

EINSTEIN-DE BROGLIE RELATIONS ON THE LATTICE

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Historically the starting point of wave mechanics is the Planck and Einstein-de Broglie relations for the energy and momentum of a particle, where the momentum is connected to the group velocity of the wave packet. We translate the arguments given by de Broglie to the case of a wave defined on the grid points of a space-time lattice and explore the physical consequences such as integral period, wave length, discrete energy, momentum and rest mass.

1 Einstein-de Broglie relations: continuous case

After Einstein applied the Planck formula $E = h\nu$ (quantization of the energy for the orbits of the harmonic oscillator) to the energy of the light waves in the photo-electric effect, de Broglie generalized this expression to relativistic momentum of a massive or massless particle.

Basically de Broglie arguments [1] are based on the transformation properties of the frequency and wave number of a plane wave and the transformations of the energy and the relativistic momentum of a particle.

Let us consider a plane wave with the wave normal \vec{n} in the xy-plane of a system S with angular frequency w , wave vector \vec{k} and phase velocity v_φ . It is described by a wave function

$$\psi(\vec{x}, t) = A \cos\left(wt - \vec{k}\vec{x}\right) \quad (1)$$

In a coordinate system S' moving in the direction of the x-axis with the velocity v relative to S the wave function will be described by

$$\psi(\vec{x}', t') = A \cos\left(w't' - \vec{k}'\cdot\vec{x}'\right) \quad (2)$$

Since the argument of both functions should be the same it follows: by elementary calculations:

$$w' = w \frac{(1 - \vec{v} \cdot \vec{n})/v_\varphi}{(1 - v^2/c^2)^{1/2}}$$

$$\vec{k}' = \vec{k} + \frac{\vec{v}}{v^2} \frac{(\vec{v} \cdot \vec{k}) \left\{ 1 - (1 - v^2/c^2)^{1/2} \right\} - v^2 k v_\varphi / c^2}{(1 - v^2/c^2)^{1/2}}$$

$$k' = k \frac{\left(1 - \frac{v^2}{c^2} + \frac{v^2 v_\varphi^2}{c^4} + \frac{(\vec{v} \cdot \vec{n})^2}{c^2} - \frac{2(\vec{v} \cdot \vec{n})v_\varphi}{c^2}\right)^{1/2}}{(1 - v^2/c^2)^{1/2}} \quad (3)$$

where $k \equiv |\vec{k}|$ is the wave number.

Suppose a particle of energy E and relativistic momentum \vec{p} is moving with respect to a coordinate system S with velocity \vec{u} . In a coordinate system S' moving in the direction of the x-axis with the velocity v relative to S , the particle will be described by the energy E' and the relativistic momentum \vec{p}' , which are related to the old coordinates by

$$\begin{aligned} E' &= E \frac{1 - \vec{v} \cdot \vec{u}/c^2}{(1 - v^2/c^2)^{1/2}} \\ \vec{p}' &= \vec{p} + \frac{\vec{v} (\vec{v} \cdot \vec{p})}{v^2} \frac{\left\{1 - (1 - v^2/c^2)^{1/2}\right\} - v^2 p/u}{(1 - v^2/c^2)^{1/2}} \\ p' &= \frac{\left\{p^2 \left(1 - \frac{v^2}{c^2}\right) + p^2 \frac{v^2}{u^2} + \frac{(\vec{v} \cdot \vec{p})^2}{c^2} - \frac{2p(\vec{v} \cdot \vec{p})}{u}\right\}^{1/2}}{(1 - v^2/c^2)^{1/2}} \end{aligned} \quad (4)$$

Comparison of formulas (3) and (4) leads to the conclusion that w, \vec{k} transform in the same way as E, \vec{p} provided \vec{k} and \vec{p} are parallel and the phase velocity v_φ is related to the velocity of the particle u by the expression [2]

$$v_\varphi = c^2/u \quad (5)$$

Following Einstein's hypothesis that the energy should be proportional to the frequency of a light quanta,

$$E = \hbar w \quad (6)$$

de Broglie made the assumption that for a particle there is an associate wave satisfying

$$E = \hbar w \quad , \quad p = \hbar \vec{k} \quad (7)$$

Since the phase velocity of the wave v_φ does not correspond to the velocity of the particle, de Broglie suggested that there is a wave packet associated with the particle, consisting of a superposition of waves with different wave vectors \vec{k} and amplitudes $\hat{\psi}(\vec{k})$

$$\psi(\vec{x}, t) = \int_{-\infty}^{\infty} d^3 k \hat{\psi}(\vec{k}) \exp i \left\{ w(\vec{k}) t - \vec{k} \cdot \vec{x} \right\} \quad (8)$$

If we suppose that the momentum vari is very little around a fixed value \vec{k}_0 , namely, $|\vec{k} - \vec{k}_0| \leq \Delta k$, then the function $w(\vec{k})$ can be expanded around $w_0 \equiv w(\vec{k}_0)$. Easy calculations gives:

$$\psi(\vec{x}, t) = \exp \left\{ i \left(w_0 t - \vec{k}_0 \cdot \vec{x} \right) \Delta k \right\} \int_{\Delta k} d^3 k \exp \left\{ i \left(w'_{0t} - x \right) \Delta k \right\}$$

This wave represent a packet with phase velocity $v_\varphi = w_0/k_0$ and group velocity $v_g = w'_0 \equiv dw/dk(k_0)$.

