sigma h \Big\rangle \\
&= \Big\langle \mathcal{L}^k (R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g), \mathcal{L}^{-k} P_\sigma h \Big\rangle
\end{aligned}$$

To justify the integration by parts we observe that near $R = 0$ we have

$$P_{\lambda_1} f(R) = R^{\frac{3}{2}} q(R^2), \qquad q \text{ analytic}$$

Then the bilinear form is given by

$$R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g = R^{\frac{3}{2}} q(R^2), \qquad q \text{ analytic}$$

which successively implies that (recall $\mathcal{L}_0 R^{\frac{3}{2}} = 0$)

$$\mathcal{L}^k (R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g) = R^{\frac{3}{2}} q(R^2), \qquad q \text{ analytic}$$

We claim that we can estimate the left-hand side here in L^2 by

$$\|\mathcal{L}^k (R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g)\|_{L^2} \lesssim \min\{\lambda_1, \lambda_2\} \max\{\lambda_1, \lambda_2\}^k \|P_{\lambda_1} f\|_{L^2} \|P_{\lambda_2} g\|_{L^2} \tag{8.9}$$

Given the above integration by parts, this implies that

$$\Big|\Big\langle R^{-\frac{3}{2}} P_{\lambda_1} f P_{\lambda_2} g, P_\sigma h \Big\rangle\Big| \lesssim \min\{\lambda_1, \lambda_2\} \max\{\lambda_1, \lambda_2\}^k \sigma^{-k} \|P_{\lambda_1} f\|_{L^2} \|P_{\lambda_2} g\|_{L^2} \|P_\sigma h\|_{L^2}$$

and further

$$\|P_\sigma(R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g)\|_{L^2} \lesssim \min\{\lambda_1,\lambda_2\}\max\{\lambda_1,\lambda_2\}^k\sigma^{-k}\|P_{\lambda_1}f\|_{L^2}\|P_{\lambda_2}g\|_{L^2}$$

thus providing the additional decay for large σ.

It remains to prove (8.9). We assume that $\lambda_1 < \lambda_2$ and use different bounds depending on whether R is small or large. Assume first that $R < \lambda_2^{-\frac{1}{2}}$. Then we start from

$$\mathcal{L}^k(R^{-\frac{3}{2}}P_{\lambda_1}f(R)\,P_{\lambda_2}g(R)) = \int_0^\infty\int_0^\infty p_{\lambda_1}(\xi)p_{\lambda_2}(\eta)\mathcal{L}^k[R^{-\frac{3}{2}}\phi(R,\xi)\phi(R,\eta)]\,\widehat{f}(\xi)\widehat{g}(\eta)\,\rho(\xi)\rho(\eta)\,d\xi d\eta \tag{8.10}$$

Next, we claim that

$$\|\mathcal{L}^k[R^{-\frac{3}{2}}\phi(R,\xi)\phi(R,\eta)]\|_{L^2(0,\lambda_2^{-\frac{1}{2}})} \lesssim \lambda_2^{k-1} \tag{8.11}$$

If true, then combining (8.10) and (8.11) via Minkowski and Cauchy-Schwarz yields

$$\|\mathcal{L}^k(R^{-\frac{3}{2}}P_{\lambda_1}f(R)\,P_{\lambda_2}g(R))\|_{L^2(0,\lambda_2^{-\frac{1}{2}})} \lesssim \lambda_2^k\lambda_1\|P_{\lambda_1}f\|_2\|P_{\lambda_2}g\|_2$$

as desired. To prove (8.11), consider first $k=0$. Then

$$\|R^{-\frac{3}{2}}\phi(R,\xi)\phi(R,\eta)\|_{L^2(0,\lambda_2^{-\frac{1}{2}})} \lesssim \left(\int_0^{\lambda_2^{-\frac{1}{2}}} R^3\,dR\right)^{\frac{1}{2}} \lesssim \lambda_2^{-1}$$

The higher k cases now follow from Proposition 4.4, which allows us to write

$$R^{-\frac{3}{2}}\phi(R,\xi)\phi(R,\eta) = R^{\frac{3}{2}}q(R^2,\xi R^2,\eta R^2), \qquad q \text{ analytic}$$

Then, following our previous discussion concerning applications of $\mathcal{L}^k$, we obtain

$$\mathcal{L}^k(R^{-\frac{3}{2}}\phi(R,\xi)\phi(R,\eta)) = \sum_{\ell+m\le k} R^{\frac{3}{2}}\xi^\ell\eta^m q_{\ell m}(R^2,\xi R^2,\eta R^2), \qquad q_{\ell m} \text{ analytic}$$

which implies (8.11).

