Structure of Polynomial Representations for Orthosymplectic Lie Superalgebras

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Abstract

Orthosymplectic Lie superalgebras are fundamental symmetries in modern physics, such as massive supergravity. However, their representations are far from being thoroughly understood. In the present paper, we completely determine the structure of their various supersymmetric polynomial representations obtained by swapping bosonic multiplication operators and differential operators in the canonical supersymmetric polynomial representations. In particular, we obtain certain new infinite-dimensional irreducible representations and new composition series of indecomposable representations for these algebras.

Keywords: representation, Lie superalgebra, weight, composition series.

1 Introduction

Lie superalgebras were introduced by physicists as the fundamental tools of studying the supersymmetry in physics (e.g., cf. [2-4], [7], [26], [29]). For instance, orthosymplectic Lie superalgebras are symmetries of massive supergravity (cf. [3], [4]). Kac [13] gave a classification of finite-dimensional Lie superalgebras. Unlike Lie algebra case, finite-dimensional modules of finite-dimensional simple Lie superalgebras may not be completely reducible and the structure of finite-dimensional irreducible modules is much more complicated due to the existence of so-called *atypical* modules (cf. [14], [15]). Serganova [25] gave a nice survey on characters of irreducible representations of simple Lie superalgebras. Indeed, representations of orthosymplectic Lie superalgebras are the most complicated among all the classical Lie superalgebras and people could so far only get partial information of them.

Palev [24] found the para-Bose and para-Fremi operators as generators of orthosymplectic Lie superalgebras. Farmer and Javis [9] constructed irreducible representations of $osp(3|2)$ and $osp(4|2)$ by superfield techniques. Moreover, they [10] enumerated finite-dimensional graded tensor representations of orthosymplectic Lie superalgebras via standard Young diagrams. Van der Jeugt [27] investigated representations of $osp(3|2)$ by means of the shift operator technique. Gould and Zhang [12] determined all the finite-dimensional unitary representations of $osp(2|2n)$. Nishiyama [21] studied unitary representations of orthosymplectic Lie superalgebras via supersymettric Heisenberg algebras. He [22] also obtained the characters and super-characters of discrete series representations for orthosymplectic Lie superalgebras. Furthermore, he [23] investigated representations of the superalgebras via super dual pairs.

Lee Shader [16] investigated certain typical representations of orthosymplectic Lie superalgebras. Moreover, Benkart, Lee Shader and Ram [1] studied the tensor product representations of orthosymplectic Lie superalgebras over the canonical represenations. Lee Shader [17] obtained certain characteristics of representations for Lie superalgebras of type C. Cheng and Zhang [6] found a combinatorial character formula for orthosymplectic Lie superalgebras via Howe duality. Dobrev and Zhang [8] classified the positive energy unitary irreducible representations of superalgebras $osp(1, 2n, \mathbb{R})$. Furthermore, Cheng, Wang and Zhang [5] presented a Fock space approach to representation theory of $osp(2|2n)$.

Lievens, Stoilova, and Van der Jeugt [18] got unitary irreducible representations of the Lie superalgebra $osp(1|2n)$ via the paraboson Fock space. Moreover, they [19] found a class of unitary irreducible representations of the Lie superalegbra $osp(1|2n)$. Zhang [29] investigated the Schrödinger equation on the superspace $\mathbb{R}^{m|2n}$ which involved a potential that varied as an inverse power of the $osp(m|2n)$-invariant distance from the origin and lead to interesting results regarding the infinite-dimensional representations of the orthosymplectic Lie superalgebra $osp(2, m+1|2n)$. In the appendix, he also presented the structure of a canonical supersymmetric polynomial representation for $osp(m|2n)$ when $m - 2n > 1$.

In this paper, we completely determine the structure of various supersymmetric polynomial representations of orthosymplectic Lie superalgebras obtained by swapping bosonic multiplication operators and differential operators in the canonical supersymmetric polynomial representations. In particular, certain new infinite-dimensional irreducible representations and new composition series of indecomposable representations for these algebras are obtained. Below we give a technical introduction to our results.

Denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the set of nonnegative integers. For convenience, we also use the following notation of indices:

$$\overline{i,j} = \{i, i+1, ..., j\}, \tag{1.1}$$

where $i \leq j$ are integers. Moreover, we also use $\{0, 1\}$ to denote $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ when the context is clear. Let $E_{i,j}$ be the square matrix whose (i, j)-entry is 1 and the others are zero. The general linear Lie superalgebra of $(m + 2n) \times (m + 2n)$ matrices $gl(m, 2n) = gl(m, 2n)_0 \oplus gl(m, 2n)_1$ with

$$gl(m,2n)_0 = \sum_{i,j=1}^{m} \mathbb{C}E_{i,j} + \sum_{s,t=1}^{2n} \mathbb{C}E_{m+s,m+t}, \; gl(m,2n)_1 = \sum_{i=1}^{m}\sum_{s=1}^{2n}(\mathbb{C}E_{i,m+s} + \mathbb{C}E_{m+s,i}) \tag{1.2}$$

and the Lie superbracket:

$$[u,v]=uv-(-1)^{\iota_1\iota_2}vu \qquad \text{for } u\in gl(m,2n)_{\iota_1},\ v\in gl(m,2n)_{\iota_2}. \tag{1.3}$$

Assume that $m=2m_1$ is an even integer. The orthosymplectic Lie superalgebra $osp(m,2n)$ is the subalgebra of $gl(m,2n)$ consisting of the matrices of the form

$$\begin{pmatrix} A & B & H & H_1 \\ C & -A^T & K & K_1 \\ K_1^T & H_1^T & D & E \\ -K^T & -H^T & F & -D^T \end{pmatrix} \tag{1.4}$$

where A, B and C are $m_1\times m_1$ matrices such that $B=-B^T$, $C=-C^T$; D, E and F are $n\times n$ matrices such that $E=E^T$, $F=F^T$; H, H_1, K and K_1 are $m_1\times n$ matrices. Let $\mathcal{A}=\mathbb{C}[x_1,\cdots,x_m,\theta_1,\cdots,\theta_{2n}]$ be the polynomial algebra in bosonic variables $x_1,\cdots,x_m$ and fermionic variables $\theta_1,\cdots,\theta_{2n}$, i.e.

$$x_ix_j=x_jx_i,\ \ \theta_p\theta_q=-\theta_q\theta_p,\ \ x_i\theta_p=\theta_px_i,\ \ i,j\in\overline{1,m},\ \ p,q\in\overline{1,2n}. \tag{1.5}$$

Taking $r\in\overline{0,m}$, we have the following supersymmetric polynomial representation of $gl(m|2n)$:

$$E_{i,j}|_{\mathcal{A}}=\begin{cases} -x_j\partial_{x_i}-\delta_{i,j} & \text{if } \ i,j\in\overline{1,r}, \\ \partial_{x_i}\partial_{x_j} & \text{if } \ i\in\overline{1,r},\ j\in\overline{r+1,m}, \\ -x_ix_j & \text{if } \ j\in\overline{1,r},\ i\in\overline{r+1,m}, \\ x_i\partial_{x_j} & \text{if } \ i,j\in\overline{r+1,m}, \end{cases}$$

$$E_{i,m+p}|_{\mathcal{A}}=\begin{cases} \partial_{x_i}\partial_{\theta_p} & \text{if } \ i\in\overline{1,r},\ p\in\overline{1,2n}, \\ x_i\partial_{\theta_p} & \text{if } \ i\in\overline{r+1,m},\ p\in\overline{1,2n}, \end{cases} \tag{1.6}$$

$$E_{m+p,j}|_{\mathcal{A}}=\begin{cases} -\theta_px_j & \text{if } \ j\in\overline{1,r},\ p\in\overline{1,2n} \\ \theta_p\partial_{x_j} & \text{if } \ j\in\overline{r+1,m},\ p\in\overline{1,2n}, \end{cases}$$

$$E_{m+p,m+q}|_{\mathcal{A}}=\theta_p\partial_{\theta_q} \text{ if } p,q\in\overline{1,2n},$$

which is obtained from the canonical supersymmetric polynomial representation ($r=0$) by swapping ∂_{x_i} and $-x_i$ for $i\in\overline{1,r}$. In particular, we have the restricted representation of $osp(m,2n)$ on $\mathcal{A}$.

For $k\in\mathbb{N}$, we define

$$\begin{aligned}\mathcal{A}_k^r=\ &\text{Span}\,\{x^\alpha\theta_{i_1}\cdots\theta_{i_t}\in\mathcal{A}^r \mid 0\le t\le 2n;\ i_1,\cdots,i_t\in\overline{1,2n};\ \alpha\in\mathbb{N}^m; \\ &t-\sum_{i=1}^r\alpha_i+\sum_{j=r+1}^m\alpha_j=k\}.\end{aligned} \tag{1.7}$$

Then subspaces $\mathcal{A}_k^r$ form $osp(m,2n)$-submodules. We can assume $r\le m_1$ by symmetry. Denote

$$\Delta=-\sum_{i=1}^r x_i\partial_{x_{m_1+i}}+\sum_{i=r+1}^{m_1}\partial_{x_i}\partial_{x_{m_1+i}}+\sum_{j=1}^n\partial_{\theta_j}\partial_{\theta_{n+j}}, \tag{1.8}$$

$$\eta=\sum_{i=1}^{r}x_{m_1+i}\partial_{x_i}+\sum_{i=r+1}^{m_1}x_ix_{m_1+i}+\sum_{j=1}^{n}\theta_j\theta_{n+j}. \tag{1.9}$$

Set

$$\mathcal{H}_k^r=\{f\in\mathcal{A}_k^r\mid\Delta(f)=0\}. \tag{1.10}$$

Denote by $\langle F\rangle$ the $osp(m,2n)$-submodule generated by a subset F.

Theorem 1. *We have the following conclusions:*

1) Assume $r=0$. If $k>2(n-m_1+1)$ or $k\le(n-m_1+1)$, then $\mathcal{A}_k^0=\mathcal{H}_k^0\oplus\eta\mathcal{A}_{k-2}^0$ and the subspace $\mathcal{H}_k^0$ is an irreducible $osp(m,2n)$-submodule. When $(n-m_1+1)<k\le 2(n-m_1+1)$,

$$\mathcal{H}_k^0\supset\eta^{k-(n-m_1+1)}\mathcal{H}_{2(n-m_1+1)-k}^0\supset\{0\} \tag{1.11}$$

is a composition series.

2) Suppose $r=m_1$. We always have $\mathcal{A}_k^{m_1}=\mathcal{H}_k^{m_1}\oplus\eta\mathcal{A}_{k-2}^{m_1}$. If $k>n$, $\mathcal{H}_k^{m_1}=\{0\}$. When $k\le n$, the subspace $\mathcal{H}_k^{m_1}$ is an irreducible $osp(m,2n)$-submodule.

3) Assume $0<r<m_1$. If $k\le n-m_1+r+1$, then $\mathcal{A}_k^r=\mathcal{H}_k^r\bigoplus\eta\mathcal{A}_{k-2}^r$ and the submodule $\mathcal{H}_k^r$ is irreducible. When $k>n-m_1+r+1$, we have the following composition series

$$\mathcal{H}_k^r\supset\eta^{k-n+m_1-r-1}\mathcal{H}_{-k+2(n-m_1+r+1)}^r\supset\{0\}\ \text{if}\ r<m_1-1; \tag{1.12}$$

$$\mathcal{H}_k^{m_1-1}\supset\langle x_{m_1}^k\rangle\supset\eta^{k-n}\mathcal{H}_{-k+2n}^{m_1-1}\supset\{0\}. \tag{1.13}$$

Take a subset T of $\overline{1,2n}$. Denote $\bar{T}=\overline{1,2n}\setminus T$. Let $osp(m,2n)$ act on $\mathcal{A}'=\mathbb{C}[x_1,\cdots,x_{2n};\theta_1,\cdots,\theta_m]$ via

$$E_{p,q}|_{\mathcal{A}'}=\theta_p\partial_{\theta_q}\qquad\text{if } p,q\in\overline{1,m},$$

$$E_{m+i,m+j}|_{\mathcal{A}'}=\begin{cases}-x_j\partial_{x_i}-\delta_{i,j} & \text{if}\quad i,j\in T,\\ \partial_{x_i}\partial_{x_j} & \text{if}\quad i\in T,\ j\in\bar{T},\\ -x_ix_j & \text{if}\quad i\in\bar{T},\ j\in T,\\ x_i\partial_{x_j} & \text{if}\quad i,j\in\bar{T},\end{cases}$$

$$E_{m+i,p}|_{\mathcal{A}'}=\begin{cases}\partial_{x_i}\partial_{\theta_p} & \text{if}\quad i\in T,\ p\in\overline{1,m},\\ x_i\partial_{\theta_p} & \text{if}\quad i\in\bar{T},\ p\in\overline{1,m},\end{cases} \tag{1.14}$$

$$E_{p,m+j}|_{\mathcal{A}'}=\begin{cases}-\theta_px_j & \text{if}\quad j\in T,\ p\in\overline{1,m},\\ \theta_p\partial_{x_j} & \text{if}\quad j\in\bar{T},\ p\in\overline{1,m}.\end{cases}$$

Then we obtain another supersymmetric polynomial representation of $osp(m,2n)$, which is obtained from the corresponding canonical one ($T=\emptyset$) by swapping ∂_{x_i} and $-x_i$ for $i\in T$. The subspace

$$\mathcal{A}'_k=\text{Span}\,\{x^\alpha\theta_{i_1}\cdots\theta_{i_t}\in\mathcal{A}'\mid t\in\overline{0,m};\ i_1,\cdots,i_t\in\overline{1,m};\ \alpha\in\mathbb{N}^{2n};\ \sum_{i\in\bar{T}}\alpha_i-\sum_{i\in T}\alpha_i=k-t\} \tag{1.15}$$

forms an $osp(m,2n)$-submodule for $k\in\mathbb{N}$. Denote

$$S_1=\{i\in\overline{1,2n}\mid i\in\bar{T},\ n+i\in\bar{T}\},\ \ T_1=\{i\in\overline{1,2n}\mid i\in T,\ n+i\in T\}. \tag{1.16}$$

Theorem 2. *The following statements hold:*

1) The submodule $\mathcal{A}'_k$ is irreducible when $S_1\cup T_1\neq\emptyset$. In particular, $\mathcal{A}'_k$ is not highest weight type if $S_1\neq\emptyset$ and $T_1\neq\emptyset$.

2) Suppose $S_1=\emptyset$ and $T_1=\emptyset$. We may assume $T=\overline{1,n}$ by symmetry.

a) The submodule $\mathcal{A}'_k$ is irreducible when $k\neq m_1$.

b) The submodule $\mathcal{A}'_{m_1}=\langle\theta_1\cdots\theta_{m_1}\rangle\oplus\langle(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1}\rangle$ is a direct sum of two irreducible submodules.

Suppose that $m=2m_1+1$ is an odd integer. The orthosymplectic Lie superalgebra $osp(m,2n)$ is the subalgebra of $gl(m,2n)$ consisting of the matrices of the form

$$\begin{pmatrix} A & B & U & H & H_1\\ C & -A^T & V & K & K_1\\ -V^T & U^T & 0 & M & M_1\\ K_1^T & H_1^T & M_1^T & D & E\\ -K^T & -H^T & -M^T & F & -D^T\end{pmatrix} \tag{1.17}$$

where A, B and C are $m_1\times m_1$ matrices such that $B=-B^T$, $C=-C^T$; D, E and F are $n\times n$ matrices such that $E=E^T$, $F=F^T$; H, H_1, K and K_1 are $m_1\times n$ matrices; U and V are $m_1\times 1$ matrices; M and M_1 are $1\times n$ matrices. Similarly, we have a representation of $osp(m,2n)$ on $\mathcal{A}=\mathbb{C}[x_1,\cdots,x_m,\theta_1,\cdots,\theta_{2n}]$ via (1.6), (1.17) and a representation of $osp(m,2n)$ on $\mathcal{A}'=\mathbb{C}[x_1,\cdots,x_{2n};\theta_1,\cdots,\theta_m]$ via (1.14), (1.17).

Theorem 3. *All the $osp(2m+1,2n)$-submodules $\mathcal{H}_k^r$ and $\mathcal{A}'_k$ are irreducible.*

In addition to the results given in the above, we have also constructed a basis for the module $\mathcal{H}_k^r$, and the submodules $\langle\theta_1\cdots\theta_{m_1}\rangle$ and $\langle(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1}\rangle$ in b) of 2) in Theorem 2.

In Section 2, we prove Theorem 1. Moreover, Theorem 2 is proved in Section 3. We give a proof of Theorem 3 in Section 4.

2 Proof of Theorem 1

In this section, we discuss the polynomial representations of $osp(2m_1,2n)$ $(m_1>0,n>0)$ defined via (1.6).

