HOFER-ZEHNDER CAPACITY AND A HAMILTONIAN CIRCLE ACTION WITH NONCONTRACTIBLE ORBITS

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ABSTRACT. Let (M,ω) be an aspherical symplectic manifold, which is closed or convex. Let U be an open set in M, which admits a circle action generated by an autonomous Hamiltonian $H \in C^\infty(U)$, such that each orbit of the circle action is not contractible in M. Under these assumptions, we prove that the Hofer-Zehnder capacity of U is bounded by the Hofer norm of H. The proof uses a variant of the energy-capacity inequality, which is proved by the theory of action selectors.

1. Introduction

1.1. **Hofer-Zehnder capacity.** First we fix some notations. We set $S^1 := \mathbb{R}/\mathbb{Z}$. For any topological space X, we set $\pi_1'(X) := C^0(S^1, X)/\sim$, where $\gamma \sim \gamma'$ means that γ and γ' are homotopic. For each $\gamma \in C^0(S^1, X)$, $\bar{\gamma} \in C^0(S^1, X)$ is defined as $\bar{\gamma}(t) := \gamma(-t)$. Since $\gamma \sim \gamma' \implies \bar{\gamma} \sim \bar{\gamma}'$, one can define $\bar{\alpha} \in \pi_1'(X)$ for any $\alpha \in \pi_1'(X)$. When X is path connected, c_X denotes the element in $\pi_1'(X)$ which consists of contractible loops on X.

We introduce a refinement of the Hofer-Zehnder capacity, taking into account free homotopy classes of periodic orbits. Let (M,ω) be a symplectic manifold. We always assume that $\partial M = \emptyset$. For $H \in C^\infty(M)$, its Hamiltonian vector field $X_H \in \mathscr{X}(M)$ is defined by the equation $\omega(X_H, \cdot\,) = -dH(\,\cdot\,)$. For any $S \subset \pi_1'(M)$, $\mathscr{H}^S_{\mathrm{HZ}}(M,\omega)$ denotes the set of $H \in C_0^\infty(M)$ which satisfies the following properties:

(1) $H \le 0$ and $\{H = 0\} \ne \emptyset$.
(2) There exists a nonempty open set $U \subset M$ such that $H|_U \equiv \min H$.
(3) Any nonconstant periodic orbit γ of X_H satisfying $[\gamma] \in S$ has period > 1.

Then we define

$$c^S_{\mathrm{HZ}}(M,\omega) := \sup\{-\min H \mid H \in \mathscr{H}^S_{\mathrm{HZ}}(M,\omega)\}.$$

Following properties are immediate from the definition:

- For any $S, S' \subset \pi_1'(M)$, $S \subset S' \implies c^S_{\mathrm{HZ}}(M,\omega) \ge c^{S'}_{\mathrm{HZ}}(M,\omega)$.
- For any nonempty open set U in M, let $i^M_U : U \to M$ denote the inclusion map, and let $(i^M_U)_* : \pi_1'(U) \to \pi_1'(M)$ denote the induced map. Then, for any $S \subset \pi_1'(M)$, $c^{(i^M_U)_*^{-1}(S)}_{\mathrm{HZ}}(U,\omega) \le c^S_{\mathrm{HZ}}(M,\omega)$.
- Abbreviateing $c_{\mathrm{HZ}}(M,\omega) := c^{\pi_1'(M)}_{\mathrm{HZ}}(M,\omega)$, $c_{\mathrm{HZ}}(M,\omega) \le c^S_{\mathrm{HZ}}(M,\omega)$ for any $S \subset \pi_1'(M)$. Moreover, $c_{\mathrm{HZ}}(U,\omega) \le c_{\mathrm{HZ}}(M,\omega)$ for any nonempty open set U in M.

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$c_{\mathrm{HZ}}(M,\omega)$ defined as above coincides with the original Hofer-Zehnder capacity ([3], [4]).

1.2. **Main result.** First we fix some terminologies. Let (M,ω) be a symplectic manifold such that $\partial M = \emptyset$.