From the Einstein-de Broglie relations follows:

$$v_\varphi = \frac{w}{k} = \frac{E}{p} = \frac{c^2}{u} \quad (9)$$

$$v_g = \frac{dw}{dk} = \frac{dE}{dp} = \frac{pc^2}{E} = u \quad (10)$$

where in the last equation we have used $E = (p^2 c^2 + m_0^2 c^4)^{1/2}$. There fore we have $v_\varphi = c^2/v_g$ in agreement with de Broglie assumption about the wave-packet.

The Einstein-de Broglie relation were used to write the wave function associated to a particle

$$\psi(\vec{x}, t) = \exp \left\{ i \left(Et - \vec{p} \cdot \vec{x} \right) / \hbar \right\} \quad (11)$$

If the energy and relativistic momentum are connected by $E^2 - p^2 c^2 = m_0^2 c^4$ the wave function satisfies

$$\left(-\frac{1}{c^2} \frac{\partial}{\partial t^2} + \Delta \right) \psi(\vec{x}, t) = \frac{m_0^2 c^4}{\hbar^2} \psi(\vec{x}, t) \quad (12)$$

2 Einstein-de Broglie relations: discrete case

If we introduce the assumption os a discrete space-time [3] we must have

$$\begin{aligned} t &= n\tau, \quad \vec{x} = \vec{j}\varepsilon, \quad n, j_1, j_2, j_3 \in Z \\ T &= N\tau, \quad \lambda = M\varepsilon, \quad N, M \in Z \\ w &= \frac{2\pi}{N\tau}, \quad k = \frac{2\pi}{M\varepsilon}, \quad \frac{1}{N}, \frac{1}{M} \in Q \end{aligned} \quad (13)$$

where ε, τ are the fundamental length and time.

From these quantities one constructs the discrete wave functions (for simplicity we use only one spacial coordinate):

$$\psi(x, t) = \exp \left\{ 2\pi i \left(\frac{n}{N} - \frac{j}{M} \right) \right\} \quad (14)$$

which is periodic in n, j with period N and wave length M .

We introduce an other wave function

$$\psi(x, t) = \left(\frac{1 + \frac{1}{2}i\frac{2\pi}{N}}{1 - \frac{1}{2}i\frac{2\pi}{N}} \right)^n \left(\frac{1 - \frac{1}{2}i\frac{2\pi}{M}}{1 + \frac{1}{2}i\frac{2\pi}{M}} \right)^j \quad (15)$$

This is a hot periodic function in n or j , but is quasi-periodic in the sense that in the limit $n \rightarrow \infty, j \rightarrow \infty, \tau \rightarrow 0, \varepsilon \rightarrow 0, n\tau = t, j\varepsilon = x$,

$$\psi(x, t) \rightarrow \exp i2\pi (wt - kx).$$

which is periodic in t and x

The arguments leading to de Broglie relations are translated into the discrete language. The integral Lorentz transformations are factorized with the help of Kac generators [4]

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

in such a way that an element of the complete Lorentz group

$$L = P_1^\alpha P_2^\beta P_3^\gamma S_4 \quad P_1^\delta P_2^\xi P_3^\eta S_4 \dots S_4 S_1^\rho S_2^\sigma S_3^\tau$$

where $P_1 = S_1 S_2 S_3 S_2 S_1, P_2 = S_2 S_3 S_2, P_3 = S_3; \alpha, \beta, \gamma, \delta, \xi, \eta, \rho, \sigma, \tau = 0, 1$.

The energy and the relativistic momentum are written

$$E = \frac{m_0 c^2 (c\Delta t)}{\left\{ (c\Delta t)^2 - (\Delta x)^2 \right\}^{1/2}}, \quad P = \frac{m_0 c \Delta x}{\left\{ (c\Delta t)^2 - (\Delta x)^2 \right\}^{1/2}} \quad (16)$$

hence

$$\frac{pc^2}{E} = \frac{\Delta x}{\Delta t} = u \quad (17)$$

As in the continuous case, if $v_\varphi = c^2/u (E, p)$ transform in the same way as (w, k) , hence

$$E = \hbar w \quad , \quad p = \hbar k \quad (18)$$

The identification of $v_g = u$ is made by the superposition of two wave functions of slightly different wave length and period [5]

$$\begin{aligned} \psi(x, t) &= \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} \right) + \cos 2\pi \left(\frac{t}{T'} - \frac{x}{\lambda'} \right) \\ &= 2 \cos \pi \left\{ t \left(\frac{1}{T} - \frac{1}{T'} \right) - x \left(\frac{1}{\lambda} - \frac{1}{\lambda'} \right) \right\} \cos \pi \left\{ t \left(\frac{1}{T} + \frac{1}{T'} \right) - x \left(\frac{1}{\lambda} + \frac{1}{\lambda'} \right) \right\} \end{aligned}$$