Recall the Lie superalgebra $osp(2m_1,2n)$ given (1.4). The even part of $osp(2m,2n)$

$$\begin{aligned}
osp(2m_1,2n)_0 = \sum_{i,j=1}^{m_1}&\big[\mathbb{C}(E_{i,j}-E_{m_1+j,m_1+i})+\mathbb{C}(E_{i,m_1+j}-E_{j,m_1+i})\\
&+\mathbb{C}(E_{m_1+i,j}-E_{m_1+j,i})\big]+\sum_{p,q=1}^{n}\big[\mathbb{C}(E_{2m_1+p,2m_1+q}-E_{2m_1+n+q,2m_1+n+p})\\
&+\mathbb{C}(E_{2m_1+p,2m_1+n+q}+E_{2m_1+q,2m_1+n+p})\\
&+\mathbb{C}(E_{2m_1+n+p,2m_1+q}+E_{2m_1+n+q,2m_1+p})\big] \qquad (2.1)
\end{aligned}$$

is a subalgebra isomorphic to $o(2m_1,\mathbb{C})\oplus sp(2n,\mathbb{C})$ and the odd part is

$$\begin{aligned}
osp(2m_1,2n)_1 \;=\; &\sum_{i=1}^{m_1}\sum_{p=1}^{n}\big[\mathbb{C}(E_{i,2m_1+p}-E_{2m_1+n+p,m_1+i})+\mathbb{C}(E_{i,2m_1+n+p}+E_{2m_1+p,m_1+i})\\
&+\mathbb{C}(E_{m_1+i,2m_1+p}-E_{2m_1+n+p,i})+\mathbb{C}(E_{m_1+i,2m_1+n+p}+E_{2m_1+p,i})\big]. \qquad (2.2)
\end{aligned}$$

Take

$$H=\sum_{i=1}^{m_1}\mathbb{C}(E_{i,i}-E_{m_1+i,m_1+i})+\sum_{j=1}^{n}\mathbb{C}(E_{2m_1+j,2m_1+j}-E_{2m_1+n+j,2m_1+n+j}) \qquad (2.3)$$

as a Cartan subalgebra of $osp(2m_1,2n)$. Let $\lambda_1,\cdots,\lambda_{m_1},\ \nu_1,\cdots,\nu_n$ be the fundamental weights of $o(2m_1,\mathbb{C})\oplus sp(2n,\mathbb{C})$. Let

$$\begin{aligned}
osp(2m_1,2n)^+ \;=\; &\sum_{1\le i<j\le m_1}\big(\mathbb{C}(E_{i,j}-E_{m_1+j,m_1+i})+\mathbb{C}(E_{i,m_1+j}-E_{j,m_1+i})\big)\\
&+\sum_{1\le p<q\le n}\mathbb{C}(E_{2m_1+p,2m_1+q}-E_{2m_1+n+q,2m_1+n+p})\\
&+\sum_{1\le p\le q\le n}\mathbb{C}(E_{2m_1+p,2m_1+n+q}+E_{2m_1+q,2m_1+n+p})\\
&+\sum_{1\le i\le m_1,1\le q\le n}\big(\mathbb{C}(E_{i,2m_1+q}-E_{2m_1+n+q,m_1+i})\\
&+\mathbb{C}(E_{i,2m_1+n+q}+E_{2m_1+q,m_1+i})\big), \qquad (2.4)
\end{aligned}$$

and $osp(2m_1,2n)^+_\sigma=osp(2m_1,2n)_\sigma\cap osp(2m_1,2n)^+$ for $\sigma=0,1$. A weight vector $f\in\mathcal{A}^r$ is called a *highest weight vector* if $osp(2m_1,2n)^+(f)=0$, and the corresponding weight is called the *highest weight*.

We first quote a useful lemma found by Xu [28].

Lemma 2.1 *Suppose $\mathcal{A}$ is a free module over a subalgebra B generated by a filtrated subspace $V=\bigcup_{r=0}^{\infty}V_r$ (i.e., $V_r\subset V_{r+1}$). Let $\mathcal{T}_1$ be a linear operator on $\mathcal{A}$ with a right inverse $\mathcal{T}_1^-$ such that*

$$\mathcal{T}_1(B),\mathcal{T}_1^-(B)\subset B,\quad \mathcal{T}_1(\eta_1\eta_2)=\mathcal{T}_1(\eta_1)\eta_2,\quad \mathcal{T}_1^-(\eta_1\eta_2)=\mathcal{T}_1^-(\eta_1)\eta_2 \qquad (2.5)$$

for $\eta_1 \in B$, $\eta_2 \in V$, *and let* $\mathcal{T}_2$ *be a linear operator on* $\mathcal{A}$ *such that*

$$\mathcal{T}_2(V_{r+1}) \subset BV_r, \quad \mathcal{T}_2(f\zeta) = f\mathcal{T}_2(\zeta) \quad for \quad 0 \le r \in \mathbb{Z}, \quad f \in B, \quad \zeta \in \mathcal{A}. \tag{2.6}$$

Then we have

$$\begin{aligned} & \{g \in \mathcal{A} \mid (\mathcal{T}_1 + \mathcal{T}_2)(g) = 0\} \\ = \quad & Span\,\{\sum_{i=0}^{\infty}(-\mathcal{T}_1^- \mathcal{T}_2)^i(hg) | g \in V, h \in B; \mathcal{T}_1(h) = 0\}, \end{aligned} \tag{2.7}$$

where the summation is finite under our assumption.

Below we discuss the $osp(2m_1, 2n)$-module structure of $\mathcal{A}_k^r$ case by case.

Case 1. $r = 0$

In this case,

$$\Delta = \sum_{i=1}^{m_1} \partial_{x_i}\partial_{x_{m_1+i}} + \sum_{j=1}^{n} \partial_{\theta_j}\partial_{\theta_{n+j}}, \quad \eta = \sum_{i=1}^{m_1} x_i x_{m_1+i} + \sum_{j=1}^{n} \theta_j\theta_{n+j}. \tag{2.8}$$

The subspaces $\mathcal{A}_k^0$ ($k \in \mathbb{N}$) are all finite dimensional and $\mathcal{A}_k^0 = 0$ when $k < 0$. Moreover, $\mathcal{A}_k^0 = \mathcal{H}_k^0 \oplus \eta\mathcal{A}_{k-2}^0$ when $m_1 = 0$ or $n = 0$.

Theorem 2.2 *If* $k > 2(n - m_1 + 1)$ *or* $k \le (n - m_1 + 1)$, *then* $\mathcal{A}_k^0 = \mathcal{H}_k^0 \oplus \eta\mathcal{A}_{k-2}^0$ *and the subspace* $\mathcal{H}_k^0$ *is an irreducible* $osp(2m_1, 2n)$*-submodule with highest weight vector* x_1^k *and the corresponding highest weight* $k\lambda_1$. *When* $(n - m_1 + 1) < k \le 2(n - m_1 + 1)$,

$$\mathcal{H}_k^0 \supset \eta^{k-(n-m_1+1)}\mathcal{H}_{2(n-m_1+1)-k}^0 \supset \{0\} \tag{2.9}$$

is a composition series. Moreover,

$$\begin{aligned} \{ \sum_{\substack{r_1,\cdots,r_{m_1-1}\in\mathbb{N},\\ l_1,\cdots,l_n\in\{0,1\}}} & \frac{(-1)^{\sum_{i=1}^{m_1-1} r_i + \sum_{j=1}^{n} l_j} \alpha_{m_1}!\alpha_{2m_1}! \prod_{i=1}^{m_1-1} r_i! \begin{pmatrix} \alpha_i \\ r_i \end{pmatrix} \begin{pmatrix} \alpha_{m_1+i} \\ r_i \end{pmatrix} \prod_{j=1}^{n} \delta_{l_j,1}(-1)^{\beta_j}\beta_j\beta_{n+j}}{(\alpha_{m_1} + \sum_{i=1}^{m_1-1} r_i + \sum_{j=1}^{n} l_j)!(\alpha_{2m_1} + \sum_{i=1}^{m_1-1} r_i + \sum_{j=1}^{n} l_j)!} \\ \times x_{m_1}^{\alpha_{m_1} + \sum_{i=1}^{m_1-1} r_i + \sum_{j=1}^{n} l_j} & x_{2m_1}^{\alpha_{2m_1} + \sum_{i=1}^{m_1-1} r_i + \sum_{j=1}^{n} l_j} \prod_{i=1}^{m_1-1} x_i^{\alpha_i - r_i} x_{m_1+i}^{\alpha_{m_1+i} - r_i} \prod_{j=1}^{n} \theta_j^{\beta_j - l_j} \beta_{n+j}^{\beta_{n+j} - l_j} \\ \mid \alpha_1, \cdots, \alpha_{2m_1} \in \mathbb{N}, l_1 \cdots, l_{2n} \in \{0,1\}; \; & \alpha_{m_1}\alpha_{2m_1} = 0; \; \sum_{i=1}^{2m_1} \alpha_i + \sum_{j=1}^{2n} l_j = k\} \end{aligned} \tag{2.10}$$

is a basis of $\mathcal{H}_k^0$ ($k > 0$).

Proof. We divide our arguments as the following steps.

(1) *The submodule* $\mathcal{H}_k^0$ *is generated by* x_1^k *for* $k > 0$.

It is well known that $\mathcal{H}_k^0 = \langle x_1^k \rangle$ when $n = 0$. Now assume $k \geq 2$. Take induction on n. For any $0 \neq f \in \mathcal{H}_k^0$, we can write

$$f = f_0 + f_1\theta_n + f_2\theta_{2n} + f_3\theta_n\theta_{2n}, \tag{2.11}$$

with

$$f_i \in \mathbb{C}[x_1, \cdots, x_{2m_1}; \theta_1 \cdots, \theta_{n-1}, \theta_{n+1}, \cdots, \theta_{2n-1}]. \tag{2.12}$$

We denote

$$\Delta' = \sum_{i=1}^{m_1} \partial_{x_i}\partial_{x_{m_1+i}} + \sum_{j=1}^{n-1} \partial_{\theta_j}\partial_{\theta_{n+j}} \tag{2.13}$$

and get

$$0 = \Delta(f) = \Delta' f_0 + (\Delta' f_1)\theta_n + (\Delta' f_2)\theta_{2n} + (\Delta' f_3)\theta_n\theta_{2n} - f_3. \tag{2.14}$$

Hence

$$\Delta'(f_1) = \Delta'(f_2) = \Delta'(f_3) = 0, \quad \Delta'(f_0) - f_3 = 0. \tag{2.15}$$

By induction,

$$f_1 = X'(x_1^{k-1}), \quad f_2 = X''(x_1^{k-1}), \quad f_3 = X(x_1^{k-2}) \tag{2.16}$$

for some $X', X'', X \in U(osp(2m_1, 2(n-1)))$. Thus

$$f_1\theta_n = \frac{1}{k}X'(E_{m_1+1,2m_1+2n} + E_{2m_1+n,1})(x_1^k), \tag{2.17}$$

$$f_2\theta_{2n} = \frac{1}{k}X''(E_{2m_1+2n,1} - E_{m_1+1,2m_1+n})(x_1^k). \tag{2.18}$$

Moreover,

$$f_3 = X(x_1^{k-2}) = \Delta'\big(\frac{1}{k-1}X(x_1^{k-1}x_{m_1+1})\big) \tag{2.19}$$

because $[\Delta', X] = 0$. So

$$\begin{aligned} 0 &= \Delta'(f_0) - f_3 = \Delta'(f_0) - \Delta'\big(\frac{1}{k-1}X(x_1^{k-1}x_{m_1+1})\big) \\ &= \Delta'(f_0 - \frac{1}{k-1}X(x_1^{k-1}x_{m_1+1})), \end{aligned} \tag{2.20}$$

which implies

$$f_0 - \frac{1}{k-1}X(x_1^{k-1}x_{m_1+1}) \in U(osp(2m_1, 2(n-1)))(x_1^k). \tag{2.21}$$

Note

$$\begin{aligned} &\frac{1}{k-1}x_1^{k-1}x_{m_1+1} + x_1^{k-2}\theta_n\theta_{2n} \\ = \ &\frac{1}{k(k-1)}(E_{m_1+1,2m_1+n} - E_{2m_1+2n,m})(E_{m_1+1,2m_1+2n} + E_{2m_1+n,1})(x_1^k). \end{aligned} \tag{2.22}$$

Thus

$$\begin{aligned} f &= (f_0 - \frac{1}{k-1}X(x_1^{k-1}x_{m_1+1})) + f_1\theta_n + f_2\theta_{2n} \\ &\quad + X(\frac{1}{k-1}x_1^{k-1}x_{m_1+1} + x_1^{k-2}\theta_n\theta_{2n}) \in \langle x_1^k \rangle. \end{aligned} \tag{2.23}$$

(2) *The submodule $\mathcal{H}_k^0$ is irreducible if $k \leq (n - m_1 + 1)$ or $k > 2(n - m_1 + 1)$.*

Denote

$$\eta_x = \sum_{i=1}^{m_1} x_i x_{m_1+i}, \quad \eta_\theta = \sum_{j=1}^{n} \theta_j \theta_{n+j}. \tag{2.24}$$

Then $\eta = \eta_x + \eta_\theta$. By direct calculation, we get all the weight vectors in $\mathcal{A}_k^0$ annihilated by $osp(2m, 2n)_0^+$ are scalar multiples of the following elements

$$\sum_{i=0}^{l} a_i x_1^{k-2l-t} \eta_x^{l-i} \eta_\theta^i \theta_1 \cdots \theta_t \tag{2.25}$$

with $l - 2l - t \geq 0$, $l \geq 0$, $0 \leq t \leq n$ and $a_i = 0$ for $i > n - t$. Suppose $\sum\limits_{i=0}^{l} a_i x_1^{k-2l-t} \eta_x^{l-i} \eta_\theta^i \theta_1 \cdots \theta_t \in \mathcal{H}_k^0$. Then

$$\begin{aligned} 0 &= \Delta(\sum_{i=0}^{l} a_i x_1^{k-2l-t} \eta_x^{l-i} \eta_\theta^i \theta_1 \cdots \theta_t) \\ &= \sum_{i=0}^{l-1} a_i (l-i)(k-l-t+m_1-1-i) x_1^{k-2l-t} \eta_x^{l-i-1} \eta_\theta^i \theta_1 \cdots \theta_t \\ &\quad - \sum_{i=1}^{l} a_i i (n-t-i+1) x_1^{k-2l-t} \eta_x^{l-i} \eta_\theta^{i-1} \theta_1 \cdots \theta_t \end{aligned} \tag{2.26}$$

by (4.31) in [28]. So

$$a_i (l-i)(k-l-t+m_1-1-i) - a_{i+1}(i+1)(n-t-i) = 0 \tag{2.27}$$

for $0 \leq i < n - t$ and $l \leq n - t$. Thus up to a scalar multiple, all the weight vectors in $\mathcal{H}_k^0$ annihilated by $osp(2m_1, 2n)_0^+$ are

$$f_{l,t} = \sum_{i=0}^{l} \frac{(n-t-i)!}{i!(l-i)!(k-l-t+m_1-1-i)!} x_1^{k-2l-t} \eta_x^{l-i} \eta_\theta^i \theta_1 \cdots \theta_t, \tag{2.28}$$

for $0 \leq t \leq n$ and $0 \leq l \leq \min\{n-t, \frac{1}{2}(k-t)\}$.

Note

$$f_{0,t-1} = (-1)^{t-1} \frac{n-t+1}{k-t+m_1} (E_{1,2m_1+t} - E_{2m_1+n+t,m_1+1})(f_{0,t}) \tag{2.29}$$

for $0 < t \leq n$ and

$$(E_{1,2m_1+n+t+1} + E_{t+1,m_1+1})(f_{l,t}) = (-1)^{t-1}(k+m_1-n-1-l) f_{l-1,t+1} \tag{2.30}$$

for $0 < l \leq \min\{n-t, \frac{1}{2}(k-t)\}$. If $k \leq (n - m_1 + 1)$, then

$$k + m_1 - n - 1 - l \leq -l < 0. \tag{2.31}$$

When $k > 2(n - m_1 + 1)$,

$$k + m_1 - n - 1 - l \geq k + m_1 - n - 1 - \frac{1}{2}(k-t) \geq \frac{1}{2}k - (n - m_1 + 1) > 0. \tag{2.32}$$

Thus x_1^k is the only highest weight vector in $\mathcal{H}_k^0$ up to a scalar multiple, which implies $\mathcal{H}_k^0$ is irreducible by (1).