- (M,ω) is called *aspherical* when $\omega|_{\pi_2(M)} = 0$.
- (M,ω) is called *convex* when there exists an increasing sequence $M_1 \subset M_2 \subset \cdots$ of compact codimension 0 submanifolds of M, which satisfies $\bigcup_i M_i = M$ and the following property for each $i \geq 1$: there exists a vector field X_i defined on some neighborhood of ∂M_i in M_i, which points strictly outwards on ∂M_i and $L_{X_i}\omega = \omega$.
- A circle action $S^1 \circlearrowleft M$ is called *Hamiltonian* action generated by $H \in C^\infty(M)$, when there holds $\frac{d}{dt}(t \cdot x)|_{t=0} = X_H(x)$ for any $x \in M$.

The main result of this note is the following:

Theorem 1.1. *Let (M,ω) be a connected aspherical symplectic manifold, which is closed or convex. Let U be an open set in M, which admits a Hamiltonian circle action generated by $H \in C^\infty(U)$. Suppose that for any $x \in U$, $\gamma^x : S^1 \to M; t \mapsto t \cdot x$ is not contractible in M, and $[\gamma^x] \in \pi_1'(M)$ does not depend on $x \in U$. Then, setting $\alpha := [\gamma^x] \in \pi_1'(M)$, $c_{\mathrm{HZ}}^{(i_U^M)_*^{-1}(\{c_M,\bar{\alpha}\})}(U,\omega) \leq \sup H - \inf H$.*

Remark 1.2. In [7], L. Macarini gives a similar upper bound of $c_{\mathrm{HZ}}(U,\omega)$, provided that U is a connected open set in a geometrically bounded symplectic manifold, and U admits a free Hamiltonian circle action, which satisfies an additional condition on "the order of the action". For precise statement, see Theorem 1.1 in [7].

Theorem 1.1 is proved in section 2. First we give the following application:

Corollary 1.3. *Let N be a compact connected Riemannian manifold, ω_N be the standard symplectic form on T^*N, and $DT^*N := \{(q,p) \in T^*N \mid |p| < 1\}$. Suppose that N admits a circle action (which may not preserve the metric), such that for any $x \in N$, $\gamma^x : S^1 \to N; t \mapsto t \cdot x$ is not contractible. Then, setting $\alpha := [\gamma^x] \in \pi_1'(N)$,*

$$c_{\mathrm{HZ}}^{\{c_N,\bar{\alpha}\}}(DT^*N,\omega_N) \leq 2 \sup_{x \in N} \mathrm{length}(\gamma^x).$$

Proof. Let Z be a vector field on N, which generates the given circle action, i.e. $Z_x := \frac{d}{dt}(t \cdot x)|_{t=0}$ $(x \in N)$. Up to reparamatrization of the action, we may assume that $\sup_{x \in N} |Z_x| \leq \sup_{x \in N} \mathrm{length}(\gamma^x) =: L$. The circle action on N naturally extends to a Hamiltonian circle action on T^*N generated by $H \in C^\infty(T^*N)$, where $H(q,p) := p(Z_q)$. Since $DT^*N \subset H^{-1}\big((-L,L)\big)$, we get

$$c_{\mathrm{HZ}}^{\{c_N,\bar{\alpha}\}}(DT^*N,\omega_N) \leq c_{\mathrm{HZ}}^{\{c_N,\bar{\alpha}\}}(H^{-1}\big((-L,L)\big),\omega_N) \leq 2L,$$

where the second inequality follows from Theorem 1.1. □

Remark 1.4. In [5], the author proved $c_{\mathrm{HZ}}(DT^*N,\omega_N) < \infty$ under same assumption as Corollary 1.3, based on the correspondence between the pair-of-pants product in Floer homology of cotangent bundles and the loop product on homology of loop spaces.

As a specific case of Corollary 1.3, we recover the following result of M. Jiang [6]:

Corollary 1.5. *Let N be a flat torus: $N := \mathbb{R}/a_1\mathbb{Z} \times \cdots \times \mathbb{R}/a_n\mathbb{Z}$, where $n \geq 1$ and $0 < a_1 \leq \cdots \leq a_n$. Then, $c_{\mathrm{HZ}}(DT^*N, \omega_N) \leq 2a_1$.*

2. Proof

2.1. **Action selector.** To prove Theorem 1.1, we use the notion of *action selectors.* Let (M, ω) be an aspherical symplectic manifold, and $\mathscr{H}(M) := C_0^\infty(M \times [0,1])$.