For large R we use the product rule to write

$$\mathcal{L}^k(R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g) = \sum_{\substack{2i+2j\le 2k-\ell-m\\ \ell,m=0,1}} W_{ij}^{\ell m}(R)\,\partial_R^\ell\mathcal{L}^iP_{\lambda_1}f\cdot\partial_R^m\mathcal{L}^jP_{\lambda_2}g$$

$$|W_{ij}^{\ell m}(R)| \lesssim R^{-2(k-i-j)+\ell+m-\frac{3}{2}}$$

Then we have

$$\|\mathcal{L}^k(R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g)\|_{L^2(\lambda_2^{-\frac{1}{2}},\infty)} \lesssim \sum_{\substack{2i+2j\le 2k-\ell-m\\ \ell,m=0,1}} \lambda_2^{(k-i-j)-\frac{\ell+m}{2}}\|R^{-\frac{3}{2}}\partial_R^\ell\mathcal{L}^iP_{\lambda_1}f\cdot\partial_R^m\mathcal{L}^jP_{\lambda_2}g\|_{L^2}$$

We use Lemmas 8.4, 8.3 to bound the first factor in L^∞ and the second in L^2. This gives

$$\|\mathcal{L}^k(R^{-\frac{3}{2}}P_{\lambda_1}fP_{\lambda_2}g)\|_{L^2(\lambda_2^{-\frac{1}{2}},\infty)} \lesssim \lambda_2^k\lambda_1\|P_{\lambda_1}f\|_{L^2}\|P_{\lambda_2}g\|_{L^2}$$

as desired. □

Finally, in order to estimate the higher order terms in the Taylor expansion of the sin and cos functions in the proposition we also prove a trilinear estimate:

Lemma 8.6. *Let $\alpha > 0$. Then*

$$\|R^{-1}fgh\|_{H^\alpha_\rho} \lesssim \|f\|_{H_\rho^{\alpha+\frac{1}{2}}}\|g\|_{H_\rho^{\alpha+\frac{1}{2}}}\|h\|_{H^\alpha_\rho}$$

for all f, g, h so that the right-hand side is finite.

Proof. The pointwise bounds above imply the following L^2 estimate,

$$\|R^{-1}P_{\lambda_1}fP_{\lambda_2}gP_{\lambda_3}h\|_{L^2} \lesssim \min_{i\neq j}\{\lambda_i^{\frac14}\lambda_j^{\frac34}\}\|P_{\lambda_1}f\|_{L^2}\|P_{\lambda_2}g\|_{L^2}\|P_{\lambda_3}h\|_{L^2} \tag{8.12}$$

which again suffices to estimate the output at frequency $\sigma \le \lambda := \max\{\lambda_1,\lambda_2,\lambda_3\}$. To see this, we write, with the summation variables $\sigma, M, \lambda_1,\lambda_2,\lambda_3 \in \{2^j\}_{j=0}^\infty$,

$$\sum_{\lambda_1,\lambda_2,\lambda_3}\sum_{\sigma\le\lambda} P_\sigma(R^{-1}P_{\lambda_1}fP_{\lambda_2}gP_{\lambda_3}h) = \sum_M \sum_{\substack{\lambda_1,\lambda_2,\lambda_3\\ \lambda\ge M}} P_{\lambda/M}\big(R^{-1}P_{\lambda_1}fP_{\lambda_2}gP_{\lambda_3}h\big)$$

On the right-hand side we distinguish the cases $\lambda_1\le\lambda_2<\lambda_3$, $\lambda_1\le\lambda_3\le\lambda_2$, $\lambda_3<\lambda_1\le\lambda_2$. We only treat the first case, the other two being similar and easier. Thus, we estimate the right-hand side for fixed M as follows:

$$\begin{aligned}\Big\|\sum_{\substack{\lambda_1\le\lambda_2<\lambda_3\\ \lambda_3\ge M}} P_{\lambda_3/M}\big(R^{-1}P_{\lambda_1}fP_{\lambda_2}gP_{\lambda_3}h\big)\Big\|_{H^\alpha_\rho}^2 &\lesssim \sum_{\lambda_3>M}\Big(\frac{\lambda_3}{M}\Big)^{2\alpha}\|P_{\lambda_3}h\|_{L^2}^2\Big(\sum_{\lambda_1\le\lambda_2}\lambda_1^{\frac34}\lambda_2^{\frac14}\|P_{\lambda_1}f\|_{L^2}\|P_{\lambda_2}g\|_{L^2}\Big)^2\\ &\lesssim \sum_{\lambda_3>M}\Big(\frac{\lambda_3}{M}\Big)^{2\alpha}\|P_{\lambda_3}h\|_{L^2}^2\Big(\sum_{\lambda_1<\lambda_2}\lambda_1^{\frac14-\alpha}\lambda_2^{-\frac14-\alpha}\|f\|_{H_\rho^{\alpha+\frac12}}\|g\|_{H_\rho^{\alpha+\frac12}}\Big)^2\\ &\lesssim M^{-2\alpha}\|f\|^2_{H_\rho^{\alpha+\frac12}}\|g\|^2_{H_\rho^{\alpha+\frac12}}\|h\|^2_{H^\alpha_\rho}\end{aligned}$$