(3) *$\mathcal{A}_k^0 = \mathcal{H}_k^0 \oplus \eta\mathcal{A}_{k-2}^0$ when $k \leq (n - m_1 + 1)$ or $k > 2(n - m_1 + 1)$.*

Since $x_1^k \notin \eta\mathcal{A}_{k-2}^0$, we get $\mathcal{H}_k^0 \cap \eta\mathcal{A}_{k-2}^0 = 0$. Now we still have to show that $\mathcal{A}_k^0 = \mathcal{H}_k^0 + \eta\mathcal{A}_{k-2}^0$. Note that $\mathcal{A}_k^0$ is a finite-dimensional $osp(2m_1, 2n)_0$-module in this case. It is sufficient to check

$$x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p}\eta_\theta^p \in \mathcal{H}_k^0 + \eta\mathcal{A}_{k-2}^0 \text{ for all } 0 \leq t \leq n;\ p \leq l \in \mathbb{N}. \tag{2.33}$$

Observe

$$x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p}\eta_\theta^p = \eta \sum_{i=0}^{n-t-p} (-1)^i x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p-i-1}\eta_\theta^{p+i} \tag{2.34}$$

if $l > n - t$. When $l \leq n - t$, we have

$$\Delta\Big(\sum_{s=0}^{l} \frac{(n-t-s)!}{s!(l-s)!(k-l-t+n-s-1)!} x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-s}\eta_\theta^s\Big) = 0, \tag{2.35}$$

$$\begin{aligned} x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p}\eta_\theta^p &= \eta\Big(\sum_{i=0} (-1)^i x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p-i-1}\eta_\theta^{p+i}\Big) \\ &\quad + (-1)^{s-p} x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-s}\eta_\theta^s \end{aligned} \tag{2.36}$$

for $s > p$ and

$$\begin{aligned} x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p}\eta_\theta^p &= \eta\Big(\sum_{i=0}^{p-s-1} (-1)^i x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p+i}\eta_\theta^{p-i-1}\Big) \\ &\quad + (-1)^{p-s} x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-s}\eta_\theta^s \end{aligned} \tag{2.37}$$

for $s < p$. Thus

$$\begin{aligned} &\Big(\sum_{s=0}^{l} \frac{(-1)^{p-s}(n-t-s)!}{s!(l-s)!(k-l-t+n-s-1)!}\Big) x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p}\eta_\theta^p \\ = &\sum_{s=0}^{l} \frac{(n-t-s)!}{s!(l-s)!(k-l-t+n-s-1)!} x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-s}\eta_\theta^s \\ &+\eta\Big(\sum_{s=0}^{p-1} \sum_{i=0}^{p-s-1} (-1)^{p-s+i} x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p+i}\eta_\theta^{p-i-1} \\ &+ \sum_{s=p+1}^{l} \sum_{i=0} (-1)^{i+s-p} x_1^{k-2l-t}\theta_1 \cdots \theta_t \eta_x^{l-p-i-1}\eta_\theta^{p+i}\Big), \end{aligned} \tag{2.38}$$

which implies (2.33) holds.

(4) *$\mathcal{H}_k^0$ has only one nonzero proper submodule $\eta^{k-(n-m_1+1)}\mathcal{H}_{2(n,m_1+1)-k}^0$ when $(n - m_1 + 1) < k \leq 2(n - m_1 + 1)$.*

According to (2.28), we have

$$\mathcal{H}_k^0 = \bigoplus_{\substack{0\leq t\leq n,\\ 0\leq l\leq \min\{n-t,\frac{1}{2}(k-t)\}}} U(osp(2m_1, 2n)_0(f_{l,t}). \tag{2.39}$$

Since

$$f_{k-(n-m_1+1),0} = \frac{1}{(k-n+m_1-1)!}\eta^{k-n+m_1-1}x_1^{2(n-m_1+1)-k} \in \mathcal{H}_k^0 \tag{2.40}$$

and $\mathcal{H}^0_{2(n-m_1+1)-k} = \langle x_1^{2(n-m_1+1)-k}\rangle$, we have

$$\eta^{k-(n-m_1+1)}\mathcal{H}^0_{2(n-m_1+1)-k} = \eta^{k-(n-m_1+1)}\langle x_1^{2(n-m_1+1)-k}\rangle \subset \mathcal{H}_k^0. \tag{2.41}$$

Note

$$\Delta\Big(\sum_{i=0}^{l-k+(n-m_1+1)} \frac{(n-i)!\eta_x^{l-k+n-m_1+1-i}\eta_\theta^i}{i!(l-k+n-m_1+1-i)!(n-l-i)!}x_1^{k-2l}\Big) = 0, \tag{2.42}$$

which implies

$$\begin{aligned}&\eta^{k-(n-m_1+1)}\Big(\sum_{i=0}^{l-k+(n-m_1+1)} \frac{(n-i)!\eta_x^{l-k+n-m_1+1-i}\eta_\theta^i}{i!(l-k+n-m_1+1-i)!(n-l-i)!}x_1^{k-2l}\Big)\\ &\in \eta^{k-(n-m_1+1)}\langle x_1^{2(n-m_1+1)-k}\rangle \subset \mathcal{H}_k^0\end{aligned} \tag{2.43}$$

for $l \geq k-(n-m_1+1)$. Observe

$$osp(2m_1, 2n)_0^+\eta^{k-(n-m_1+1)}\Big(\sum_{i=0}^{l-k+(n-m_1+1)} \frac{(n-i)!\eta_x^{l-k+n-m_1+1-i}\eta_\theta^i}{i!(l-k+n-m_1+1-i)!(n-l-i)!}x_1^{k-2l}\Big) = 0. \tag{2.44}$$

Thus we get

$$\eta^{k-(n-m_1+1)}\Big(\sum_{i=0}^{l-k+(n-m_1+1)} \frac{(n-i)!\eta_x^{l-k+n-m_1+1-i}\eta_\theta^i}{i!(l-k+n-m_1+1-i)!(n-l-i)!}x_1^{k-2l}\Big) = c_l f_{l,0} \tag{2.45}$$

for some $c_l \in \mathbb{C}$ by (4.14). Hence $f_{l,0} \in \eta^{k-(n-m_1+1)}\mathcal{H}^0_{2(n-m_1+1)-k}$. Consequently, $f_{l,t} \in \eta^{k-(n-m_1+1)}\mathcal{H}^0_{2(n-m_1+1)-k}$ by (2.30) for all $l > k-(n-m_1+1)$.

Suppose that W is a nonzero submodule of $\mathcal{H}_k^0$. If there exists $f_{l,t} \in W$ for some $l < k-(n-m_1+1)$, then $f_{0,l+t} \in W$ by (2.30), which implies $x_1^k \in W$ by (2.29); otherwise $W \subset \eta^{k-(n-m_1+1)}\mathcal{H}^0_{2(n-m_1+1)-k}$, which means $W = \eta^{k-(n-m_1+1)}\mathcal{H}^0_{2(n-m_1+1)-k}$ since $\eta^{k-(n-m_1+1)}\mathcal{H}^0_{2(n-m_1+1)-k}$ is an irreducible $osp(2m_1, 2n)$-submodule. □

Remark. In the appendix of [29], Zhang presented the structure of a canonical supersymmetric polynomial representation for $osp(m|2n)$ when $m-2n > 1$. Our above theorem give a complete answer to the structure of the representation.

Case 2. $r = m_1$.

In this case,

$$\Delta = -\sum_{i=1}^{m_1} x_i \partial_{x_{m_1+i}} + \sum_{j=1}^{n} \partial_{\theta_j}\partial_{\theta_{n+j}}, \quad \eta = \sum_{i=1}^{m_1} x_{m_1+i}\partial_{x_i} + \sum_{j=1}^{n} \theta_j\theta_{n+j}. \tag{2.46}$$

Furthermore, $\mathcal{H}_k^{m_1} = 0$ when $k \geq n$.

By similar arguments as those in Case 1, we can obtain that $\mathcal{H}_k^{m_1}$ is irreducible and $\mathcal{A}_k^{m_1} = \mathcal{H}_k^{m_1} \oplus \eta\mathcal{A}_{k-2}^{m_1}$ when $k \leq n$. Set

$$\begin{aligned} h(\vec{k},\vec{l},\vec{s}) \;=\; & \prod_{t=1}^{m_1} x_t^{k_t} \prod_{1\leq i<j\leq m_1} (x_i x_{m_1+j} - x_j x_{m_1+i})^{k_{i,j}} \prod_{j=1}^{n} \theta_j^{l_j}\theta_{n+j}^{l_{n+j}} \prod_{1\leq i<j\leq n} (\theta_i\theta_{n+i} \\ & -\theta_j\theta_{n+j})^{l_{i,j}} \prod_{1\leq p\leq m_1, 1\leq q\leq n} (x_{m_1+p} - x_p\theta_q\theta_{n+q})^{s_{p,q}} \end{aligned} \tag{2.47}$$

where

$$\vec{k} = (k_1, \cdots, k_{m_1}; k_{1,2}, k_{1,3}, \cdots, k_{1,m_1}, k_{2,3}\cdots, k_{m_1-1,m_1}) \in \mathbb{N}^{\frac{m_1(m_1+1)}{2}}, \tag{2.48}$$

$$\vec{l} = (l_1, \cdots, l_n; l_{1,2}, \cdots, l_{1,n}, l_{2,3}, \cdots, l_{2,n}, \cdots, l_{n-1,n}) \in \{0,1\}^{\frac{n(n+1)}{2}}, \tag{2.49}$$

$$\vec{s} = (s_{1,1}, \cdots, s_{1,n}, \cdots, s_{m_1,n}) \in \{0,1\}^{m_1 n}. \tag{2.50}$$

Denote

$$\begin{aligned} I \;=\; & \{(\vec{k},\vec{l},\vec{s}) \mid l_t + l_{n+t} + \sum_{1\leq i<t} l_{i,t} + \sum_{t<j\leq n} l_{t,j} + \sum_{p=1}^{m_1} s_{p,t} \leq 1 \text{ for } t \in \overline{1,n}; \\ & k_{i,j}k_t = 0 \text{ for } i<j<t;\; k_{i,j}k_{i',j'} = 0 \text{ for } i>i' \text{ and } j<j'; \\ & k_t l_{i,j} = 0 \text{ for } t \in \overline{1,m_1}, 1\leq i<j\leq n;\; k_t s_{p,q} = 0 \text{ for } t<p; \\ & k_{i,j}s_{p,q} = 0 \text{ for } i<j<p;\; s_{p,q}s_{p',q'} = 0 \text{ for } p>p' \text{ and } q<q'; \\ & l_{i,j} = 0 \text{ if } l_t = l_{n+t} = \sum_{p=1}^{m_1} s_{p,t} = \sum_{i'<t} l_{i',t} + \sum_{j'>t} l_{t,j'} = 0 \text{ for some } i<t<j; \\ & l_{i,j}l_{i',j'} = 0 \text{ if } i<i'<j<j';\; l_{i,j}s_{p,q} = 0 \text{ if } i<j<q\}. \end{aligned} \tag{2.51}$$

Theorem 2.3 *If $k \leq n$, the subspace $\mathcal{H}_k^{m_1}$ is an irreducible highest weight submodule. The highest weight is $-2\lambda_{m_1} + \nu_k$ (resp. $k\lambda_{m_1-1} - (k+2)\lambda_{m_1}$) if $k>0$ (resp. $k\leq 0$) and a corresponding highest weight vector is $\theta_1\cdots\theta_k$ (resp. $x_{m_1}^{-k}$). Moreover, $\mathcal{A}_k^{m_1} = \mathcal{H}_k^{m_1} \oplus \eta\mathcal{A}_{k-2}^{m_1}$ and*

$$\{h(\vec{k},\vec{l},\vec{s}) \mid (\vec{k},\vec{l},\vec{s}) \in I;\; \sum_{t=1}^{n}(l_t + l_{n+t}) + 2\sum_{1\leq i<j\leq n} l_{i,j} + \sum_{1\leq p\leq m_1, 1\leq q\leq n} s_{p,q} - \sum_{t=1}^{m_1} k_t = k;\}$$
$$\tag{2.52}$$

forms a basis for $\mathcal{H}_k^{m_1}$.

Proof. Denote by V the subspace spanned by (2.52). It is easy to check $\mathcal{H}_k^{m_1} \supset V$. For the reverse inclusion, we prove it by induct on n. Take any $0 \neq f \in \mathcal{H}_k^{m_1}$. We write

$$f = f_0 + f_1\theta_n + f_2\theta_{2n} + f_3\theta_n\theta_{2n} \tag{2.53}$$

where

$$f_i \in \mathbb{C}[x_1, \cdots, x_{2m_1}; \theta_1, \cdots, \theta_{n-1}, \theta_{n+1} \cdots, \theta_{2n-1}]. \tag{2.54}$$

Let

$$\Delta' = -\sum_{i=1}^{m_1} x_i \partial_{x_{m_1+i}} + \sum_{j=1}^{n-1} \partial_{\theta_j}\partial_{\theta_{n+j}}. \tag{2.55}$$

Since

$$\Delta(f) = 0 = \Delta'(f_0) + \Delta'(f_1)\theta_n + \Delta'(f_2)\theta_n + \Delta'(f_3)\theta_n\theta_{2n} - f_3, \tag{2.56}$$

we get

$$\Delta'(f_0) - f_3 = \Delta'(f_1) = \Delta'(f_2) = \Delta'(f_3) = 0. \tag{2.57}$$

Thus by inductive assumption, we obtain $f_1\theta_n, f_2\theta_{2n} \in V$. We may assume $f_3 = h(\vec{k}, \vec{l}, \vec{s})$ with $(\vec{k}, \vec{l}, \vec{s}) \in I$ and

$$\sum_{t=1}^{n}(l_t + l_{n+t}) + 2\sum_{1\leq i<j\leq n} l_{i,j} + \sum_{1\leq p\leq m_1, 1\leq q\leq n} s_{p,q} - \sum_{t=1}^{m_1} k_t = k-2, \tag{2.58}$$

$$l_n = l_{2n} = l_{1,n} = \cdots = l_{n-1,n} = s_{1,n} = \cdots = s_{m_1,n} = 0. \tag{2.59}$$

Suppose that there exists some $t \in \overline{1, m_1}$ such that $k_t > 0$. Let

$$t_0 = \min\{t \mid k_t > 0\}, \quad f_0' = -x_{t_0}^{-1} x_{m_1+t_0} f_3. \tag{2.60}$$

We have

$$\Delta'(f_0') = f_3 \tag{2.61}$$

and

$$0 = \Delta(f_0 + f_3\theta_n\theta_{2n}) = \Delta'(f_0 - f_0') + (\Delta'(f_0') - f_3). \tag{2.62}$$

So $f_0 - f_0' \in V$ by inductive assumption and

$$f_0' + f_3\theta_n\theta_{2n} = \Delta'(f_0) - f_3 = -x_{t_0}^{-1} f_3(x_{m_1+t_0} - x_{t_0}\theta_n\theta_{2n}) \in V. \tag{2.63}$$

Thus $f \in V$.

Now we assume $k_t = 0$ for all $t \in \overline{1, m_1}$. Then there must exist $j \in \overline{1, n-1}$ such that

$$l_j = l_{n+j} = \sum_{1\leq i<j} l_{i,j} = \sum_{j<t\leq n} l_{j,t} = \sum_{p=1}^{m_1} s_{p,j} = 0 \tag{2.64}$$

because

$$\sum_{t=1}^{n-1}\big(l_t + l_{n+t} + \sum_{1\leq i<t} l_{i,t} + \sum_{t<j\leq n} l_{t,j} + \sum_{p=1}^{m_1} s_{p,t}\big) = k-2 < n-1. \tag{2.65}$$

Set

$$j_0=\max\{j \mid l_j=l_{n+j}=\sum_{1\leq i<j} l_{i,j}=\sum_{j<t\leq n} l_{j,t}=\sum_{p=1}^{m_1} s_{p,j}=0\} \tag{2.66}$$

and $f_0''=-f_3\theta_{j_0}\theta_{n+j_0}$. We have $\Delta'(f_0'')=f_3$ and

$$0=\Delta(f_0+f_3\theta_n\theta_{2n})=\Delta'(f_0-f_0')+\Delta'(f_0'')-f_3. \tag{2.67}$$

So

$$f_0-f_0''\in V,\ \ f_0''+f_3\theta_n\theta_{2n}=-f_3(\theta_{j_0}\theta_{n+j_0}-\theta_n\theta_{2n})\in V, \tag{2.68}$$

which implies $f\in V$.

