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*Branislav Boričić**

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DICTATORSHIP, LIBERALISM AND THE PARETO RULE: POSSIBLE AND IMPOSSIBLE[1]

ABSTRACT: *The current economic crisis has shaken belief in the capacity of neo-liberal 'free market' policies. Numerous supports of state intervention have arisen, and the interest for social choice theory has revived. In this paper we consider three standard properties for aggregating individual into social preferences: dictatorship, liberalism and the Pareto rule, and their formal negations. The context of the pure first-order classical logic makes it possible to show how some combinations of the above mentioned conditions, under the hypothesis of unrestricted domain, form simple and reasonable examples of possible or impossible social choice systems. Due to their simplicity, these examples, including the famous 'liberal paradox', could have a particular didactic value.*

KEY WORDS: *(non)dictatorship, (non) liberalism, (non)Pareto rule, (in)consistency, (im)possibility*

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* Faculty of Economics, University of Belgrade

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1. Introduction

The working title of this paper was 'The Impossibility of a non-Paretian Dictator', in the spirit of the famous A. Sen's 'impossibility of a Paretian liberal' (see Sen (1970)), but the final results have redefined it. Let us note that the central problem in social choice theory, as introduced by K. Arrow (1951) and developed by A. Sen (1970), (1993) and (1995), is how to define a procedure generating a social preference relation from a given finite system of individual preference relations. The first question arising here is whether it is possible, or impossible, to define such a procedure. A pure logical analysis of Arrow's impossibility theorem proof has been made in logical literature by Routley (1979). The economic literature has rarely been dealt with the logical status (dependence, consistency,...) of social choice basis (see Sen (1970 and 1993)). Most authors have concentrated on the problem of how to improve Arrow's impossibility and get a possible decision system. In particular this was the case during the last decades of the twentieth century with the appearance of the fuzzy systems methodology approach. On the other hand, the tradition of impossibility results has captured an honorary place in the rich history of mathematics (see Boričić (2007a)).

Our intention in this paper is not to deal with justification or disputation of the semantical basis of particular axioms, but to present some pure syntactic arguments regarding their consistency and interdependence. More accurately, we consider relationships between dictatorship **D**, liberalism **L**, Pareto condition **P**, and their negations non–dictatorship **ND**, non–liberalism **NL** and non–Pareto rule **NP**, the basic axioms appearing in Arrow–Sen theory, and prove that some combinations of them, under the hypothesis of unrestricted domain **U**, form possible or impossible social choice systems. These combinations seem to be approachable examples having a didactic value illustrating possibility and impossibility statements.

Let us mention here a quite interesting and relevant result, obtained by G. Chichilnisky (1982), on topological equivalence between **P** and **D**.

In the first part of this paper we present the formal mathematical framework precisely, including the original formulations of axioms as introduced by Arrow (1951) and Sen (1970 and 1995). Let us note that the original axioms do not belong to the first–order language of mathematics, as individuals ($i \in V$) and alternatives ($x, y \in X$) share the same level in the quantified formulae. In order to enable a pure formal treatment of axioms, we translate them into the first–order language. This operation results in the replacement of *one* original non–dictatorship axiom **ND** by a *finite number* (number of all individuals) of the corresponding non–dictatorship axioms of the first–order language, and, similarly, *one* original liberalism axiom **L** has been replaced by a *finite number* of the corresponding first–order liberalism axioms. We also propose a new form for **L** and justify it by the adequate equivalence proof. This context provides quite natural and pure proofs and arguments for the relationships, inter–deducibility and (in)consistency between the axioms. In other words, by using pure formal, but not complex tools, we find out and justify some new semantical properties of the basic social choice theory axioms.

In the sequel of the paper we prove that each dictatorship is Paretian ($\mathbf{D} \to \mathbf{P}$), that each non–Paretian society is non–dictatorial ($\mathbf{NP} \to \mathbf{ND}$), and, as a consequence of Sen's 'impossibility of Paretian liberal', we also infer that any liberalism is non–Paretian ($\mathbf{L} \to \mathbf{NP}$) and that each Paretian society is non–liberal ($\mathbf{P} \to \mathbf{NL}$). From here we conclude immediately that each liberal society is non–dictatorial ($\mathbf{L} \to \mathbf{ND}$), and that each dictatorial society is non–liberal ($\mathbf{D} \to \mathbf{NL}$). We obtain two new simple examples of inconsistent (impossible) sets of axioms: $\{\mathbf{D}, \mathbf{NP}\}$ and $\{\mathbf{D}, \mathbf{L}\}$, as well as, the 'Paretian liberal': $\{\mathbf{L}, \mathbf{P}\}$. Furthermore, we establish a sequence of relative consistency (possibility) and relative inconsistency (impossibility) conclusions, such as the following: if a dictatorial system is possible, then its extension by the Pareto rule is possible as well, and if a liberal system is possible, then its extension by non–dictatorship requirement is possible, or, on the other hand, if a non–dictatorial system is impossible, then its extension by non–Pareto rule is impossible as well. These relationships complete the picture regarding logical properties of the axioms under consideration.