The summation with respect to M is trivial.
For higher frequency outputs we need some additional decay,

$$\|P_\sigma(R^{-1}P_{\lambda_1}fP_{\lambda_2}gP_{\lambda_3}h)\|_{L^2} \lesssim \min_{i\neq j}\{\lambda_i^{\frac14}\lambda_j^{\frac34}\}\max\{\lambda_1,\lambda_2,\lambda_3\}^k\sigma^{-k}\|P_{\lambda_1}f\|_{L^2}\|P_{\lambda_2}g\|_{L^2}\|P_{\lambda_3}h\|_{L^2}$$

This in turn is a consequence of the estimate

$$\|\mathcal{L}^k(R^{-1}P_{\lambda_1}fP_{\lambda_2}gP_{\lambda_3}h)\|_{L^2} \lesssim \min_{i\neq j}\{\lambda_i^{\frac14}\lambda_j^{\frac34}\}\max\{\lambda_1,\lambda_2,\lambda_3\}^k\|P_{\lambda_1}f\|_{L^2}\|P_{\lambda_2}g\|_{L^2}\|P_{\lambda_3}h\|_{L^2}$$

which is proved in the same manner as (8.9). □

These lemmas now imply Proposition 8.2. Indeed, we express the cosine-map in (8.7) in the form

$$R^{-\frac12}(\cos(2\tilde\varepsilon R^{-\frac12})-1) = R^{-\frac32}\tilde\varepsilon^2 q(R^{-1}\tilde\varepsilon^2) \qquad q \text{ entire}$$

The first factor is bounded by

$$\|R^{-\frac32}\tilde\varepsilon^2\|_{H^\alpha_\rho} \lesssim \|\tilde\varepsilon\|^2_{H_\rho^{\alpha+\frac12}}$$

while for q we use its Taylor series together with Lemma 8.6, which shows that as a multiplication operator the factor $R^{-1}\tilde\varepsilon^2$ can be bounded by

$$\|R^{-1}\tilde\varepsilon^2\|_{H^\alpha_\rho\to H^\alpha_\rho} \lesssim \|\tilde\varepsilon\|^2_{H_\rho^{\alpha+\frac12}} \tag{8.13}$$

Similarly, we write the sine-map from (8.8) in the form

$$R^{-\frac32}(\sin(2\tilde\varepsilon R^{-\frac12}) - 2\tilde\varepsilon R^{-\frac12}) = R^{-3}\tilde\varepsilon^3 q(R^{-1}\tilde\varepsilon^2)$$

For the first factor we apply Lemma 8.5 twice to estimate

$$\|R^{-3}\tilde\varepsilon^3\|_{H^\alpha_\rho} \lesssim \|R^{-\frac32}\tilde\varepsilon^2\|_{H_\rho^{\alpha+\frac14}}\|\tilde\varepsilon\|_{H_\rho^{\alpha+\frac12}} \lesssim \|\tilde\varepsilon\|^3_{H_\rho^{\alpha+\frac12}}$$

while for the q factor we use again (8.13).

9. Proof of the main theorem.

Here we summarize how to assemble together the elements of the proof. Fixing $\nu > \frac{1}{2}$ we begin with the approximate solution u_{2k-1} given by Theorem 2.1 and with the corresponding error e_{2k-1}. The index k is chosen sufficiently large, depending on ν. Apriori both u_{2k-1} and e_{2k-1} are defined only inside the cone $\{r \leq t\}$. We can extend them to functions with similar regularity supported in a double cone $\{r \leq 2t\}$. This extension is done crudely, without any reference to the equation but insuring the matching on the cone for all derivatives which are meaningful.
With these choices for u_{2k-1} and e_{2k-1} we seek to solve (3.2) backward in τ and find a solution $\tilde{\varepsilon}$ so that

$$\|\tilde{\varepsilon}(\tau)\|_{H_\rho^{\alpha+\frac{1}{2}}} \lesssim \tau^{2-N}, \qquad \left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}\right)\tilde{\varepsilon}(\tau)\right\|_{H_\rho^\alpha} \lesssim \tau^{1-N}, \qquad N \leq 2k \tag{9.1}$$