Next we check the linear independence of (2.52). Again induct on n. Suppose

$$\sum_{\substack{(\vec{k},\vec{l},\vec{s})\in I\\ \sum_{t=1}^{n}(l_t+l_{n+t})+2\sum_{1\leq i<j\leq n} l_{i,j}+\sum_{1\leq p\leq m_1,1\leq q\leq n} s_{p,q}-\sum_{t=1}^{m_1} k_t=k}} a_{\vec{k},\vec{l},\vec{s}}h(\vec{k},\vec{l},\vec{s})=0. \tag{2.69}$$

We write

$$h(\vec{k},\vec{l},\vec{s})=h'(\vec{k},\vec{l},\vec{s})(\theta_i\theta_{n+i}-\theta_n\theta_{2n}) \text{ if } l_{i,n}=1, \tag{2.70}$$
$$h(\vec{k},\vec{l},\vec{s})=h''(\vec{k},\vec{l},\vec{s})(x_{m_1+p}-x_p\theta_n\theta_{2n}) \text{ if } s_{p,n}=1. \tag{2.71}$$

Thus (2.69) becomes

$$\begin{aligned}&\sum_{(\vec{k},\vec{l},\vec{s})\in I;l_n=1} a_{\vec{k},\vec{l},\vec{s}}h(\vec{k},\vec{l},\vec{s})+\sum_{(\vec{k},\vec{l},\vec{s})\in I;l_{2n}=1} a_{\vec{k},\vec{l},\vec{s}}h(\vec{k},\vec{l},\vec{s})+\sum_{\substack{(\vec{k},\vec{l},\vec{s})\in I;\\ l_n=l_{2n}=\sum l_{i,n}=\sum s_{p,n}=0}} a_{\vec{k},\vec{l},\vec{s}}h(\vec{k},\vec{l},\vec{s})\\ &+\sum_{i=1}^{n-1}\sum_{(\vec{k},\vec{l},\vec{s})\in I;l_{i,n}=1} a_{\vec{k},\vec{l},\vec{s}}h'(\vec{k},\vec{l},\vec{s})(\theta_i\theta_{n+i}-\theta_n\theta_{2n})\\ &+\sum_{p=1}^{m_1}\sum_{(\vec{k},\vec{l},\vec{s})\in I;s_{p,n}=1} a_{\vec{k},\vec{l},\vec{s}}h''(\vec{k},\vec{l},\vec{s})(x_{m_1+p}-x_p\theta_n\theta_{2n})=0.\end{aligned} \tag{2.72}$$

We get

$$a_{\vec{k},\vec{l},\vec{s}}=0 \text{ if } l_n=l_{2n}=\sum l_{i,n}=\sum s_{p,n}=0 \text{ or } l_n=1 \text{ or } l_{2n}=1, \tag{2.73}$$

and

$$\sum_{i=1}^{n-1}\sum_{(\vec{k},\vec{l},\vec{s})\in I;l_{i,n}=1} a_{\vec{k},\vec{l},\vec{s}}h'(\vec{k},\vec{l},\vec{s})+\sum_{p=1}^{m_1}\sum_{(\vec{k},\vec{l},\vec{s})\in I;s_{p,n}=1} a_{\vec{k},\vec{l},\vec{s}}x_ph''(\vec{k},\vec{l},\vec{s})=0. \tag{2.74}$$

Since $h'(\vec{k},\vec{l},\vec{s})$ and $x_ph''(\vec{k},\vec{l},\vec{s})$ are linearly independent by inductive assumption, we get $a_{\vec{k},\vec{l},\vec{s}}=0$ for $l_{i,n}=1$ or $s_{p,n}=1$. □

Case 3. $0<r<m_1$

This case is a little more complicated. To study the structure of the submodules $\mathcal{A}_k^r$ and $\mathcal{H}_k^r$, we first introduce some subalgebras of $osp(2m_1, 2n)$. Set

$$\begin{aligned} L_1 = & \sum_{r+1\le i,j\le m_1} \mathbb{C}(E_{i,j} - E_{m_1+j,m_1+i}) + \sum_{r+1\le i<j\le m_1} \big(\mathbb{C}(E_{i,m_1+j} - E_{j,m_1+i}) \\ & +\mathbb{C}(E_{m_1+j,i} - E_{m_1+i,j})\big) \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} L_1' = & \sum_{r+1\le i,j\le m_1-1} \mathbb{C}(E_{i,j} - E_{m_1+j,m_1+i}) + \sum_{r+1\le i<j\le m_1-1} \big(\mathbb{C}(E_{i,m_1+j} - E_{j,m_1+i}) \\ & +\mathbb{C}(E_{m_1+j,i} - E_{m_1+i,j})\big). \end{aligned} \tag{2.76}$$

Denote $L_1^+ = osp(2m_1, 2n)^+ \cap L_1$ and $L_1'^+ = osp(2m_1, 2n)^+ \cap L_1'$. We treat $L_1' = 0$ when $r = m_1 - 1$. Let

$$\begin{aligned} L_2 \;\; = \;\; & \sum_{i,j=1}^{r} \big(\mathbb{C}(E_{i,j} - E_{m_1+j,m_1+i}) + \mathbb{C}(E_{i,m_1+j} - E_{j,m_1+i}) \\ & +\mathbb{C}(E_{m_1+i,j} - E_{m_1+j,i})\big) + \sum_{p,q=1}^{n} \big(\mathbb{C}(E_{2m_1+p,2m_1+q} - E_{2m_1+n+q,2m_1+n+p}) \\ & +\mathbb{C}(E_{2m_1+p,2m_1+n+q} + E_{2m_1+q,2m_1+n+p}) + \mathbb{C}(E_{2m_1+n+p,2m_1+q} + E_{2m_1+n+q,2m_1+p})\big) \\ & + \sum_{i\in\overline{1,r};p\in\overline{1,n}} \big(\mathbb{C}(E_{i,2m_1+p} - E_{2m_1+n+p,m_1+i}) + \mathbb{C}(E_{i,2m_1+n+p} + E_{2m_1+p,m_1+i}) \\ & +\mathbb{C}(E_{m_1+i,2m_1+p} - E_{2m_1+n+p,i}) + \mathbb{C}(E_{m_1+i,2m_1+n+p} + E_{2m_1+p,i}) \end{aligned} \tag{2.77}$$

and

$$\begin{aligned} L_2^+ \;\; = \;\; & L_2 \cap osp(2m_1, 2n)_0^+ + \sum_{1\le i\le m, 1\le p\le n} \big(\mathbb{C}(E_{m_1+i,2m_1+n+p} + E_{2m_1+p,i}) \\ & +\mathbb{C}(E_{i,2m_1+n+p} + E_{2m_1+p,m_1+i})\big). \end{aligned} \tag{2.78}$$

We have the following result:

Theorem 2.4 *When $0 < r < m_1$ and $k \le n - m_1 + r + 1$, the submodule $\mathcal{H}_k^r$ is irreducible and $\mathcal{A}_k^r = \mathcal{H}_k^r \bigoplus \eta\mathcal{A}_{k-2}^r$. If $k > n - m_1 + r + 1$, we have the following composition series*

$$\mathcal{H}_k^r \supset \eta^{k-n+m_1-r-1}\mathcal{H}^r_{-k+2(n-m_1+r+1)} \supset \{0\} \;\; \textit{if}\; r < m_1 - 1; \tag{2.79}$$

$$\mathcal{H}_k^{m_1-1} \supset \langle x_{m_1}^k \rangle \supset \eta^{k-n}\mathcal{H}^{m_1-1}_{-k+2n} \supset \{0\}. \tag{2.80}$$

The subspace $\mathcal{H}_k^r$ $(k \in \mathbb{Z})$ has a basis

$$\{\sum_{\substack{r_1,\cdots,r_{m_1-1}\in\mathbb{N};\\ l_1,\cdots,l_n\in\{0,1\}}} \frac{(-1)^{\sum_{i=r+1}^{m_1-1} r_i+\sum_{j=1}^{n} l_j}\prod_{i=1}^{r}\binom{\alpha_{m_1+i}}{r_i}\prod_{i=r+1}^{m_1}\binom{\alpha_i}{r_i}\binom{\alpha_{m_1+i}}{r_i}\prod_{j=1}^{n}\binom{\beta_j}{l_j}\binom{\beta_{n+j}}{l_j}}{\binom{\alpha_{m_1}+\sum_{i=1}^{m_1-1} r_i+\sum_{j=1}^{n} l_j}{\alpha_{m_1}}\binom{\alpha_{2m_1}+\sum_{i=1}^{m_1-1} r_i+\sum_{j=1}^{n} l_j}{\alpha_{2m_1}}\binom{\sum_{i=1}^{m_1-1} r_i+\sum_{j=1}^{n} l_j}{r_1,\cdots,r_{m_1-1}}\prod_{i=1}^{r} r_i!}$$

$$\times\prod_{i=1}^{r} x_i^{\alpha_i+r_i}x_{m_1+i}^{\alpha_{m_1+i}-r_i}\prod_{i=r+1}^{m_1} x_i^{\alpha_i-r_i}x_{m_1+i}^{\alpha_{m_1+i}-r_i}\prod_{j=1}^{n}(-1)^{\beta_j l_j}\theta_j^{\beta_j-l_j}\theta_{n+j}^{\beta_{n+j}-l_j}x_{m_1}^{\alpha_{m_1}+\sum_{i=1}^{m_1-1} r_i+\sum_{j=1}^{n} l_j}$$

$$\times x_{2m_1}^{\alpha_{2m_1}+\sum_{i=1}^{m_1-1} r_i+\sum_{j=1}^{n} l_j} \mid \alpha_i\in\mathbb{N},\beta_j\in\{0,1\};\sum_{j=1}^{2n}\beta_j-\sum_{i=1}^{r}\alpha_i+\sum_{i=r+1}^{2m_1}\alpha_i=k;\alpha_{m_1}\alpha_{2m_1}=0\}. \tag{2.81}$$

Proof. (1) The subspace $\mathcal{H}_k^{m_1-1}$ is generated by

$$f_{l,p,s}=\sum_{i=0}^{l}\frac{l!(l+p)!}{i!(l-i)!(l+p-i)!}x_{m_1}^{p}(x_{m_1}x_{2m_1})^{l-i}x_{m_1-1}^{s-i}x_{2m_1-1}^{i}\theta_1\cdots\theta_n \tag{2.82}$$

and

$$g_{l,p,s}=\sum_{i=0}^{l}\frac{l!(l+p)!}{i!(l-i)!(l+p-i)!}x_{2m_1}^{p}(x_{m_1}x_{2m_1})^{l-i}x_{m_1-1}^{s-i}x_{2m_1-1}^{i}\theta_1\cdots\theta_n \tag{2.83}$$

as an (L_1+L_2)-submodule with $l,p,s\in\mathbb{N}$, $p+2l-s=k$ and $l\le s$.

Using Lemma 2.1, we obtain that the subspace $\mathcal{H}_k^{m_1-1}$ is spanned by

$$\sum_{i=0}^{\infty}(-\int_{(m_1)}\int_{(2m_1)})^i x_{m_1}^{\alpha_m}x_{2m_1}^{\alpha_{2m_1}}(-\sum_{j=1}^{m_1-1}x_j\partial_{x_{m_1+j}}+\sum_{j=1}^{n}\partial_{\theta_j}\partial_{\theta_{n+j}})^i(g), \tag{2.84}$$

where

$$\alpha_{m_1}\alpha_{2m_1}=0,\qquad g\in\mathbb{C}[x_1,\cdots,x_{m_1-1},x_{m_1+1},\cdots,x_{2m_1-1};\theta_1,\cdots,\theta_{2n}] \tag{2.85}$$

and

$$\int_{(t)}x^\alpha=\frac{x_t x^\alpha}{\alpha_t+1}\quad\text{for }\ t=m_1,2m_1. \tag{2.86}$$

According to Theorem 2.3, we can write $g=\sum_l X_l(x_{m_1-1}^{s-l}x_{2m_1-1}^{l}\theta_1\cdots\theta_n)$, where $X_l\in U(L_2)$, $2l-s+\alpha_{m_1}+\alpha_{2m_1}+n=k$. Thus $\mathcal{H}_k^{m_1-1}$ is spanned by

$$X_l\sum_{i=0}^{l}\frac{p!l!}{i!(l-i)!(p+l-i)!}x_{m_1}^{p}(x_{m_1}x_{2m_1})^{l-i}x_{m_1-1}^{s-i}x_{2m_1-1}^{i}\theta_1\cdots\theta_n=X_l\frac{p!}{(l+p)!}f_{l,p,s} \tag{2.87}$$

and

$$X_l\sum_{i=0}^{l}\frac{p!l!}{i!(l-i)!(p+l-i)!}x_{2m_1}^{p}(x_{m_1}x_{2m_1})^{l-i}x_{m_1-1}^{s-i}x_{2m_1-1}^{i}\theta_1\cdots\theta_n=X_l\frac{p!}{(l+p)!}g_{l,p,s}, \tag{2.88}$$

where $2l+p+n-s=k$. Consequently, $\mathcal{H}_k^{m_1-1}$ is generated by (2.82) and (2.83) as an L_2-submodule.

(2) As an (L_1+L_2)-submodule, $\mathcal{H}_k^r$ $(r<m_1-1)$ is generated by

$$h_{l,p,s}=\sum_{i=0}^{l}\frac{l!(l+p+m_1-r-1)!}{i!(l-i)!(l+p+m_1-r-1)!}x_{r+1}^p(\sum_{j=r+1}^{m_1}x_jx_{m_1+j})^{l-i}x_r^{s-i}x_{m_1+r}^i\theta_1\cdots\theta_n \quad (2.89)$$

for $2l+p+n-s=k$ and $l\leq s$.

Again by Lemma 2.1, we obtain that $\mathcal{H}_k^r$ $(r<m_1-1)$ is spanned by

$$X_{l,p,s,t}\sum_{i=0}^{l}(-\int_{(m_1)}\int_{(2m_1)})^i x_{m_1}^{\alpha_{m_1}}x_{2m_1}^{q-\alpha_{m_1}}(-x_r\partial_{x_{m_1+r}}+\sum_{j=r+1}^{m_1-1}\partial_{x_j}\partial_{x_{m_1+j}})^i(x_{r+1}^{\alpha_{r+1}}x_{m_1+r+1}^{p-\alpha_{r+1}}$$
$$\times u'^{l-t}x_r^{s-t}x_{m_1+r}^t\theta_1\cdots\theta_n), \quad (2.90)$$

where

$$u'=\sum_{j=r+1}^{m_1-1}x_jx_{m_1+j},\quad \alpha_{m_1}\in\{0,q\},\quad \alpha_{r+1}\in\{0,p\},\quad 2l+p+q-s+n=k \quad (2.91)$$

and $X_{l,p,s,t}\in U(L_1'+L_2)$. So $\mathcal{H}_k^r$ is generated by

$$\begin{aligned} g_{p,q,s,t,\alpha_{r+1},\alpha_{m_1}} &= \sum_{i=0}^{l}(-\int_{(m_1)}\int_{(2m_1)})^i x_{m_1}^{\alpha_{m_1}}x_{2m_1}^{q-\alpha_{m_1}}(-x_r\partial_{x_{m_1+r}}+\sum_{j=r+1}^{m_1-1}\partial_{x_j}\partial_{x_{m_1+j}})^i \\ &\quad (x_{r+1}^{\alpha_{r+1}}x_{m_1+r+1}^{p-\alpha_{r+1}}u'^{l-t}x_r^{s-t}x_{m_1+r}^t\theta_1\cdots\theta_n) \end{aligned} \quad (2.92)$$

as an $(L_1'+L_2)$-submodule. Denote by $\lambda_{p,q,s,\alpha_{r+1},\alpha_{m_1}}$ the weight of $g_{p,q,s,t,\alpha_{r+1},\alpha_{m_1}}$. Note

$$L'^+_1(g_{p,q,s,t,\alpha_{r+1},\alpha_{m_1}})=L_2^+(g_{p,q,s,t,\alpha_{r+1},\alpha_{m_1}})=0. \quad (2.93)$$

Hence

$$\mathcal{H}_k^r=\sum U(L_1'+L_2)(g_{p,q,s,t,\alpha_{r+1},\alpha_{m_1}}) \quad (2.94)$$

and

$$\begin{aligned} &(\mathcal{H}_k^r)_{\lambda_{p,q,s,\alpha_{r+1},\alpha_{m_1}}}\cap\text{Span}\{g\theta_1\cdots\theta_n\mid g\in\mathbb{C}[x_{r+1},x_{m_1+r+1},u',x_{m_1},x_{2m_1}]\} \\ =\ &\text{Span}\{g_{p,q,s,t,\alpha_{r+1},\alpha_{m_1}}\mid 0\leq t\leq\min\{s,l\}\}. \end{aligned} \quad (2.95)$$

Note

$$\dim\text{Span}\,\{g_{l,p,s,t,\alpha_{r+1},\alpha_{m_1}}\mid 0\leq t\leq\min\{s,l\}\}=\begin{cases} l+1 \text{ if } l\leq s,\\ s+1 \text{ if } l>s.\end{cases} \quad (2.96)$$

On the other hand,

$$(\sum_{t=r+1}^{m_1}\partial_{x_t}\partial_{x_{m_1+t}})(\sum_{j=0}^{d}\frac{(-1)^j}{j!(d-j)!(d+p+m_1-r-2-j)!(q+j)!}x_{r+1}^{\alpha_{r+1}}x_{m_1+r+1}^{p-\alpha_{r+1}}$$
$$\times x_{m_1}^{\alpha_{m_1}}x_{2m_1}^{q-\alpha_{m_1}}u'^{d-j}(x_{m_1}x_{2m_1})^j)=0, \quad (2.97)$$

which means

$$\sum_{j=0}^{d}\frac{(-1)^j}{j!(d-j)!(d+p+m_1-r-2-j)!(q+j)!}x_{r+1}^{\alpha_{r+1}}x_{m_1+r+1}^{p-\alpha_{r+1}} \times x_{m_1}^{\alpha_{m_1}}x_{2m_1}^{q-\alpha_{m_1}}u'^{d-j}(x_{m_1}x_{2m_1})^j \in U(L_1)(x_{r+1}^{p+q+2d}). \tag{2.98}$$

So we get

$$\sum_{i=0}^{l-d}\frac{l!(l+d+p+m_1-r-1)!}{i!(l-d-i)!(l+d+p+m_1-r-1)!}\Big(\sum_{j=0}^{d}\frac{(-1)^j}{j!(d-j)!(q+j)!(p+d+m_1-r-2-j)!} \times u'^{d-j}(x_{m_1}x_{2m_1})^j\Big)x_{r+1}^{\alpha_{r+1}}x_{m_1+r+1}^{p-\alpha_{r+1}}x_{m_1}^{\alpha_{m_1}}x_{2m_1}^{q-\alpha_{m_1}}(u'+x_{m_1}x_{2m_1})^{l-d-i}x_r^{s-i}x_{m_1+r}^{i}\theta_1\cdots\theta_n \in U(L_1)(h_{l-d,p+q+2d,s})\cap \text{Span}\,\{g_{l,p,s,t,\alpha_{r+1},\alpha_{m_1}}\mid 0\le t\le \min\{s,l\}\}, \tag{2.99}$$

where $0\le d\le \min\{l,s\}$. Thus

$$\dim\Big(\bigoplus_{d=0}^{\min\{l,s\}}U(L_1)(h_{l-d,p+q+2d,s})\cap \text{Span}\,\{g_{l,p,s,t,\alpha_{r+1},\alpha_{m_1}}\mid 0\le t\le \min\{s,l\}\}\Big) = \begin{cases} l+1 \text{ if } l\le s,\\ s+1 \text{ if } l>s.\end{cases} \tag{2.100}$$

Therefore, we have

$$\bigoplus_{d=0}^{\min\{l,s\}}U(L_1)(h_{l-d,p+q+2d,s})\cap \text{Span}\,\{g_{l,p,s,t,\alpha_{r+1},\alpha_{m_1}}\mid 0\le t\le \min\{s,l\}\} = \text{Span}\,\{g_{l,p,s,t,\alpha_{r+1},\alpha_{m_1}}\mid 0\le t\le \min\{s,l\}\}, \tag{2.101}$$

which implies

$$g_{l,p,s,t,\alpha_{r+1},\alpha_{m_1}} \in \bigoplus_{d=0}^{\min\{l,s\}}U(L_1)(h_{l-d,p+q+2d,s}). \tag{2.102}$$

Hence we are done.