For any $H \in \mathscr{H}(M)$ and $t \in [0,1]$, $H_t \in C_0^\infty(M)$ is defined as $H_t(x) := H(t,x)$. For any $H \in \mathscr{H}(M)$, $(\varphi_H^t)_{0 \leq t \leq 1}$ is the flow generated by $(X_{H_t})_{0 \leq t \leq 1}$. i.e. $\varphi_H^t : M \to M$ is defined as

$$\varphi_H^0 = \mathrm{id}_M, \qquad \tfrac{d}{dt}\varphi_H^t = X_{H_t}(\varphi_H^t)\,(0 \leq t \leq 1).$$

For any $H, K \in \mathscr{H}(M)$, we define $\bar{H}, H * K \in \mathscr{H}(M)$ as

$$\bar{H}(t,x) := -H(t, \varphi_H^t(x)), \qquad H * K(t,x) := H(t,x) + K\big(t, (\varphi_H^t)^{-1}(x)\big).$$

It is easy to verifty the following properties:

- $\varphi_{\bar{H}}^t = (\varphi_H^t)^{-1}$, $\varphi_{H*K}^t = \varphi_H^t \circ \varphi_K^t$ for any $0 \leq t \leq 1$.
- $(\mathscr{H}(M), *)$ is a group. The unit element is 0, and the inverse of H is $\bar{H}$.

For any $H \in \mathscr{H}(M)$ and $x \in \mathrm{Fix}(\varphi_H^1)$, we define $\gamma_H^x : S^1 \to M$ as $\gamma_H^x(t) := \varphi_H^t(x)$. We define $\mathscr{P}(H) := \{\gamma_H^x \mid x \in \mathrm{Fix}(\varphi_H^1)\}$. $\mathscr{P}^\circ(H)$ denotes the set of $\gamma \in \mathscr{P}(H)$ which is contractible in M. Setting $D := \{z \in \mathbb{C} \mid |z| \leq 1\}$, for any contractible $\gamma : S^1 \to M$, we take $\bar{\gamma} : D \to M$ so that $\bar{\gamma}(e^{2\pi i t}) = \gamma(t)$ and define

$$\mathscr{A}_H(\gamma) := \int_D \omega - \int_{S^1} H_t(\gamma(t))dt.$$

It is well-defined since we have assumed that (M, ω) is aspherical. Then we define

$$\Sigma^\circ(H) := \{\mathscr{A}_H(\gamma) \mid \gamma \in \mathscr{P}^\circ(H)\}.$$

It is well-known that $\Sigma^\circ(H)$ is a nowhere dence subset in $\mathbb{R}$ (see Proposition 3.7 in [8]). Finally, for any $H \in \mathscr{H}(M)$, we set

$$E_-(H) := -\int_0^1 \min H_t\, dt, \quad E_+(H) := \int_0^1 \max H_t\, dt, \quad \|H\| := E_-(H) + E_+(H).$$

Definition 2.1. Let (M, ω) be a connected aspherical symplectic manifold. An *action selector* for (M, ω) is a map $\sigma : \mathscr{H}(M) \to \mathbb{R}$ which satisfies the following axioms:

(AS1) $\sigma(H) \in \Sigma^\circ(H)$ for any $H \in \mathscr{H}(M)$.
(AS2) For any $H \in \mathscr{H}_{\mathrm{HZ}}^{\{c_M\}}(M, \omega)$, $\sigma(H) = -\min H$.
(AS3) $\sigma(H) \leq E_-(H)$ for any $H \in \mathscr{H}(M)$.
(AS4) σ is continuous with respect to the C^0-topology of $\mathscr{H}(M)$.
(AS5) $\sigma(H * K) \leq \sigma(H) + E_-(K)$ for any $H, K \in \mathscr{H}(M)$.

Remark 2.2. The above set of axioms for action selectors follows that in [2], although our sign conventions are different from [2]. Moreover, our notion of Hofer-Zehnder admissible Hamiltonians is wider than that in [2].