For a contemporary more detailed and extended treatment of social choice theory we refer the reader to Hodge and Klima (2005), Schofield (2003) and Taylor (1995). Note that a quite different approach to the social choice analysis has been developed during the last decades (see Kuran (1995) and Boričić and Pešić (2004)).

2. General framework

First of all we have to point out that the terms 'consistency' and 'possibility' will be used as synonyms. Namely, in the context of Logics the term 'consistent system of conditions' denotes the set of statements from which we can not derive the statement $A \wedge \neg A$. Equivalently, it means that this set of conditions has a model, or, actually, that there exists a mathematical structure satisfying all considered conditions simultaneously. On the other hand, in the context of social choice theory the term 'possible system of conditions' has the same meaning exactly. Otherwise, when the conditions are inconsistent, i.e. there is no model for them, we will say that the system is 'impossible'. Obviously, any extension of an impossible system is *a fortiori* impossible. Also, if X implies Y, then: (a) any system containing X and $\neg Y$ will be impossible, and (b) if a system containing X is possible, then its extension by Y will be possible as well. On the other hand, if the system consisting of X and Y is impossible, then X implies $\neg Y$ and Y implies $\neg X$ (see Rubin (1990)).

We also point out that if X implies Y, then $X \wedge Z$ implies Y, for any sentences X, Y and Z. This logical rule is known as the *antecedent weakening rule.*

Our second remark is related to the usual descriptive interpretation of the basic notions, such as preference relation, dictatorship, liberalism and Pareto rule.

By a (strict) preference relation we mean any asymmetric, transitive, linear (i. e. complete) and acyclic binary relation. By P_i and P, respectively, we denote the individual preference relation corresponding to the person i and to the social preference relation, generated by $(P_i)_{1\leq i\leq n}$. The intended meaning of xPy is 'x is preferred to y'.

Also, a general condition **U** of unrestricted domain, requiring that the procedure of generating a social preference P can be applied to any configuration of individual preferences P_i, is supposed to hold.

The basic conditions of the Arrow–Sen theory, non–dictatorship, liberalism and the Pareto rule, respectively, are usually presented as follows (see Arrow (1951), Routley (1979) and Sen (1970) and (1995)):

$$\neg(\exists i \in V)(\forall x, y \in X)(xP_iy \rightarrow xPy)$$
$$(\forall i \in V)(\exists x, y \in X)(x \neq y \wedge (xP_iy \rightarrow xPy) \wedge (yP_ix \rightarrow yPx))$$
$$(\forall x, y \in X)((\forall i \in V)xP_iy \rightarrow xPy)$$

supposing that V and X are finite sets of individuals and alternatives, and P_i and P are preference relations on X.

Note that if we omit the condition $x \neq y$ in the liberalism axiom, then, for $x = y$, this axiom would hold universally, for trivial reasons, because neither condition xP_iy, nor yP_ix would be satisfied. Also, let us note that the given axioms are not expressed in the first–order language.

The non–dictatorship axiom states that there is no person i, dictator, having such power that, for each two alternatives x and y, if i prefers

x to y, the society must prefer x to y as well. The liberalism condition provides that each individual is decisive over at least one pair of distinct alternatives. The Pareto rule says that, if every individual prefers x to y, then the society must prefer x to y.

The non–dictatorship and Pareto rule were originally introduced by K. Arrow (see Arrow (1951), Routley (1979) and Sen (1995)), and the liberalism axiom was formally introduced by A. Sen (see Sen (1970) or (1995)) in the spirit of J. S. Mill's liberalism comprehension.

3. (Non)dictatorship, (non)liberalism and (non)Pareto rule

As a result of applying the abstraction operation to the original conditions of the Arrow–Sen theory, by ignoring the real nature of the symbols, we obtain dictatorship **D**, liberalism **L** and Pareto rule **P**, respectively, in the following form:

$$\textbf{D} \qquad (\forall x, y)(xP_iy \to xPy),\ \text{for some } i\ (1 \le i \le n)$$

$$\textbf{L} \qquad (\exists x, y)(x \ne y \land xP_iy \land xPy),\ \text{for each } i\ (1 \le i \le n)$$

$$\textbf{P} \qquad (\forall x, y)(\bigwedge_{1 \le i \le n} xP_iy \to xPy)$$

as well as their logical negations, non–dictatorship **ND**, non–liberalism **NL** and non–Pareto rule **NP**, as follows:

$$\textbf{ND} \qquad \neg(\forall x, y)(xP_iy \to xPy),\ \text{for each } i\ (1 \le i \le n)$$

$$\textbf{NL} \qquad (\forall x, y)(x = y \lor \neg xP_iy \lor \neg xPy),\ \text{for some } i\ (1 \le i \le n)$$

$$\textbf{NP} \qquad (\exists x, y)((\bigwedge_{1 \le i \le n} xP_iy) \land \neg xPy)$$

These axioms, expressed as formulae of the first–order language, enable us to make a pure formal logical analysis of some relationships between the conditions under consideration.