(3) We claim that

$$\mathcal{H}_k^r=\begin{cases}\langle x_r^{-k}\rangle & \textit{if}\quad k\le 0,\\ \langle x_{r+1}^{k}\rangle & \textit{if}\quad k>0, r<m_1-1,\\ \langle x_{m_1}^{k}\rangle+\langle x_{2m_1}^{k}\rangle & \textit{if}\quad k>0, r=m_1-1.\end{cases} \tag{2.103}$$

When $r=m_1-1$, we have

$$\begin{aligned} f_{l,p,s} &= \sum_{i=0}^{l}\frac{l!(l+p)!}{i!(l-i)!(l+p-i)!}x_{m_1}^{p}(x_{m_1}x_{2m_1})^{l-i}x_{m_1-1}^{s-i}x_{2m_1-1}^{i}\theta_1\cdots\theta_n\\ &= (E_{2m_1-1,m_1}-E_{2m_1,m_1-1})^l(x_{m_1-1}^{s-l}x_{m_1}^{p+l}\theta_1\cdots\theta_n) \end{aligned} \tag{2.104}$$

and

$$\begin{aligned} g_{l,p,s} &= \sum_{i=0}^{l} \frac{l!(l+p)!}{i!(l-i)!(l+p-i)!} x_{2m_1}^{p} (x_{m_1} x_{2m_1})^{l-i} x_{m_1-1}^{s-i} x_{2m_1-1}^{i} \theta_1 \cdots \theta_n \\ &= (E_{2m_1-1,2m_1} - E_{m_1,m_1-1})^l (x_{m_1-1}^{s-l} x_{2m_1}^{p+l} \theta_1 \cdots \theta_n). \end{aligned} \tag{2.105}$$

It is straightforward to check

$$x_{m_1-1}^{s-l} x_{m_1}^{p+l} \theta_1 \cdots \theta_n,\ x_{m_1-1}^{s-l} x_{2m_1}^{p+l} \theta_1 \cdots \theta_n \in \begin{cases} \langle x_{m_1}^k \rangle + \langle x_{2m_1}^k \rangle & \text{if } k > 0, \\ \langle x_{m_1-1}^{-k} \rangle & \text{if } k \leq 0. \end{cases} \tag{2.106}$$

Now we assume $r < m_1 - 1$. Then $h_{0,p,p+n-k} \in \langle x_{r+1}^k \rangle$ (or $\langle x_r^{-k} \rangle$). Note

$$\begin{aligned} l(p+1)h_{l,p,s} = & \sum_{j=r+2}^{m_1} (E_{m_1+j,r+1} - E_{m_1+r+1,j})(-E_{j,r} + E_{m_1+r,m_1+j})(h_{l-1,s-1,p+1}) \\ & + (m_1 - r)(E_{m_1+r,r+1} - E_{m_1+r+1,r})(h_{l-1,p+1,s-1}). \end{aligned} \tag{2.107}$$

So $h_{l,p,s} \in \langle x_{r+1}^k \rangle$ (or $\langle x_r^{-k} \rangle$) by inductive assumption. Therefore, (2.103) holds.

(3) When $k \leq n - m_1 + r + 1$, the submodule $\mathcal{H}_k^r$ is irreducible and $\mathcal{A}_k^r = \mathcal{H}_k^r \oplus \eta \mathcal{A}_{k-2}^r$.

We may assume $r < m_1 - 1$ and $k > 0$. The proof for $r = m_1 - 1$ or $k \leq 0$ is similar. For any submodule $W \subset \mathcal{H}_k^r$, there should be some weight vector $g \in W$ such that $L_1^+(g) = 0$ and $L_2^+(g) = 0$. Thus $g = \sum_{i=0}^{l} a_i x_{r+1}^p u^{l-i} x_r^{s-i} x_{m_1+r}^i \theta_1 \cdots \theta_n$ for some $a_i \in \mathbb{C}$. Since

$$\begin{aligned} 0 &= \Delta(g) = -\sum_{i=1}^{l} a_i i x_{r+1}^p u^{l-i} x_r^{s-i+1} x_{m_1+r}^{i-1} \theta_1 \cdots \theta_n \\ &\quad + \sum_{i=0}^{l-1} x_{r+1}^p u^{l-i} x_r^{s-i} x_{m_1+r}^i \theta_1 \cdots \theta_n \end{aligned} \tag{2.108}$$

we get

$$a_{i+1}(i+1) = a_i(l-i)(p+l+m_1-r-i-1). \tag{2.109}$$

Thus g is a scalar multiple of

$$h_{l,p,s} = \sum_{i=0}^{l} \frac{l!(l+p+m_1-r-1)!}{i!(l-i)!(l+p+m_1-r-1-i)!} x_{r+1}^p u^{l-i} x_r^{s-i} x_{m_1+r}^i \theta_1 \cdots \theta_n. \tag{2.110}$$

If $l > 0$, we have

$$(E_{r,m_1+r+1} - E_{r+1,m_1+r})(h_{l,p,s}) = -l(l+p+m_1-r-1-s)h_{l-1,p+1,s-1}, \tag{2.111}$$

and

$$l + m_1 - r - 1 + p - s = k - n - l + m_1 - r - 1 \leq -l < 0, \tag{2.112}$$

which implies $h_{l-1,p+1,s-1}, \cdots, h_{0,p+l,s-l} \in W$. It is easy to see $x_{r+1}^k \in \langle h_{0,p+l,s-l} \rangle$. Hence $W = \mathcal{H}_k^r$.

By the similar arguments as those in (3) of the proof of Theorem 2.2, we obtain $\mathcal{A}_k^r = \mathcal{H}_k^r \oplus \eta \mathcal{A}_{k-2}^r$.

(4)(2.79) and (2.80) are composition series.

According to (2),

$$\mathcal{H}_k^{m_1-1} = \bigoplus_{l,p,s=0,p+2l-s=k,l\leq s}^{\infty} (U(L_2)f_{l,p,s} \oplus U(L_2)g_{l,p,s}) \tag{2.113}$$

and

$$\mathcal{H}_k^r = \bigoplus_{l,p,s=0,p+2l-s=k,l\leq s}^{\infty} U(L_1+L_2)h_{l,p,s}. \tag{2.114}$$

Again by the similar arguments as those in (4) of the proof of Theorem 2.2, we can get the composition series. □

3 Proof of Theorem 2

In this section, we discuss the $osp(2m_1, 2n)$-module $\mathcal{A}'_k$ defined in (1.14) and (1.15). The following facts will be used.

Proposition 3.1 *If $n = 0$, then the subspace $\mathcal{A}'_k$ $(k \neq m_1)$ is an irreducible $so(2m_1, \mathbb{C})$-submodule and $\mathcal{A}'_{m_1} = \langle \theta_1 \cdots \theta_{m_1} \rangle \oplus \langle \theta_1 \cdots \theta_{m_1-1}\theta_{2m_1} \rangle$.*

If $m_1 = 0$, then the subspace $\mathcal{A}'_k$ is an irreducible $sp(2n, \mathbb{C})$-submodule when $S_1 \cup T_1 \neq \emptyset$ or $k \neq 0$. When $S_1 \cup T_1 = \emptyset$, we can assume $T = \overline{1,n}$ by symmetry. In this case, $\mathcal{A}'_0 = \langle 1 \rangle \oplus \langle x_{n-1}x_{2n} - x_n x_{2n-1} \rangle$. (cf. [20])

Now let us deal with the general case with $m_1 > 0$ and $n > 0$.

In fact, if $S_1 \neq \emptyset$, we can take a $j_0 \in S_1$ and $0 \neq f \in \mathcal{A}'_k$. Since

$$E_{2m_1+j_0, 2m_1+n+j_0} |_{\mathcal{A}'} = x_{j_0}\partial_{x_{n+j_0}}, \tag{3.1}$$

we can assume $\partial_{x_{n+j_0}}(f) = 0$. Applying

$$(E_{m_1+i, 2m_1+n+j_0} + E_{2m_1+j_0, i}) |_{\mathcal{A}'} = \theta_{m_1+i}\partial_{x_{n+j_0}} + x_{j_0}\partial_{\theta_i} \tag{3.2}$$

and

$$(E_{i, 2m_1+n+j_0} + E_{2m_1+j_0, m_1+i}) |_{\mathcal{A}'} = \theta_i\partial_{x_{n+j_0}} + x_{j_0}\partial_{\theta_{m_1+i}} \tag{3.3}$$

$(i \in \overline{1, m_1})$, we get a nonzero element $f' = f'(x_1, \cdots, \hat{x_{n+j_0}}, \cdots, x_{2n}) \in \langle f \rangle$. Since

$$\text{Span}\,\{x^\alpha \mid x^\alpha \in \mathcal{A}'_k\} \tag{3.4}$$

is an irreducible $sp(2n, \mathbb{C})$-submodule according to Proposition 3.1, we obtain

$$\text{Span}\,\{x^\alpha \mid x^\alpha \in \mathcal{A}'_k\} \subset \langle f \rangle. \tag{3.5}$$

Observe

$$(E_{i,2m_1+j_0}-E_{2m_1+n+j_0,m_1+i})|_{\mathcal{A}'}=\theta_i\partial_{x_{j_0}}-x_{n+j_0}\partial_{\theta_{m_1+i}} \tag{3.6}$$

and

$$(E_{m_1+i,2m_1+j_0}-E_{2m_1+n+j_0,i})|_{\mathcal{A}'}=\theta_{m_1+i}\partial_{x_{j_0}}-x_{n+j}\partial_{\theta_i}. \tag{3.7}$$

Thus by induction on t, we can obtain $x^\alpha\theta_{i_1}\cdots\theta_{i_t}\in\langle f\rangle$ for all $i_1,\cdots,i_t\in\overline{1,2m_1}$ and $\sum_{i\in\bar{T}}\alpha_i-\sum_{j\in T}\alpha_j=k-t$. So $\langle f\rangle=\mathcal{A}'_k$, which implies that $\mathcal{A}'_k$ is irreducible. It can be similarly proved when $T_1\neq\emptyset$.

Theorem 3.2 *1) The submodule $\mathcal{A}'_k$ is irreducible when $S_1\cup T_1\neq\emptyset$. In particular, $\mathcal{A}'_k$ is not highest weight type if $S_1\neq\emptyset$ and $T_1\neq\emptyset$.*
2) If $S_1=\emptyset$ and $T_1=\emptyset$, we may assume $T=\overline{1,n}$ by symmetry.
a) The submodule $\mathcal{A}'_k$ is irreducible and of highest weight type when $k\neq m_1$. A highest weight vector is $x_n^{m_1-k}\theta_1\cdots\theta_m$ (resp. $x_{2n}^{k-m_1}\theta_1\cdots\theta_m$) if $k>m_1$ (resp. $k<m_1$).
b) The submodule $\mathcal{A}'_{m_1}=\langle\theta_1\cdots\theta_{m_1}\rangle\oplus\langle(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1}\rangle$ is a sum of two irreducible submodules.

Proof. Assume $T=\overline{1,n}$.

a) We claim that $\mathcal{A}'_k=\langle x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\rangle$ when $k<m_1$.

In fact, we have

$$x_n^{l-k}\theta_1\cdots\theta_l=\frac{(l-k)!}{(m_1-k)!}\prod_{t=l+1}^{m_1}(-1)^{t-1}(E_{m_1+t,2m_1+2n}+E_{2m_1+n,t})(x_n^{m_1-k}\theta_1\cdots\theta_{m_1}) \tag{3.8}$$

for $l\in\overline{1,m_1}$. Thus we get

$$\text{Span}\,\{x^\alpha\theta_{i_1}\cdots\theta_{i_l}\mid i_1,\cdots,i_l\in\overline{1,2m};\ l+\sum_{i\in\bar{T}}\alpha_i-\sum_{i\in T}\alpha_i=k\}\subset\langle x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\rangle \tag{3.9}$$

for $k<l<m_1$ and $\theta_{i_1}\cdots\theta_{i_k}\in\langle x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\rangle$ by applying $so(2m_1,,\mathbb{C})$ and $sp(2n,\mathbb{C})$ to $x_n^{l-k}\theta_1\cdots\theta_l$ (We treat $\theta_1\cdots\theta_k=0$ if $k\leq 0$). Since

$$\begin{aligned}&(E_{2m_1,2m_1+2n}+E_{2m_1+n,m_1})(-E_{2m_1+2n,2m_1+n})(x_n^{m_1-k}\theta_1\cdots\theta_{m_1})\\ =\;&x_n^{m_1-k+1}\theta_{2m_1}\theta_1\cdots\theta_{m_1}+(-1)^{m_1-1}(m_1-k+1)x_n^{m_1-k}x_{2n}\theta_1\cdots\theta_{m_1-1},\end{aligned} \tag{3.10}$$

we get $x_n^{m_1-k+1}\theta_1\cdots\theta_{m_1}\theta_{2m_1}\in\langle x_n^{m_1-1}\theta_1\cdots\theta_{m_1}\rangle$. Now we have

$$\begin{aligned}&x_n^{m_1-k}\theta_1\cdots\theta_{m_1-1}\theta_{2m_1}\\ =\;&\frac{(-1)^{m_1-1}}{m_1+1-k}(E_{2m_1,2m_1+2n}+E_{2m_1+n,m_1})(x_n^{m_1+1-k}\theta_1\cdots\theta_{m_1}\theta_{2m_1}).\end{aligned} \tag{3.11}$$

Applying

$$(E_{m_1+t,2m_1+n}-E_{2m_1+2n,t})|_{\mathcal{A}'}=-x_n\theta_{m_1+t}-x_{2n}\partial_{\theta_t} \tag{3.12}$$

and taking induction on l, we obtain

$$\text{Span}\,\{x^\alpha\theta_{i_1}\cdots\theta_{i_l}\mid i_1,\cdots,i_l\in\overline{1,2m};\ l+\sum_{i\in\bar{T}}\alpha_i-\sum_{i\in T}\alpha_i=k\}\subset\langle x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\rangle \quad (3.13)$$

for $l\geq m_1$. Since

$$\begin{aligned}&(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_k\\ =\ &(-1)^k(E_{m_1+k+1,2m_1+n-1}-E_{2m_1+2n-1,k+1})(x_n\theta_1\cdots\theta_{k+1})\\ &-(-1)^k(E_{m_1+k+1,2m_1+n}-E_{2m_1+2n,k+1})(x_{n-1}\theta_1\cdots\theta_{k+1}),\end{aligned} \quad (3.14)$$

we get

$$\text{Span}\,\{x^\alpha\theta_{i_1}\cdots\theta_{i_l}\mid i_1,\cdots,i_l\in\overline{1,2m};\ l+\sum_{i\in\bar{T}}\alpha_i-\sum_{i\in T}\alpha_i=k\}\subset\langle x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\rangle \quad (3.15)$$

for $l=k$. Now by induction on $k-l$ for $l\leq k$ and (3.12), we attain

$$\text{Span}\,\{x^\alpha\theta_{i_1}\cdots\theta_{i_l}\mid i_1,\cdots,i_l\in\overline{1,2m};\ l+\sum_{i\in\bar{T}}\alpha_i-\sum_{i\in T}\alpha_i=k\}\subset\langle x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\rangle \quad (3.16)$$

for $0\leq l<k$. Hence $\mathcal{A}'_k=\langle x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\rangle$.