Our proof of Theorem 1.1 is based on the following result:

Theorem 2.3 ([8], [1])**.** *Let (M, ω) be a connected aspherical symplectic manifold.*

(1) *When M is closed, there exists an action selector for (M, ω).*
(2) *When M is convex, there exists an action selector for (M, ω).*

(1) was proved by M. Schwarz in [8], based on the Piunkhin-Salamon-Schwarz isomorphism. (2) was proved by U. Frauenfelder and F. Schlenk in [1], based on [8] and Floer theory for convex symplectic manifolds ([9]).

2.2. **A variant of the energy-capacity inequality.** First we prove the following result, which can be considered to be a variant of the energy-capacity inequality:

Theorem 2.4. *Let (M, ω) be a connected aspherical symplectic manifold, which is closed or convex. Let U be an open set in M and $H \in \mathscr{H}(M)$ such that:*

(1) $\varphi_H^1|_U = \mathrm{id}_U$.
(2) *For any $x \in U$, $\gamma_H^x : S^1 \to M; t \mapsto \varphi_H^t(x)$ is not contractible. Moreover, $[\gamma_H^x] \in \pi_1'(M)$ does not depend on $x \in U$.*

Then, setting $\alpha := [\gamma_H^x] \in \pi_1'(M)$, $c_{\mathrm{HZ}}^{(i_U^M)_*^{-1}(\{c_M, \bar{\alpha}\})}(U, \omega) \le \|H\|$.

The proof is similar to the proof of the energy-capacity inequality in [2] (section 2.1 in [2]). In the following, $\sigma : \mathscr{H}(M) \to \mathbb{R}$ denotes an action selector for (M, ω), which exists due to Theorem 2.3.

Suppose that U, H, α are as in Theorem 2.4. We have to show $-\min K \le \|H\|$ for any $K \in \mathscr{H}_{\mathrm{HZ}}^{\{c_M, \bar{\alpha}\}}(M)$, $\mathrm{supp} K \subset U$. Since $K \in \mathscr{H}_{\mathrm{HZ}}^{\{c_M\}}(M)$, $\sigma(K) = -\min K$ by (AS2). Hence it is enough to show $\sigma(K) \le \|H\|$. First notice the following lemma:

Lemma 2.5. *For any $\chi \in C^\infty([0,1])$ satisfying $\int_0^1 \chi(t)\, dt = 1$, we set $K_\chi \in \mathscr{H}(M)$ by $K_\chi(x, t) := K(x)\chi(t)$. Then, $\sigma(K_\chi) = \sigma(K)$.*

Proof. For $0 \le s \le 1$, set $\chi_s := s\chi + (1 - s)$. Then, it is easy to verify that $\Sigma^\circ(K_{\chi_s}) = \Sigma^\circ(K)$ for any s. By (AS1), $\sigma(K_{\chi_s}) \in \Sigma^\circ(K)$ for any $0 \le s \le 1$. On the otherhand, $\sigma(K_{\chi_s})$ depends continuously on s by (AS4). Since $\Sigma^\circ(K)$ is nowhere dence, $[0,1] \to \mathbb{R}; s \mapsto \sigma(K_{\chi_s})$ is a constant function. Hence $\sigma(K_\chi) = \sigma(K)$. □

Remark 2.6. The above lemma is same as Lemma 2.2 in [2]. We have included the proof for the convenience of the reader.

Take $\chi \in C^\infty([0,1])$ so that $\int_0^1 \chi(t)\, dt = 1$ and $\mathrm{supp}\chi \subset (1/2, 1)$. By Lemma 2.5, it is enough to show $\sigma(K_\chi) \le \|H\|$. After reparametrizing in t, we may assume that $H_t \equiv 0$ for $1/2 \le t \le 1$. Then

$$aK_\chi * H(t, x) = \begin{cases} H(t, x) & (0 \le t \le 1/2) \\ aK_\chi(t, x) & (1/2 \le t \le 1) \end{cases}.$$