Note that the axiom **L** has a quite different form from the original liberalism axiom given in the preceding section, but this fact will not be of crucial importance for our work.

Lemma. *The axiom* **L** *is equivalent to the original liberalism axiom.*

Proof. Let us consider the following two formulas:

$$(1) \qquad (xP_iy \to xPy) \land (yP_ix \to yPx)$$

and

(2) $$xP_iy \wedge xPy$$

on condition that $x \neq y$. If (1) and xP_iy, then, from the first conjunct of (1), by *modus ponens,* we infer xPy, i. e. (2), for some alternatives x and y. If (1) and $\neg xP_iy$, then, by completeness and asymmetry of P_i, we get yP_ix, when, from the second conjunct of (1), by *modus ponens,* we infer yPx, i. e. $yP_ix \wedge yPx$, for some alternatives y and x, wherefrom, by renaming, we obtain **L**. Conversely, if we suppose (2), then, obviously, for the alternatives x and y, the first conjunct of (1) must hold, due to the fact that both its antecedent and its consequent are true, but as xP_iy is true, then yP_ix is false, and so the second conjunct of (1) must hold as well.□

In the sequel we present some facts regarding **D**, **ND**, **L**, **NL**, **P** and **NP**, under the hypothesis **U**. For instance, we establish some reasonable examples of possible or impossible social choice systems based on these axioms. In addition, we also complete a picture of their logical interdependence.

Obviously, by definition, we have that, *if* **X** *is any of axioms* **D**, **L** *or* **P**, *then the system containing* **NX** *and* **X** *is impossible.*

Lemma. *If* **D**, *then* **P**.

Proof. This is the result of a pure formal deduction of **P** from **D**, by weakening the antecedent of the formula $xP_iy \rightarrow xPy$ n–times.□

Consequently, if **NP**, then **ND**. This means that each dictatorial society is Paretian, or, equivalently, that each non–Paretain society is non–dictatorial.

Now we give some relative consistency and inconsistency results:

Corollary. *If the system containing* **D** *is possible, then the system obtained by substituting* **D** *by* **P** *is possible. Consequently, if the system containing* **D** *is possible, then its extension by* **P** *is possible.*

Corollary. *If the system containing* **P** *is impossible, then the system obtained by substituting* **P** *by* **D** *is impossible. Consequently, if the system containing* **P** *is impossible, then its extension by* **D** *is impossible.*

Corollary. *If the system containing* **NP** *is possible, then the system obtained by substituting* **NP** *by* **ND** *is possible. Consequently, if the system containing* **NP** *is possible, then its extension by* **ND** *is possible.*

Corollary. *If the system containing* **ND** *is impossible, then the system obtained by substituting* **NP** *by* **ND** *is impossible. Consequently, if the system containing* **ND** *is impossible, then its extension by* **ND** *is impossible.*

Also, we obtain a few absolute inconsistency results including 'the impossibility of a non–Paretian dictator':

Corollary. *The system containing* **D** *and* **NP** *is impossible. Consequently, the system containing* **L**, **D** *and* **NP** *is impossible, and the system containing* **NL**, **D** *and* **NP** *is impossible.*

In order to complete this puzzle, let us cite the famous Sen's result (see Sen (1970 or 1995)) known as 'impossibility of Paretian liberal' or 'liberal paradox':

Lemma. *The system containing* **L** *and* **P** *is impossible.*

Proof. Let suppose that the set V of all individuals consists of $n(\geq 2)$ persons and that, for the alternatives $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$, the particular cases of liberalism axiom