Note that all the weight vectors annihilated by $osp(2m_1,2n)_0^+$ are scalar multiples of $x_n^{m_1-k}\theta_1\cdots\theta_{m_1-1}\theta_{2m_1}$, $x_n^{i-k}\theta_1\cdots\theta_i$ $(k\leq i\leq m_1)$, $x_{2n}^{k-l}\theta_1\cdots\theta_l$ $(0\leq l<k)$ and $(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_k$. Since

$$\prod_{t=l+1}^{k}(E_{t,2m_1+2n}+E_{2m_1+n,m_1+t})(x_{2n}^{k-l}\theta_1\cdots\theta_l)=(-1)^{l(k-l)}(k-l)!\theta_1\cdots\theta_k, \quad (3.17)$$

$$\begin{aligned}&(E_{m_1,2m_1+2n}+E_{2m_1+n,2m_1})(x_n^{m_1-k}\theta_1\cdots\theta_{m_1-1}\theta_{2m_1})\\ &=(-1)^{m_1-1}(m_1-k)x_n^{m_1-k-1}\theta_1\cdots\theta_{m_1-1},\end{aligned} \quad (3.18)$$

$$\begin{aligned}&(E_{k+1,2m_1+2n-1}+E_{2m_1+n-1,m_1+k+1})\big((x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_k\big)\\ =\ &(-1)^{k+1}x_n\theta_1\cdots\theta_{k+1}\end{aligned} \quad (3.19)$$

and

$$\prod_{j=i+1}^{m_1}(-1)^j(E_{j,2m_1+n}-E_{2m_1+2n,m_1+j})(x_n^{i-k}\theta_1\cdots\theta_i)=x_n^{m_1-k}\theta_1\cdots\theta_{m_1}, \quad (3.20)$$

we get that up to a scalar multiple, $\mathcal{A}'_k$ has only one highest weight vector $x_n^{m_1-k}\theta_1\cdots\theta_{m_1}$ and thus it is irreducible.

It can be similarly proved that $\mathcal{A}'_k=\langle x_{2n}^{k-m_1}\theta_1\cdots\theta_{m_1}\rangle$ is irreducible when $k>m_1$.

b) Assume $k=m_1$. We claim that for any nonzero submodule V of $\mathcal{A}'_{m_1}$, we have

$$\theta_1\cdots\theta_{m_1}\in V\ \text{ or }(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1}\in V. \quad (3.21)$$

In fact, there should be at least one weight vector $f \in V$ such that $osp(2m_1, 2n)_0^+(f) = 0$. Thus we can assume that f is of the form

$$(x_{n-1}x_{2n} - x_n x_{2n-1})^l \theta_1 \cdots \theta_{m_1-1} \theta_{m_1}^{l_{m_1}} \theta_{2m_1}^{l_{2m_1}} \tag{3.22}$$

with $l, l_{m_1}, l_{2m_1} \in \{0,1\}$ such that $l_{m_1} l_{2m_1} = 0$ or

$$a x_n^{m_1-i} \theta_1 \cdots \theta_{m_1} \theta_{m_1+i+1} \cdots \theta_{2m_1} + b x_{2n}^{m_1-i} \theta_1 \cdots \theta_i \tag{3.23}$$

with $0 \leq i < m_1$ and $a, b \in \mathbb{C}$.

i) If

$$f = a x_n^{m_1-i} \theta_1 \cdots \theta_{m_1} \theta_{m_1+i+1} \cdots \theta_{2m_1} + b x_{2n}^{m_1-i} \theta_1 \cdots \theta_i, \tag{3.24}$$

then

$$\begin{aligned} f_1 &= \prod_{j=i+1}^{m_1-1} (E_{j,2m_1+2n} + E_{2m_1+n,m_1+j})(f) \\ &= a' x_n \theta_1 \cdots \theta_{m_1} \theta_{2m_1} + b' x_{2n} \theta_1 \cdots \theta_{m_1-1} \in \langle f \rangle. \end{aligned} \tag{3.25}$$

If $a' \neq b'$, then

$$(E_{m_1,2m_1+2n} + E_{2m_1+n,2m_1})(f_1) = (-1)^{m_1}(a' - b')\theta_1 \cdots \theta_{m_1} \in V. \tag{3.26}$$

Otherwise, $a' = b'$ and

$$(E_{m_1,2m_1+2n-1} + E_{2m_1+n-1,2m_1})(f_1) = (-1)^{m_1} a' (x_{n-1}x_{2n} - x_n x_{2n-1})\theta_1 \cdots \theta_{m_1} \in V. \tag{3.27}$$

ii) If $f = \theta_1 \cdots \theta_{m_1-1}\theta_{2m_1}$, then

$$\begin{aligned} & (E_{m_1,2m_1+2n-1} + E_{2m_1+n-1,2m_1})(E_{m_1,2m_1+n} - E_{2m_1+2n,2m_1})(f) \\ =\; & (-1)^{m_1}(E_{m_1,2m_1+2n-1} + E_{2m_1+n-1,2m_1})(x_n \theta_1 \cdots \theta_{m_1} \theta_{2m_1} + x_{2n}\theta_1 \cdots \theta_{m_1-1}) \\ =\; & (x_{n-1}x_{2n} - x_n x_{2n-1})\theta_1 \cdots \theta_{m_1} \in V. \end{aligned} \tag{3.28}$$

iii) When $f = (x_{n-1}x_{2n} - x_n x_{2n-1})\theta_1 \cdots \theta_{m_1-1}\theta_{2m_1}$, we have

$$\begin{aligned} & (E_{m_1,2m_1+2n} + E_{2m_1+n,2m_1})(E_{m_1,2m_1+2n-1} + E_{2m_1+n-1,2m_1})(f) \\ =\; & (-1)^{m_1}(E_{m_1,2m_1+2n} + E_{2m_1+n,2m_1})(x_n \theta_1 \cdots \theta_{m_1} \theta_{2m_1} - x_{2n}\theta_1 \cdots \theta_{m_1-1}) \\ =\; & -2\theta_1 \cdots \theta_{m_1} \in V. \end{aligned} \tag{3.29}$$

Using Lemma 3.1, we get

$$x^\alpha \theta_1 \cdots \theta_{m_1} \in \langle \theta_1 \cdots \theta_{m_1} \rangle + \langle (x_{n-1}x_{2n} - x_n x_{2n-1})\theta_1 \cdots \theta_{m_1} \rangle \tag{3.30}$$

for $\alpha \in \mathbb{N}^{2n}$ such that $\sum_{i=1}^{m_1} \alpha_i = \sum_{i=1}^{m_1} \alpha_{m_1+i}$. Now it is straightforward to check

$$\mathcal{A}'_{m_1} \subset \langle \theta_1 \cdots \theta_{m_1} \rangle + \langle (x_{n-1}x_{2n} - x_n x_{2n-1})\theta_1 \cdots \theta_{m_1} \rangle. \tag{3.31}$$

$\square$

We can get basis for $\langle\theta_1\cdots\theta_{m_1}\rangle$ and $\langle(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1}\rangle$ by the following way. Set

$$\Delta=-\sum_{i=1}^{n}x_i\partial_{x_{n+i}}+\sum_{j=1}^{m_1}\partial_{\theta_j}\partial_{\theta_{m_1+j}} \tag{3.32}$$

and

$$\eta=\sum_{i=1}^{n}x_{n+i}\partial_{x_i}+\sum_{j=1}^{m_1}\theta_j\theta_{m_1+j}. \tag{3.33}$$

Let L be the subalgebra of $osp(2m_1,2n)$ consisting of the matrices of the form

$$\begin{pmatrix} A & 0 & 0 & H_1\\ 0 & -A^T & J & 0\\ 0 & H_1^T & D & 0\\ -J^T & 0 & 0 & -D^T\end{pmatrix}. \tag{3.34}$$

It is straightforward to check

$$g\Delta=\Delta g,\quad g\eta=\eta g\ \ \text{for } g\in L. \tag{3.35}$$

Set

$$\mathcal{A}_{s,t}=\text{Span }\{x^\alpha\prod_{j=1}^{2m_1}\theta_j^{l_j}\mid\alpha\in\mathbb{N}^{2n};\ l_j\in\{0,1\};\ \sum_{j=1}^{m_1}l_{m_1+j}-\sum_{i=1}^{n}\alpha_i=s,$$
$$\sum_{j=1}^{m_1}l_j+\sum_{i=1}^{n}\alpha_{n+i}=t\} \tag{3.36}$$

for $s\in\mathbb{Z}$, $t\in\mathbb{N}$ and

$$\mathcal{H}_{s,t}=\{f\in\mathcal{A}_{s,t}\mid\Delta(f)=0\}. \tag{3.37}$$

We have

$$\mathcal{A}_{s,t}=\mathcal{H}_{s,t}\oplus\eta\mathcal{A}_{s-1,t-1}=\bigoplus_{l=0}^{t}\eta^{t-l}\mathcal{H}_{s-t+l,l}. \tag{3.38}$$

Thus

$$\mathcal{A}'_{m_1}=\bigoplus_{l=0}^{\infty}\mathcal{A}_{m_1-l,l}=\bigoplus_{l=0}^{\infty}\bigoplus_{q=0}^{l}\eta^{l-q}\mathcal{H}_{m_1-2l+q,q}=\bigoplus_{t,q=0}^{\infty}\eta^t\mathcal{H}_{m_1-2t-q,q}. \tag{3.39}$$

By similar arguments as those in 1) of the proof of Theorem 2.2, we get

$$\mathcal{H}_{m_1-2t-q,q}=\begin{cases} U(L)\big(x_n^{2t}\theta_1\cdots\theta_q\theta_{m_1+q+1}\cdots\theta_{2m_1}\big) & \text{if}\quad q\le m_1,\\ U(L)\big(x_n^{2t}(x_{n-1}x_{2n}-x_nx_{2n-1})^{q-m_1}\theta_1\cdots\theta_{m_1}\big) & \text{if}\quad q>m_1.\end{cases} \tag{3.40}$$

Since

$$\eta^t(x_n^{2t}\theta_1\cdots\theta_q\theta_{m_1+q+1}\cdots\theta_{2m_1})=\frac{(2t)!}{t!}x_n^tx_{2n}^t\theta_1\cdots\theta_q\theta_{m_1+q+1}\cdots\theta_{2m_1}$$
$$=\frac{(2t)!}{t!}(-E_{2n,n})^t(\theta_1\cdots\theta_q\theta_{m_1+q+1}\cdots\theta_{2m_1})$$
$$\in\begin{cases}\langle\theta_1\cdots\theta_{m_1}\rangle & \text{if}\quad m_1-q \text{ is even},\\ \langle\theta_1\cdots\theta_{m_1-1}\theta_{2m_1}\rangle=\langle(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1}\rangle & \text{if}\quad m_1-q\text{ is odd}\end{cases} \tag{3.41}$$

when $q \leq m_1$ and

$$\begin{aligned}
& \eta^t(x_n^{2t}(x_{n-1}x_{2n} - x_n x_{2n-1})^{q-m_1}\theta_1 \cdots \theta_{m_1}) \\
= \ & \frac{(2t)!}{t!} x_n^t x_{2n}^t (x_{n-1}x_{2n} - x_n x_{2n-1})^{q-m_1}\theta_1 \cdots \theta_{m_1} \\
\in \ & \begin{cases} U(sp(2n,\mathbb{C}))(1)\theta_1 \cdots \theta_{m_1} & \text{if } \ m_1 - q \text{ is even,} \\ U(sp(2n,\mathbb{C}))(x_{n-1}x_{2n} - x_n x_{2n-1})\theta_1 \cdots \theta_{m_1} & \text{if } \ m_1 - q \text{ is odd} \end{cases}
\end{aligned} \tag{3.42}$$

when $q > m_1$ (cf. [20]), we get

$$\eta^t \mathcal{H}_{m_1 - 2t - q, q} \subset \begin{cases} \langle \theta_1 \cdots \theta_{m_1} \rangle & \text{if } \ m_1 - q \text{ is even,} \\ \langle (x_{n-1}x_{2n} - x_n x_{2n-1})\theta_1 \cdots \theta_{m_1} \rangle & \text{if } \ m_1 - q \text{ is odd.} \end{cases} \tag{3.43}$$

Therefore,

$$\langle \theta_1 \cdots \theta_{m_1} \rangle = \bigoplus_{q \in \mathbb{N};\ m_1 - q \text{ is even}} \eta^t \mathcal{H}_{m_1 - 2t - q, q} \tag{3.44}$$

and

$$\langle (x_{n-1}x_{2n} - x_n x_{2n-1})\theta_1 \cdots \theta_{m_1} \rangle = \bigoplus_{q \in \mathbb{N};\ m_1 - q \text{ is odd}} \eta^t \mathcal{H}_{m_1 - 2t - q, q}. \tag{3.45}$$

Like in Theorem 2.3, we denote

$$\begin{aligned}
h(\vec{k}, \vec{l}, \vec{s}) \ = \ & \prod_{t=1}^{n} x_t^{k_t} \prod_{1 \leq i < j \leq n} (x_i x_{n+j} - x_j x_{n+i})^{k_{i,j}} \prod_{j=1}^{m_1} \theta_j^{l_j} \theta_{m_1+j}^{l_{m_1+j}} \prod_{1 \leq i < j \leq m_1} (\theta_i \theta_{m_1+i} \\
& -\theta_j \theta_{m_1+j})^{l_{i,j}} \prod_{1 \leq p \leq n, 1 \leq q \leq m_1} (x_{n+p} - x_p \theta_q \theta_{m_1+q})^{s_{p,q}}
\end{aligned} \tag{3.46}$$

where

$$\vec{k} = (k_1, \cdots, k_n; k_{1,2}, k_{1,3}, \cdots, k_{1,n}, k_{2,3} \cdots, k_{n-1,n}) \in \mathbb{N}^{\frac{n(n+1)}{2}}, \tag{3.47}$$

$$\vec{l} = (l_1, \cdots, l_{m_1}; l_{1,2}, \cdots, l_{1,m_1}, l_{2,3}, \cdots, l_{2,m_1}, \cdots, l_{m_1-1,m_1}) \in \{0,1\}^{\frac{m_1(m_1+1)}{2}}, \tag{3.48}$$

$$\vec{s} = (s_{1,1}, \cdots, s_{1,m_1}, \cdots, s_{n,m_1}) \in \{0,1\}^{m_1 n}. \tag{3.49}$$

Set

$$\begin{aligned}
I \ = \ & \{(\vec{k}, \vec{l}, \vec{s}) \mid l_t + l_{m_1+t} + \sum_{1 \leq i < t} l_{i,t} + \sum_{t < j \leq m_1} l_{t,j} + \sum_{p=1}^{n} s_{p,t} \leq 1 \text{ for } t \in \overline{1, m_1}; \\
& k_{i,j} k_t = 0 \text{ for } i < j < t;\ k_{i,j} k_{i',j'} = 0 \text{ for } i > i' \text{ and } j < j'; \\
& k_t l_{i,j} = 0 \text{ for } t \in \overline{1,n}, 1 \leq i < j \leq m_1;\ k_t s_{p,q} = 0 \text{ for } t < p; \\
& k_{i,j} s_{p,q} = 0 \text{ for } i < j < p;\ s_{p,q} s_{p',q'} = 0 \text{ for } p > p' \text{ and } q < q'; \\
& l_{i,j} = 0 \text{ if } l_t = l_{m_1+t} = \sum_{p=1}^{n} s_{p,t} = \sum_{i' < t} l_{i',t} + \sum_{j' > t} l_{t,j'} = 0 \text{ for some } i < t < j; \\
& l_{i,j} l_{i',j'} = 0 \text{ if } i < i' < j < j';\ l_{i,j} s_{p,q} = 0 \text{ if } i < j < q\}.
\end{aligned} \tag{3.50}$$

Then the subspace $\mathcal{H}_{m_1-2t-q,q}$ has a basis

$$B_{t,q}=\{h(\vec{k},\vec{l},\vec{s})\mid \sum_{j=1}^{m_1}l_{m_1+j}+\sum_{1\le i<j\le m_1}l_{i,j}-\sum_{i=1}^{n}k_i-\sum_{1\le i\le j\le n}k_{i,j}=m_1-2t-q;$$

$$\sum_{j=1}^{m_1}l_j+\sum_{1\le i<j\le m_1}l_{i,j}+\sum_{p=1}^{n}\sum_{j=1}^{m_1}s_{p,j}+\sum_{1\le i\le j\le n}k_{i,j}=q;\ (\vec{k},\vec{l},\vec{s})\in I\}. \tag{3.51}$$

Note

$$\{f\in\mathcal{A}_{s,t}\mid \eta(f)=0\}=0 \tag{3.52}$$

when $s+t<m_1$. Hence we have:

Theorem 3.3 *The set*

$$\bigcup_{t,q\in\mathbb{N};m_1-q\text{ is even}}\eta^tB_{t,q} \tag{3.53}$$

forms a basis for $\langle\theta_1\cdots\theta_{m_1}\rangle$ *and the set*

$$\bigcup_{t,q\in\mathbb{N};m_1-q\text{ is odd}}\eta^tB_{t,q} \tag{3.54}$$

forms a basis for $\langle(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1}\rangle$.