We claim that $\Sigma^\circ(aK_\chi * H) \subset \Sigma^\circ(H)$ for any $0 \le a \le 1$. Let $x \in \mathrm{Fix}(\varphi^1_{aK_\chi * H})$ such that $\gamma^x_{aK_\chi * H}$ is contractible in M. We distinguish two cases:

- Suppose $x \in U$. Since $\varphi^1_H|_U = \mathrm{id}_U$, $x \in \mathrm{Fix}(\varphi^1_{aK_\chi})$. Since $0 \le a \le 1$ and $K \in \mathscr{H}^{\{\bar\alpha\}}_{\mathrm{HZ}}(M)$, $[\gamma^x_{aK_\chi}] \ne \bar\alpha$. On the otherhand, $\gamma^x_{aK_\chi * H}$ is a concatanation of γ^x_H and $\gamma^x_{aK_\chi}$, and $[\gamma^x_H] = \alpha$. Hence $\gamma^x_{aK_\chi * H}$ is not contractible: it contradicts our assumption.
- Suppose $x \notin U$. Since $\varphi^1_{aK_\chi}|_{M\setminus U} = \mathrm{id}_{M\setminus U}$, $x \in \mathrm{Fix}(\varphi^1_H)$. Using $\mathrm{supp} K \subset U$, it is easy to verify that $\gamma^x_{aK_\chi * H} = \gamma^x_H$, $aK_\chi * H(t, \gamma^x_{aK_\chi * H}(t)) = H(t, \gamma^x_H(t))$. Hence $\mathscr{A}_{aK_\chi * H}(\gamma^x_{aK_\chi * H}) = \mathscr{A}_H(\gamma^x_H) \in \Sigma^\circ(H)$.

Hence we have verified $\Sigma^\circ(aK_\chi * H) \subset \Sigma^\circ(H)$ for $0 \le a \le 1$. By (AS4), $\sigma(aK_\chi * H)$ depends continuously on a. Since $\Sigma^\circ(H)$ is nowhere dence, $[0,1] \to \mathbb{R}; a \mapsto \sigma(aK_\chi * H)$ is a constant function. Hence $\sigma(H) = \sigma(K_\chi * H)$. Finally, we obtain $\sigma(K_\chi) \le \|H\|$ by

$$\begin{aligned}\sigma(K_\chi) &= \sigma(K_\chi * H * \bar H) \le \sigma(K_\chi * H) + E_-(\bar H) \\ &= \sigma(H) + E_+(H) \le E_-(H) + E_+(H) = \|H\|.\end{aligned}$$

□

2.3. **Proof of Theorem 1.1.** Finally we prove Theorem 1.1:

For any $S \subset \pi'_1(U)$, it is easy to verify that $c^S_{\mathrm{HZ}}(U,\omega) = \sup_V c^{(i^U_V)^{-1}_*(S)}_{\mathrm{HZ}}(V,\omega)$, where V runs over all open sets of U with compact closures. Hence it is enough to show

$$c^{(i^M_V)^{-1}_*(\{c_M,\bar\alpha\})}_{\mathrm{HZ}}(V,\omega) \le \sup H - \inf H$$

for any open $V \subset U$ with a compact closure. Since $H \in C^\infty(U)$ generates the given circle action on U, $V' := \bigcup_{0 \le t \le 1} \varphi^t_H(V)$ is invariant under the circle action, and it is again an open set of U with compact closure. Then there exists $\rho \in C^\infty_0(U)$ such that $0 \le \rho \le 1$ and $\rho|_{V'} \equiv 1$. Then $\rho H \in C^\infty_0(U)$ extends to M, and $\varphi^t_{\rho H}|_V = \varphi^t_H|_V$ for any $0 \le t \le 1$. By adding constant to $H \in C^\infty(U)$ if necessary, we may assume that $\inf H \le 0 \le \sup H$. Then $\inf \rho H \ge \inf H$, $\sup \rho H \le \sup H$, hence $\|\rho H\| \le \|H\|$. Finally we get

$$c^{(i^M_V)^{-1}_*(\{c_M,\bar\alpha\})}_{\mathrm{HZ}}(V,\omega) \le \|\rho H\| \le \|H\|.$$

The first inequality follows from Theorem 2.4 (applied to V and ρH). □

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