$$x_i \neq y_i \wedge x_i P_i y_i \wedge x_i P y_i$$

hold, for all i $(1 \leq i \leq n)$, wherefrom we infer

$$x_1 P y_1 \wedge x_2 P y_2 \wedge \cdots \wedge x_n P y_n$$

Now, we analyze the preferences of the first two individuals only, having in mind that, for alternatives x_1, y_1, x_2 and y_2, $x_1 P y_1 \wedge x_2 P y_2$ holds. In the case when $y_1 = x_2$ or $x_1 = y_2$, it is possible to suppose that $y_2 P_i x_1$, for all i $(1 \leq i \leq n)$, or $y_1 P_i x_2$, for all i $(1 \leq i \leq n)$, respectively, wherefrom, by the Pareto rule, we can infer $y_2 P x_1$ or $y_1 P x_2$, violating acyclicity of P in each case. If $y_1 \neq x_2 \wedge x_1 \neq y_2$, then it is possible to suppose that, for all i $(1 \leq i \leq n)$, $y_2 P_i x_1 \wedge y_1 P_i x_2$, wherefrom, by the Pareto rule, we can infer $y_2 P x_1 \wedge y_1 P x_2$, violating acyclicity of P in each case as well.□

Note that, essentially, we followed the spirit of Sen's original proof (see Sen (1970 or 1995)), based on the minimal liberalism argument, meaning that there are at least two persons decisive over two existing pairs of distinct alternatives.

Corollary. *If* **L**, *then* **NP**.

Consequently, if **P**, then **NL**. This means that each liberal society is non–Paretian, or, equivalently, that each Paretian society is non–liberal.

Corollary. *If the system containing* **P** *is possible, then the system obtained by substituting* **P** *by* **NL** *is possible. Consequently, if the system containing* **P** *is possible, then its extension by* **NL** *is possible.*

Corollary. *If the system containing* **L** *is possible, then the system obtained by substituting* **L** *by* **NP** *is possible. Consequently, if the system containing* **L** *is possible, then its extension by* **NP** *is possible.*

Finally, we can infer the following conclusions as well:

Corollary. *The system containing* **D** *and* **L** *is impossible. Consequently, the system containing* **D**, **L** *and* **P** *is impossible, and the system containing* **ND**, **L** *and* **P** *is impossible.*

Analogously, the just presented consequence could be considered as 'the impossibility of a liberal dictator'.

Corollary. *If* **L**, *then* **ND**.

Consequently, if **D**, then **NL**. It means that each liberal society is non–dictatorial, or, equivalently, that each dictatorial society is non–liberal.

Corollary. *If the system containing* **D** *is possible, then the system obtained by substituting* **D** *by* **NL** *is possible. Consequently, if the system containing* **D** *is possible, then its extension by* **NL** *is possible.*

Corollary. *If the system containing* **L** *is possible, then the system obtained by substituting* **L** *by* **ND** *is possible. Consequently, if the system containing* **L** *is possible, then its extension by* **ND** *is possible.*

4. Concluding remarks

Bearing in mind the glorious history of independence results in mathematics (see Boričić (2007a)) and, particularly, in mathematical economics (see Arrow (1951) and Sen (1995)), as well as the importance and the complexity of this kind of result, we were looking for simple, approachable, logically pure and didactically eligible examples of impossibilities. So, in this paper we have shown, by using elementary logical tools, that the three simplest axioms appearing in social choice theory, dictatorship, liberalism and the Pareto rule, present an appropriate structure to find, in their combinations, examples of both possible and impossible decision systems. It is shown that the systems containing **D** and **NP** ('non–Paretian dictator'), **D** and **L** ('liberal dictator'), or **P** and **L** ('Paretian liberal', as proved by A. Sen) are impossible. On the other hand, we justify that an extension, by **ND**, of a possible system containing **NP**, is possible, as well as that each possible system containing **D**, extended by **P**; containing **P**, extended by **NL**; containing **L**, extended by **NP**; containing **D**, extended by **NL**, or containing **L**, extended by **ND**, is possible.

Let us conclude that the logical dependence between the axioms above can be contracted informally in the following six basic schemata: $\mathbf{L} \rightarrow \mathbf{NP}$, $\mathbf{NP} \rightarrow \mathbf{ND}$, $\mathbf{L} \rightarrow \mathbf{ND}$, $\mathbf{D} \rightarrow \mathbf{P}$ (meaning that each dictatorship is Paretian), $\mathbf{P} \rightarrow \mathbf{NL}$ and $\mathbf{D} \rightarrow \mathbf{NL}$.

Finally, we emphasize that, under the hypotheses concerning individual and social rationality, independence of irrelevant alternatives and unrestricted domain, by the famous Arrow's general impossibility theorem (see Arrow (1951) and Sen (1995)), we also have that the set of axioms $\{\mathbf{P}, \mathbf{ND}\}$ is impossible, having as its immediate consequences that $\mathbf{P} \rightarrow \mathbf{D}$ and $\mathbf{ND} \rightarrow \mathbf{NP}$, which, together with our conclusion that $\mathbf{D} \rightarrow \mathbf{P}$, roughly corresponds to the basic result presented by Chichilnisky (1982).

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