4 Proof of Theorem 3

Recall the Lie superalgebra

$$osp(2m_1+1,2n)=osp(2m_1+1,2n)_0\oplus osp(2m_1+1,2n)_1 \tag{4.1}$$

where

$$osp(2m_1+1,2n)_0=\sum_{i,j=1}^{m_1}\big(\mathbb{C}(E_{i,j}-E_{m_1+j,m_1+i})+\mathbb{C}(E_{i,m_1+j}-E_{j,m_1+i})$$

$$+\mathbb{C}(E_{m_1+i,j}-E_{m_1+j,i})\big)+\sum_{i=1}^{m_1}\big(\mathbb{C}(E_{i,2m_1+1}-E_{2m_1+1,m_1+i})+\mathbb{C}(E_{m_1+i,2m_1+1}-E_{2m_1+1,i})\big)$$

$$+\sum_{p,q=1}^{n}\big(\mathbb{C}(E_{2m_1+1+p,2m_1+1+q}-E_{2m_1+1+n+q,2m_1+1+n+p})+\mathbb{C}(E_{2m_1+1+p,2m_1+1+n+q}$$

$$+E_{2m_1+1+q,2m_1+1+n+p})+\mathbb{C}(E_{2m_1+1+n+p,2m_1+1+q}+E_{2m_1+1+n+q,2m_1+1+p})\big) \tag{4.2}$$

and

$$osp(2m_1+1,2n)_1=\sum_{i\in\overline{1,m_1};p\in\overline{1,n}}\big(\mathbb{C}(E_{i,2m_1+1+p}-E_{2m_1+1+n+p,m_1+i})$$

$$+\mathbb{C}(E_{i,2m_1+1+n+p}+E_{2m_1+1+p,m_1+i})+\mathbb{C}(E_{m_1+i,2m_1+1+p}-E_{2m_1+1+n+p,i})$$

$$+\mathbb{C}(E_{m_1+i,2m_1+1+n+p}+E_{2m_1+1+p,m_1+i})\big)+\sum_{p=1}^{n}\big(\mathbb{C}(E_{2m_1+1,2m_1+1+p}-E_{2m_1+1+n+p,2m_1+1})$$

$$+\mathbb{C}(E_{2m_1+1,2m_1+1+n+p}+E_{2m_1+1+p,2m_1+1})\big). \tag{4.3}$$

Take

$$H=\sum_{i=1}^{m_1}\mathbb{C}(E_{i,i}-E_{m_1+i,m_1+i})+\sum_{j=1}^{n}\mathbb{C}(E_{2m_1+1+j,2m_1+1+j}-E_{2m_1+1+n+j,2m_1+1+n+j}) \tag{4.4}$$

as a Cartan subalgebra of $osp(2m_1+1,2n)$. We still denote by $\lambda_1,\cdots,\lambda_{m_1},\nu_1,\cdots,\nu_n$ the fundamental weights. Let

$$\begin{aligned}osp(2m_1+1,2n)^+ \;=\;& \sum_{1\leq i<j\leq m}\big(\mathbb{C}(E_{i,j}-E_{m_1+j,m_1+i})+\mathbb{C}(E_{i,m_1+j}-E_{j,m_1+i})\big)\\ &+\sum_{1\leq p<q\leq n}\mathbb{C}(E_{2m_1+1+p,2m_1+1+q}-E_{2m_1+1+n+q,2m_1+1+n+p})\\ &+\sum_{1\leq p\leq q\leq n}\mathbb{C}(E_{2m_1+1+p,2m_1+1+n+q}+E_{2m_1+1+q,2m_1+1+n+p})\\ &+\sum_{1\leq i\leq m,1\leq q\leq n}\big(\mathbb{C}(E_{i,2m_1+1+q}-E_{2m_1+1+n+q,m_1+i})\\ &+\mathbb{C}(E_{i,2m_1+1+n+q}+E_{2m_1+1+q,m_1+i})\big)\\ &+\sum_{i=1}^{m_1}\mathbb{C}(E_{i,2m_1+1}-E_{2m_1+1,m_1+i})\\ &+\sum_{p=1}^{n}\mathbb{C}(E_{2m_1+1,2m_1+1+n+p}+E_{2m_1+1+p,2m_1+1}),\end{aligned} \tag{4.5}$$

and $osp(2m_1+1,2n)^+_\sigma=osp(2m_1+1,2n)_\sigma\cap osp(2m_1+1,2n)^+$ for $\sigma=0,1$. We redefine Δ and η by

$$\Delta_x=-2\sum_{i=1}^{r}x_i\partial_{x_{m_1+i}}+2\sum_{i=r+1}^{m_1}\partial_{x_i}\partial_{x_{m_1+i}}+\partial^2_{x_{2m_1+1}}, \tag{4.6}$$

$$\Delta_\theta=\sum_{j=1}^{n}\partial_{\theta_j}\partial_{\theta_{n+j}},\quad \Delta=\Delta_x+2\Delta_\theta, \tag{4.7}$$

$$\eta_x=2\sum_{i=1}^{r}x_{m_1+i}\partial_{x_i}+2\sum_{i=r+1}^{m_1}x_ix_{m_1+i}+x^2_{2m_1+1}, \tag{4.8}$$

$$\eta_\theta=\sum_{j=1}^{n}\theta_j\theta_{n+j},\quad \eta=\eta_x+2\eta_\theta. \tag{4.9}$$

Recall the notions in (1.7) and (1.10).

Theorem 4.1 *The subspace $\mathcal{H}^r_k$ $(0\leq r\leq m_1)$ is an irreducible highest weight $osp(2m_1+1,2n)$-module with highest weights and corresponding vectors are listed as follows:*

r	k	*vector*	*weight*
$r=0$	$k>0$	x_1^k	$k\lambda_1$
$r<m_1$	$k>0$	x_{r+1}^k	$-(k+1)\lambda_r+k\lambda_{r+1}$
	$k\leq 0$	x_r^{-k}	$k\lambda_{r-1}-(k+1)\lambda_r$
$r=m_1$	$k\leq 0$	$x_{m_1}^{-k}$	$k\lambda_{m_1-1}-(k-1)\lambda_{m_1}$
	$0<k\leq n$	$\theta_1\cdots\theta_k$	ν_k *if* $k<n$ $2\nu_n$ *if* $k=n$
	$n<k\leq 2n$	$\sum_{i=0}^{k-n-1}\frac{(k-n-1-i)!}{2^i i!(2k-2n-1-2i)!}x_{2m_1+1}^{2k-2n-1-2i}\eta_\theta^i\theta_1\cdots\theta_{k-n-1}$	$2\nu_n$ *if* $k=n+1$ ν_{k-n-1} *if* $k>n+1$
	$k>2n$	$\sum_{j=0}^{n}\sum_{i=0}^{k-2n-1}\frac{1}{i!(k-2n-1-i)!(2k-2n-1-2i-2j)!!j!}$ $\times x_{m_1}^{k-2n-1-i}x_{2m_1}^i x_{2m_1+1}^{2k-2n-1-2i-2j}\eta_\theta^j$	$(2n-k+1)\lambda_{m_1-1}$ $+(k-2n-2)\lambda_{m_1}$

Moreover, $\mathcal{A}_k^r=\mathcal{H}_k^r\oplus\eta\mathcal{A}_{k-2}^r$.

Proof. (1) Assume $r=0$.

First we can get $H_k^0=\langle x_1^k\rangle$ by induction on n. If

$$osp(2m_1+1,2n)_0^+(g)=0, \tag{4.10}$$

then up to a scalar multiple, g must be of the form

$$\sum_{i=0}^{l}a_i x_1^{k-2l-t}\eta_x^{l-i}\eta_\theta^i\theta_1\cdots\theta_t \tag{4.11}$$

with $l-2l-t\geq 0$, $l\geq 0$, $0\leq t\leq n$ and $a_i=0$ for $i>n-t$. Since

$$\begin{aligned}0 &= \Delta(\sum_{i=0}^{l}a_i x_1^{k-2l-t}\eta_x^{l-i}\eta^i\theta_1\cdots\theta_t)\\ &= 2\sum_{i=0}^{l-1}a_i(l-i)(2k-2l-2t+2m_1-1-2i)x_1^{k-2l-t}\eta_x^{l-i-1}\eta^i\theta_1\cdots\theta_t\\ &\quad -2\sum_{i=1}^{l}a_i i(n-t-i+1)x_1^{k-2l-t}\eta_x^{l-i}\eta^{i-1}\theta_1\cdots\theta_t\end{aligned} \tag{4.12}$$

we get that

$$a_i(l-i)(2k-2l-2t+2m_1-1-2i)=a_{i+1}(i+1)(n-t-i) \tag{4.13}$$

for $0\leq i<l$ and $l\leq n-t$. Thus all the vectors satisfying (4.10) should be a scalar multiple of

$$f_{l,t}=\sum_{i=0}^{l}\frac{(n-t-i)!}{i!(l-i)!(2k-2l-2t+2m_1-1-2i)!}x_1^{k-2l-t}\eta_x^{l-i}\eta_\theta^i\theta_1\cdots\theta_t, \tag{4.14}$$

where $0\leq t\leq n$ and $0\leq l\leq\min\{n-t,\frac{1}{2}(k-t)\}$.

Take any $0\neq f\in\mathcal{H}_k^0$. Then there should be some $f_{l_0,t_0}\in\langle f\rangle$. If $l_0=0$, then

$$x_1^k=\prod_{i=1}^{t}(-1)^{t-1}(E_{1,2m_1+1+i}-E_{2m_1+1+n+i,m_1+1})(f_{0,t_0}), \tag{4.15}$$

and so $\mathcal{H}_k^0 = \langle x_1^k \rangle \subset \langle f_{0,t_0} \rangle \subset \langle f \rangle$. Now assume $l_0 > 0$. Observe that

$$(E_{1,2m_1+n+t+1}+E_{2m_1+1+t+1,m_1+1})(f_{l,t}) = (-1)^{t-1}\big(2(n-t-k+l+t-m)+1\big)f_{l-1,t+1} \quad (4.16)$$

for $0 < l \leq \min\{n-t, \frac{1}{2}(k-t)\}$. Since $2(n-t-k+l+t-m)+1$ is odd, we obtain $f_{l_0-1,t_0+1}, \cdots, f_{0,l_0+t_0} \in \langle f \rangle$ by (4.16). Hence $\mathcal{H}_k^0 = \langle f_{0,l_0+t_0} \rangle \subset \langle f \rangle$.

(2) Suppose $r < m_1$.

Taking induction on n, we obtain that the submodule $\mathcal{H}_k^r$ is generated by x_{r+1}^k if $k \geq 0$ or x_r^{-k} if $k < 0$.

Let W be a nonzero submodule of $\mathcal{H}_k^r$. Then W should contain some weight vector f annihilated by $osp(2m_1+1, 2n)_0^+$. Indeed, f should be of the form

$$\sum_{i=0}^{l} a_i \eta_x^{l-i} x_{r+1}^{k-2l-t} \eta_\theta^i \theta_1 \cdots \theta_t \quad (4.17)$$

or

$$\sum_{i=0}^{l} a_i \eta_x^{l-i} x_r^{2l+t-k} \eta_\theta^i \theta_1 \cdots \theta_t \quad (4.18)$$

because $osp(2m_1+1, 2n)_0 = so(2m_1+1) \oplus sp(2n)$, where $0 \leq t \leq n$ and $0 \leq l \leq n-t$.

If

$$f = \sum_{i=0}^{l} a_i \eta_x^{l-i} x_{r+1}^{k-2l-t} \eta_\theta^i \theta_1 \cdots \theta_t, \quad (4.19)$$

then by the similar argument as in (1), we obtain $x_{r+1}^k \in \langle f \rangle \subset W$, which implies $W = \mathcal{H}_k^r$. Now we assume

$$f = \sum_{i=0}^{l} a_i \eta_x^{l-i} x_r^{2l+t-k} \eta_\theta^i \theta_1 \cdots \theta_t. \quad (4.20)$$

Since

$$\begin{aligned} 0 = \Delta(f) \quad &= \quad \sum_{i=0}^{l-1} 2(l-i)(2l-2p+2m_1-2r-1-2i) a_i \eta_x^{l-i-1} x_r^p \eta_\theta^i \theta_1 \cdots \theta_t \\ &\quad - \sum_{i=1}^{l} 2i(n-t-i+1) a_i \eta_x^{l-i} x_r^p \eta_\theta^{i-1} \theta_1 \cdots \theta_t, \end{aligned} \quad (4.21)$$

we get

$$a_{i+1}(i+1)(n-t-i) = a_i(l-i)(2l-2p+2m_1-2r-1-2i). \quad (4.22)$$

So f is a scalar multiple of

$$f_{l,t} = \sum_{i=0}^{l} \frac{(n-t-i)!(2l-2p+2m_1-2r-1)!!}{i!(l-i)!(2l-2p+2m_1-2r-1-2i)!!} \eta_x^{l-i} x_r^{2l+t-k} \eta_\theta^i \theta_1 \cdots \theta_t, \quad (4.23)$$

where

$$\frac{(2l-2p+2m_1-2r-1)!!}{(2l-2p+2m_1-2r-1-2i)!!} = \prod_{j=0}^{i-1}(2l-2p+2m_1-2r-1-2j). \quad (4.24)$$

Note

$$\begin{aligned}&(E_{r,2m_1+1+n+t+1}+E_{2m_1+1+t+1,m_1+r})(f_{l,t})\\ =\quad&(-1)^{t+1}(2l+t-k)(2l-2p+2m-2r-1-2n+2t)f_{l-1,t+1}\end{aligned}\tag{4.25}$$

for $2l+t>k$, $t<n$ and $l>0$. Thus we have $f_{l-1,t+1},f_{l-2,t+2},\cdots,f_{0,l+t}\in W$ when $k\leq l+t\leq n$ or $f_{l-1,t+1},f_{l-2,t+2},\cdots,f_{k-l-t,2(l+t)-k}\in W$ when $l+t<k$. For the later case, we can get $x_{r+1}^k\in W$. Now suppose $f_{0,l+t}\in W$. Applying

$$E_{r,2m_1+1+q}-E_{2m_1+1+n+q,m_1+r})|_{\mathcal{A}_k^r}=\partial_{x_r}\partial_{\theta_q}-\theta_{n+q}\partial_{x_{m_1+r}}\tag{4.26}$$

to $f_{0,l+t}$, we obtain $x_r^{-k}\in W$ or $\theta_1\cdots\theta_k\in W$. Applying

$$(E_{r+1,2m_1+1+q}-E_{2m_1+1+n+q,m_1+r+1})|_{\mathcal{A}_k^r}=x_{r+1}\partial_{\theta_q}-\theta_{n+q}\partial_{x_{m_1+r+1}}\tag{4.27}$$

to $\theta_1\cdots\theta_k$, we get $x_{r+1}^k\in W$. Thus $W=\mathcal{H}_k^r$.

(3) Assume $r=m_1$.

Denote

$$g_{p,q}=\sum_{s=0}^{q}\frac{1}{s!(q-t)!(p+q-2s)!!}x_{2m_1+1}^{p+q-2s}x_{m_1}^{q-t}x_{2m_1}^{s}\tag{4.28}$$

for $p>q\geq0$. By induction on n, we can obtain

$$\mathcal{H}_k^{m_1}=\begin{cases}\langle x_{m_1}^{n-k}\theta_1\cdots\theta_n\rangle & \text{if } k\leq n,\\ \langle g_{k-n,k-n-1}\theta_1\cdots\theta_n\rangle & \text{if } k>n.\end{cases}\tag{4.29}$$

Let W be a nonzero submodule of $\mathcal{H}_k^r$. Then W should contain some weight vector f annihilated by $osp(2m_1+1,2n)_0^+$. Indeed, f should be of the form

$$\sum_{i=0}^{l}a_i\eta_x^{l-i}x_{m_1}^{2l+t-k}\eta_\theta^i\theta_1\cdots\theta_t\tag{4.30}$$

or

$$\sum_{i=0}^{l'}b_i\eta_x^{l'-i}g_{k-2l'-t',k-2l'-t'-1}\eta_\theta^i\theta_1\cdots\theta_{t'}\tag{4.31}$$

because $osp(2m_1+1,2n)_0=so(2m_1+1)\oplus sp(2n)$, where $0\leq t,t'\leq n$, $0\leq l\leq n-t$, $0\leq l'\leq n-t'$, $2l+t\geq k$ and $2l'+t'<k$.