Here the exponent α is chosen so that

$$\frac{1}{4} < \alpha < \frac{\nu}{2}$$

The first bound is solely dictated by estimates for the cubic term in the nonlinearity. The second one is a consequence of the regularity of e_{2k-1}; namely, e_{2k-1} has a singularity of type $(1-a)^{\nu-\frac{1}{2}}\log^m(1-a)$ on the cone. Therefore, if $\alpha < \nu/2$, then for e_{2k-1} we have the bound

$$\|\lambda^{-2}R^{\frac{1}{2}}e_{2k-1}(t(\tau),\lambda^{-1}R)\|_{H_\rho^\alpha} \lesssim \tau^{-2k+2}$$

Using the transference identity we recast (3.2) for $\tilde{\varepsilon}$ in the form (6.1) with $x = \mathcal{F}\tilde{\varepsilon}$. By virtue of Propositions 6.1, 6.3, 5.2 we can solve (6.1) using the contraction principle with respect to the norm

$$\|x\|_{L^{\infty,N-2}L_\rho^{2,\alpha+\frac{1}{2}}} + \|(\partial_\tau - \frac{\lambda_\tau}{\lambda})x\|_{L^{\infty,N-1}L_\rho^{2,\alpha}}$$

Using again the transference identity and Proposition 5.2 we return back to $\tilde{\varepsilon}$, which has the regularity (9.1). Now eventually we have to return to the original coordinates (t,r) as well as the function $\varepsilon(t,r)$. For this we define the map

$$u(R) \to Tu(R,\theta) = e^{i\theta}R^{-\frac{1}{2}}u(R)$$

where the right hand side is interpreted as a function in $\mathbb{R}^2$ expressed in polar coordinates (R,θ). It is easy to see that this is an isometry

$$T : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^2)$$

Then for the corresponding Sobolev spaces we have

Lemma 9.1. *For any $\alpha \geq 0$ we have*

$$\|u\|_{H_\rho^{\alpha/2}(\mathbb{R}^+)} \asymp \|Tu\|_{H^\alpha(\mathbb{R}^2)}$$

in the sense that if one side is finite then the other is finite and they have comparable sizes.

Proof. The spaces $H_\rho^\beta(\mathbb{R}^+)$ are defined using fractional powers of the operator $\mathcal{L}$. However, we can also define them using fractional powers of the operator $\mathcal{L}_0$ since the difference $\mathcal{L} - \mathcal{L}_0$ is bounded in L^2 and also in any H_ρ^β. This is easily seen if β is an integer, and for noninteger values it follows by interpolation.

Then the conclusion of the lemma follows from the identity

$$\Delta Tu = T\mathcal{L}_0 u$$

which is valid whenever $u \in L^2$ and $\mathcal{L}_0 u \in L^2$. □

To pass from $u(\tau,R)$, or alternatively $u(t,r)$, to the co-rotational wave map in terms of the ambient coordinates of $\mathbb{R}^3 \supset S^2$, observe that these coordinates are given by $\phi \circ T(u)$, where $\phi : \mathbb{R}^2 \longrightarrow S^2 \subset \mathbb{R}^3$ is given by

$$\phi(\rho e^{i\theta}) = (\cos\rho, \sin\rho\cos\theta, \sin\rho\sin\theta)$$

It is then easily seen that $\phi \circ T(u) \in H^{2\alpha+1}(\mathbb{R}^2)$, interpreted component-wise. We have now constructed a wave map on the cone $r \leq t$, $0 < t < t_0$, which is of class $H^{1+\nu-}$ on the closure of the cone. To get a solution on all of $\mathbb{R}^{2+1}$, extend the solution $\partial_t u(t_0,.), u(t_0,.)$ at time $t = t_0$ to all of $\mathbb{R}^2$ within the same smoothness and equivariance class. Call the corresponding wave map $\tilde{u}(t,r)$. We claim that this wave map extends to

$(0,t_0]\times\mathbb{R}^2$ and is of class $H^{1+\nu-}$ until breakdown at time $t=0$. Indeed, by finite propagation speed $\tilde{u}(t,r)$ is given by $u(t,r)$ on the light cone $r\le t, 0<t\le t_0$. Furthermore, the $\tilde{u}$ does not develop singularities on the interval $0<t<t_0$, as this could only happen outside the light cone, where energy concentration is precluded by the equivariance condition. The fact that singularity formation is tantamount to an energy concentration scenario is a consequence of [Sh-Tah], [Tao], [Tat]. This concludes the proof of Theorem 1.1.

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