To this wave it corresponds a phase velocity

$$v_\varphi = \frac{\frac{1}{T} + \frac{1}{T'}}{\frac{1}{\lambda} + \frac{1}{\lambda'}} = \frac{\tilde{\Delta} w}{\tilde{\Delta} k} \quad (19)$$

and a group velocity

$$v_g = \frac{\frac{1}{T} - \frac{1}{T'}}{\frac{1}{\lambda} - \frac{1}{\lambda'}} = \frac{\Delta w}{\Delta k} \quad (20)$$

where Δ and $\tilde{\Delta}$ are the difference and average operator

From (16) (17) and (18) we have

$$v_\varphi = \frac{\tilde{\Delta} w}{\tilde{\Delta} k} = \frac{\tilde{\Delta} E}{\tilde{\Delta} p} = \frac{c^2}{u} \quad (21)$$

$$v_g = \frac{\Delta w}{\Delta k} = \frac{\Delta E}{\Delta p} = u \quad (22)$$

To prove the last equality in (22) we take the expression $E^2/c^2 - p^2 = m_0^2 c^2$ and apply the total difference to both sides. From the definition $\Delta f \equiv f(x + \Delta x) - f(x)$ we get $\Delta(E^2) = (E + \Delta E)^2 - E^2$ and $\Delta(p^2) = (p + \Delta p)^2 - p^2$. Therefore

$$\left\{ 2E\Delta E + (\Delta E)^2 \right\} / c^2 - 2p\Delta p - (\Delta p)^2 = 0 \quad (23)$$

The difference of momentum in two consecutive events of the particle Δp_μ is also a 4-vector. In order to calculate the invariant $(\Delta E)^2/c^2 - (\Delta p)^2$ we

take an inertial system such that $\Delta p = 0$ which means that the momentum is the same for two consecutive events, and the velocity and energy are the same. Hence $\Delta E = 0$, so that in an arbitrary inertial frame.

$$(\Delta E)^2/c^2 - (\Delta p)^2 = 0$$

Inserting this result in (23) and using (21) we obtain

$$\Delta E = u\Delta p \quad (24)$$

To prove the last equality of (21) we apply the identity $\Delta(fg) = \Delta f \tilde{\Delta}g + \tilde{\Delta}f \Delta g$ to the expression $E^2/c^2 - p^2 = m_0^2c^2$, as in the continuous case.

3 Physical consequences

If we accept the assumption of a discrete space and time as a consequence of the interaction of fundamental entities [6] we may conceive a vibration on this network, similar to the waves propagating on a discrete string. The plane waves satisfy the properties of section 2 therefore, we can talk of phase velocity and group velocity of the packet. Those properties can be associated to a particle whose structure is attached to the wave packet, but with experimental data given by the Einstein-de Broglie relations.

In particular we have the following physical consequences:

- i) the frequency and the wave number are discrete, because the period and wave length are integral multiple of fundamental time and length
- ii) the energy and relativistic momentum are discrete due to the Einstein-de Broglie relations

$$\frac{m_0c^2c\Delta t}{\{(c\Delta t)^2 - (\Delta x)^2\}^{1/2}} = \frac{h}{N\tau} \quad , \quad N \text{ integer} \quad (25)$$

$$\frac{m_0c\Delta x}{\{(c\Delta t)^2 - (\Delta x)^2\}^{1/2}} = \frac{h}{M\varepsilon} \quad , \quad M \text{ integer} \quad (26)$$

iii) in the rest system

$$m_0 = \frac{h}{c^2} \frac{1}{N\tau}$$

we have a discrete mass spectrum.

iv) the wave equation on the lattice reads [7]:

$$\left(-\frac{1}{c^2} \frac{1}{\tau^2} \Delta_n \nabla_n \tilde{\Delta}_j \tilde{\nabla}_j + \frac{1}{\varepsilon^2} \Delta_j \nabla_j \tilde{\Delta}_n \tilde{\nabla}_n \right) \psi(x, t) = \frac{m_0^2 c^2}{\hbar^2} \tilde{\Delta}_j \tilde{\nabla}_j \tilde{\Delta}_n \tilde{\nabla}_n \psi(x, t)$$

where the solutions are given by (14) or (15) provided the dispersion relations are satisfied

$$\frac{1}{c^2} \frac{1}{\tau^2} \tan^2 \frac{\pi}{N} - \frac{4}{\varepsilon^2} \tan^2 \frac{\pi}{M} = \frac{m_0^2 c^2}{\hbar^2}$$

in the first case and

$$\frac{1}{c^2} \left(\frac{1}{N\tau} \right)^2 - \left(\frac{1}{M\varepsilon} \right)^2 = \frac{m_0^2 c^2}{\hbar^2}$$

in the second case.

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