(i) Suppose $f=\sum_{i=0}^{l'}b_i\eta_x^{l'-i}g_{k-2l'-t',k-2l'-t'-1}\eta_\theta^i\theta_1\cdots\theta_{t'}$. Since

$$\begin{aligned}0=\Delta(f)\;=\;&\sum_{i=0}^{l'-1}2b_i(2k-2t'-2l'-1-2i)(l-i)\eta_x^{l'-i-1}g_{k-2l'-t',k-2l'-t'-1}\eta_\theta^i\theta_1\cdots\theta_{t'}\\&-\sum_{i=1}^{l'}2b_ii(n-t'-i+1)\eta_x^{l'-i}g_{k-2l'-t',k-2l'-t'-1}\eta_\theta^{i-1}\theta_1\cdots\theta_{t'},\end{aligned}\tag{4.32}$$

we get

$$b_{i+1}(i+1)(n-t'-i)=b_i(l'-i)(2k-2t'-2l'-1-2i).\tag{4.33}$$

Thus f should be a scalar multiple of

$$h_{l,t} = \sum_{i=0}^{l} \frac{(n-t-i)!(2k-2t-2l-1)!!}{i!(l-i)!(2k-2t-2l-1-2i)!!} \eta_x^{l-i} g_{k-2l-t,k-2l-t-1} \eta_\theta^i \theta_1 \cdots \theta_t. \tag{4.34}$$

Note

$$(E_{m_1,2m_1+1+n+t+1} + E_{2m_1+1+t+1,2m_1})(h_{l,t}) = \frac{(-1)^t(n-t-l)}{(2k-4l-2t-1)(2k-2t-2l+1)} h_{l,t+1} \tag{4.35}$$

if $l+t<n$ and $k-2l-t>1$. So we can assume $l+t=n$ or $k-2l-t=1$.

(a) If $l+t=n$, then

$$h_{n-t,t} = \sum_{i=0}^{n-t} \frac{(2k-2n-1)!!}{i!(2k-2n-1-2i)!!} \eta_x^{n-t-i} g_{k-2n+t,k-2n+t-1} \eta_\theta^i \theta_1 \cdots \theta_t. \tag{4.36}$$

We have

$$\begin{aligned} & (E_{2m_1,2m_1+1+n+t+1} + E_{2m_1+1+t+1,m_1})(h_{n-t,t}) \\ = \; & (-1)^{t+1}(k-2n+t)(2k-4n+2t+1) h_{n-t-1,t+1}. \end{aligned} \tag{4.37}$$

Thus $h_{n-t-1,t+1}, h_{n-t-2,t+2}, \cdots, h_{0,n} \in W$. Since $h_{0,n} = g_{k-n,k-n-1}\theta_1 \cdots \theta_n$, we get $W = \mathcal{H}_k^{m_1}$ by (4.29).

(b) Assume $k-2l-t=1$ and $l+t<n$. Then

$$h_{l,t} = h_{l,k-2l-1} = \sum_{i=0}^{l} \frac{(n-k+2l+1-i)!(2l-1)!!}{i!(l-i)!(2l-1-2i)!!} x_{2m_1+1}^{2l-2i+1} \eta_\theta^i \theta_1 \cdots \theta_{k-2l-1}. \tag{4.38}$$

Now we take induction on l. When $l=0$, we have

$$(E_{2m_1+1,2m_1+1+n+k} + E_{2m_1+1+k,2m_1+1})(h_{0,k-1}) = (-1)^{k-1}\theta_1 \cdots \theta_k \tag{4.39}$$

and

$$x_{m_1}^{n-k}\theta_1 \cdots \theta_n = (-1)^{(k+1)(n-k)} \prod_{j=k+1}^{n} (E_{2m,2m_1+1+n+j} + E_{2m_1+1+j,m_1})(\theta_1 \cdots \theta_k). \tag{4.40}$$

Thus $W = \mathcal{H}_k^{m_1}$.

Now we assume $l>0$. Observe that $k-2l = t+1 \le t+l < n$ and

$$\begin{aligned} & (E_{2m_1+1,2m_1+1+n+k-2l+1} + E_{2m_1+1+k-2l+1,2m_1+1})(E_{2m_1+1,2m_1+1+n+k-2l} \\ & + E_{2m_1+1+k-2l,2m_1+1})(h_{l,k-2l-1}) \\ = \; & -(n-k+l+1)(2l-1)(2n-2k+2l-1) h_{l-1,k-2l+1}. \end{aligned} \tag{4.41}$$

Since $n-k+l+1 = n-l-t > 0$, we get $h_{l-1,k-2l+1} \in W$. Therefore $W = \mathcal{H}_k^{m_1}$ by inductive assumption.

(ii) Assume $f=\sum_{i=0}^{l}a_i\eta_x^{l-i}x_{m_1}^{2l+t-k}\eta_\theta^i\theta_1\cdots\theta_t$. Then

$$\begin{aligned}0=\Delta(f) &= \sum_{i=0}^{l-1}2a_i(2k-2t-2l-1-2i)(l-i)\eta_x^{l-i-1}x_{m_1}^{2l+t-k}\eta_\theta^i\theta_1\cdots\theta_t\\ &\quad -\sum_{i=1}^{l}2a_ii(n-t-i+1)\eta_x^{l-i}x_{m_1}^{2l+t-k}\eta_\theta^{i-1}\theta_1\cdots\theta_t.\end{aligned}\tag{4.42}$$

Thus we get that f should be a scalar multiple of

$$f_{l,t}=\sum_{i=0}^{l}\frac{(n-t-i)!(2k-2t-2l-1)!!}{i!(l-i)!(2k-2t-2l-1-2i)!!}\eta_x^{l-i}x_{m_1}^{2l+t-k}\eta_\theta^i\theta_1\cdots\theta_t,\tag{4.43}$$

where we treat

$$\frac{(2k-2t-2l-1)!!}{(2k-2t-2l-1-2i)!!}=(2k-2t-2l-1)(2k-2t-2l-3)\cdots(2k-2t-2l+1-2i).\tag{4.44}$$

We will show $W=\mathcal{H}_k^{m_1}$ by induction on l. When $l=0$, we have $f_{0,t}=x_{m_1}^{t-k}\theta_1\cdots\theta_t\in W$ and

$$x_{m_1}^{n-k}\theta_1\cdots\theta_n=(-1)^{(k+1)(n-k)}\prod_{j=k+1}^{n}(E_{2m,2m_1+1+n+j}+E_{2m_1+1+j,m_1})(f_{l,k-2l}).\tag{4.45}$$

Thus $W=\mathcal{H}_k^{m_1}$ because of (4.29). Now assume $l>0$. Note

$$(E_{m_1,2m_1+1+n+t+1}+E_{2m_1+1+t+1,2m_1})(f_{l,t})=(-1)^t(2l+t-k)(2n-2k+2l+1)f_{l-1,t+1}.\tag{4.46}$$

If $2l+t\neq k$, then we get $f_{l-1,t+1}\in W$, which implies $W=\mathcal{H}_k^{m_1}$ by inductive assumption.

If $2l+t=k$, then

$$f_{l,t}=f_{l,k-2l}=\sum_{i=0}^{l}\frac{(2l-1)!!(n-k+2l-i)!}{i!(l-i)!(2l-1-2i)!!}x_{2m_1+1}^{2l-2i}\eta_\theta^i\theta_1\cdots\theta_{k-2l}.\tag{4.47}$$

When $k-2l=n-1$, we have $l=1$, $k=n+1$ and

$$\begin{aligned}&(E_{2m_1+1,2m_1+1+2n}+E_{2m_1+1+n,2m_1+1})(f_{1,n-1})\\ =\ &(-1)^kx_{2m_1+1}\theta_1\cdots\theta_n.\end{aligned}\tag{4.48}$$

Therefore, $W=\mathcal{H}_k^{m_1}$ due to (4.29).

Now we assume $k-2l<n-1$. Observe that

$$\begin{aligned}&(E_{2m_1+1,2m_1+1+n+k-2l+2}+E_{2m_1+1+k-2l+2,2m_1+1})(E_{2m_1+1,2m_1+1+n+k-2l+1}\\ &+E_{2m_1+1+k-2l+1,2m_1+1})(f_{l,k-2l})\\ =\ &-(2n-2k+2l+1)(2l-1)(n-k+l)f_{l-1,k-2l+2}.\end{aligned}\tag{4.49}$$

We get $W=\mathcal{H}_k^{m_1}$ by inductive assumption if $n-k+l\neq 0$.

When $k = n + l$, we have

$$f_{l,t} = f_{k-n,2n-k} = \sum_{i=0}^{k-n} \frac{(2k-2n-1)!!}{i!(2k-2n-1-2i)!!} x_{2m_1+1}^{2(k-n-i)} \eta_\theta^i \theta_1 \cdots \theta_{2n-k} \tag{4.50}$$

and

$$(E_{2m_1,2m_1+1+3n-k+1} + E_{2m_1+1+2n-k+1,m_1})(f_{k-n,2n-k}) = h_{k-n-1,2n-k+1}. \tag{4.51}$$

Thus we get $h_{k-n-1,2n-k-1} \in W$, which implies $W = \mathcal{H}_k^{m_1}$ by (i). □

Let us go for the submodule $\mathcal{A}'_k$ (cf. (1.15)). We know that the submodule $\mathcal{A}'_k$ is irreducible when $n = 0$. Thus it is not difficult to check the irreducibility of $\mathcal{A}'_k$.

Theorem 4.2 *The* $osp(2m_1 + 1, 2n)$*-submodules* $\mathcal{A}'_k$ *are irreducible.*

Proof. (1) Assume $S_1 \neq \emptyset$. Take $i \in S_1$ and $0 \neq f \in \mathcal{A}'_k$. If f is not independent of θ_{2m_1+1}, we apply

$$E_{2m_1+1+i,2m_1+1+n+i}|_{\mathcal{A}'} = x_i \partial_{x_{n+i}} \tag{4.52}$$

to f and get some $0 \neq f_1 \in \langle f \rangle$ satisfying $\partial_{x_{n+i}}(f_1) = 0$. If f_1 is not independent of θ_{2m_1+1}, we have

$$f_2 = (E_{2m_1+1,2m_1+1+n+i} + E_{2m_1+1+i,2m_1+1})|_{\mathcal{A}'}(f_1) = x_i \partial_{2m_1+1}(f_1). \tag{4.53}$$

Anyway there is some nonzero $f' = f'(x_1, \cdots, x_{2n}; \theta_1 \cdots, \theta_{2m_1}) \in \langle f \rangle$. By Theorem 3.2, we obtain

$$\text{Span}\,\{x^\alpha \theta_{i_1} \cdots \theta_{i_t} \in \mathcal{A}'_k \mid 0 \leq t \leq 2m_1;\ i_1, \cdots, i_t \in \overline{1, 2m_1}\} \subset \langle f \rangle. \tag{4.54}$$

Now for any $x^\alpha \theta_{i_1} \cdots \theta_{i_t} \theta_{2m_1+1} \in \mathcal{A}'_k$, we have

$$x^\alpha \theta_{i_1} \cdots \theta_{i_t} \theta_{2m_1+1} = \frac{(-1)^t}{\alpha_i + 1}(E_{2m_1+1,2m_1+1+i} - E_{2m_1+1+n+i,2m_1+1})(x_i x^\alpha \theta_{i_1} \cdots \theta_{i_t}) \in \langle f \rangle. \tag{4.55}$$

Therefore $\langle f \rangle = \mathcal{A}'_k$.

It can be similarly proved when $T_1 \neq \emptyset$.

(2) Suppose $S_1 \cup T_1 = \emptyset$. We can assume $T = \overline{1, n}$ by symmetry. We claim that $\mathcal{A}'_k = \langle x_n^{m_1-k} \theta_1 \cdots \theta_{m_1} \rangle$ when $k \leq m_1$.

Using Theorem 3.2, we get

$$\text{Span}\,\{x^\alpha \theta_{i_1} \cdots \theta_{i_t} \in \mathcal{A}'_k \mid 0 \leq t \leq 2m_1;\ i_1, \cdots, i_t \in \overline{1, 2m_1}\} \subset \langle x_n^{m_1-k} \theta_1 \cdots \theta_{m_1} \rangle \tag{4.56}$$

when $k < m_1$. Since

$$x^\alpha \theta_{i_1} \cdots \theta_{i_t} \theta_{2m_1+1} = \frac{(-1)^t}{\alpha_{n+i} + 1}(E_{2m+1+1,2m_1+n+i} + E_{2m_1+1+i,2m_1+1})(x_{n+i} x^\alpha \theta_{i_1} \cdots \theta_{i_t}), \tag{4.57}$$

we obtain $\mathcal{A}'_k = \langle x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\rangle$ when $k < m_1$. Now we assume $k = m_1$. Since

$$\begin{aligned}&(E_{2m_1+1,2m_1+1+n} - E_{2m_1+1+2n,2m_1+1})(E_{2m_1+1,2m_1+n} - E_{2m_1+2n,2m_1+1})(\theta_1\cdots\theta_{m_1})\\ =\ & x_{n-1}x_{2n}\theta_1\cdots\theta_{m_1} \qquad (4.58)\end{aligned}$$

and

$$\begin{aligned}&(E_{2m_1+1,2m_1+n} - E_{2m_1+2n,2m_1+1})(E_{2m_1+1,2m_1+1+n} - E_{2m_1+1+2n,2m_1+1})(\theta_1\cdots\theta_{m_1})\\ =\ & x_nx_{2n-1}\theta_1\cdots\theta_{m_1}, \qquad (4.59)\end{aligned}$$

we have

$$\text{Span}\,\{x^\alpha\theta_{i_1}\cdots\theta_{i_t} \in \mathcal{A}'_{m_1} \mid 0\le t\le 2m_1;\ i_1,\cdots,i_t\in\overline{1,2m_1}\} \subset \langle x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\rangle. \qquad (4.60)$$

Again by (4.57), we get $\mathcal{A}'_{m_1} = \langle\theta_1\cdots\theta_{m_1}\rangle$.

Now for any $0\neq f\in\mathcal{A}'_k$, we can write $f = f_0 + f_1\theta_{2m_1+1}$ with f_0, f_1 independent of θ_{2m_1+1}. Applying $osp(2m_1,2n)^+$ to f, we get some $0\neq g = g_0 + g_1\theta_{2m_1+1}\in\langle f\rangle$ satisfying $osp(2m_1,2n)^+(g_0) = 0$ and $osp(2m_1,2n)^+(g_1) = 0$.

(i) $k < m_1$. According to Theorem 3.2, we can write

$$g = ax_n^{m_1-k}\theta_1\cdots\theta_{m_1} + bx_n^{m_1-k+1}\theta_1\cdots\theta_{m_1}\theta_{2m_1+1} \qquad (4.61)$$

where $a,b\in\mathbb{C}$. Thus $x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\in\langle f\rangle$ or $x_n^{m_1-k+1}\theta_1\cdots\theta_{m_1}\theta_{2m_1+1}\in\langle f\rangle$ because they have different weights. If $x_n^{m_1-k+1}\theta_1\cdots\theta_{m_1}\theta_{2m_1+1}\in\langle f\rangle$, we have

$$x_n^{m_1-k}\theta_1\cdots\theta_{m_1} = \frac{(-1)^{m_1}}{m_1-k+1}(E_{2m_1+1,2m_1+1+2n}+E_{2m_1+1+n,2m_1+1})(x_n^{m_1-k+1}\theta_1\cdots\theta_{m_1}\theta_{2m_1+1}). \qquad (4.62)$$

Anyway $x_n^{m_1-k}\theta_1\cdots\theta_{m_1}\in\langle f\rangle$, which implies $\langle f\rangle = \mathcal{A}'_k$.

(ii) $k = m_1$. We can write

$$g = a\theta_1\cdots\theta_{m_1} + b(x_{n-1}x_{2n} - x_nx_{2n-1})\theta_1\cdots\theta_{m_1} + cx_n\theta_1\cdots\theta_{m_1}\theta_{2m_1+1}, \qquad (4.63)$$

where $a,b,c\in\mathbb{C}$. Thus $\theta_1\cdots\theta_{m_1}\in\langle f\rangle$ or $(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1}\in\langle f\rangle$ or $x_n\theta_1\cdots\theta_{m_1}\theta_{2m_1+1}\in\langle f\rangle$. If $x_n\theta_1\cdots\theta_{m_1}\theta_{2m_1+1}\in\langle f\rangle$, we have $\theta_1\cdots\theta_{m_1}\in\langle f\rangle$ by (4.62). If $(x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1}\in\langle f\rangle$, we have

$$\begin{aligned}&(E_{2m_1+1,2m_1+2n} + E_{2m_1+n,2m_1+1})((x_{n-1}x_{2n}-x_nx_{2n-1})\theta_1\cdots\theta_{m_1})\\ =\ & -(-1)^{m_1}x_n\theta_1\cdots\theta_{m_1}\theta_{2m_1+1}. \qquad (4.64)\end{aligned}$$

Thus $\theta_1\cdots\theta_{m_1}\in\langle f\rangle$. □

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