ON THE COHOMOLOGY OF LOOP SPACES FOR SOME THOM SPACES

ANDREW BAKER

Abstract. In this paper we identify conditions under which the cohomology $H^*(\Omega M\xi;\Bbbk)$ for the loop space $\Omega M\xi$ of the Thom space $M\xi$ of a spherical fibration $\xi\downarrow B$ can be a polynomial ring. We use the Eilenberg-Moore spectral sequence which has a particularly simple form when the Euler class $e(\xi)\in H^n(B;\Bbbk)$ vanishes, or equivalently when an orientation class for the Thom space has trivial square. As a consequence of our homological calculations we are able to show that the suspension spectrum $\Sigma^\infty\Omega M\xi$ has a local splitting replacing the James splitting of $\Sigma\Omega M\xi$ when $M\xi$ is a suspension.

Introduction

In [1], topological methods were used to prove the algebraic Ditter's conjecture on quasi-symmetric functions, which is equivalent to the assertion that $H^*(\Omega\Sigma\mathbb{C}P^\infty;\mathbb{Z})$ is a polynomial ring (infinitely generated but of finite type). Most of the ingredients of the proof given there are essentially formal within algebraic topology, the exception being James's splitting of $\Sigma\Omega\Sigma\mathbb{C}P^\infty$. The purpose of this paper is to identify circumstances in which the cohomology $H^*(\Omega M\xi;\Bbbk)$ of the loop space $\Omega M\xi$ of the Thom space $M\xi$ of a spherical fibration $\xi\downarrow B$ can be a polynomial ring. In place of the James splitting we use the Eilenberg-Moore spectral sequence which has a particularly simple form when the Euler class $e(\xi)\in H^n(B;\Bbbk)$ vanishes, or equivalently when an orientation class for the Thom space has trivial square. As a consequence of our homological calculations we are able to show that the suspension spectrum $\Sigma^\infty\Omega M\xi$ has a local splitting generalizing that for $\Sigma\Omega M\xi$ when $M\xi$ is a suspension. Our results appear to be more general and essentially formal in that only generic properties of the Eilenberg-Moore spectral sequence are used; however, the above stable splitting is a weaker result than the James splitting.

Although our examples are all associated with vector bundles, our methods are valid for arbitrary spherical fibrations, and even more generally they apply to p-local or p-complete spherical fibrations. We hope to consider examples associated with p-compact groups in future work.

We were very influenced by the discussion of the cohomology of $\Omega\Sigma X$ in Smith's article [15]. Massey's paper [5] provides a useful background to our work. Although we do not make direct use of it, Ray's paper [8] has ideas that might allow generalizations to other mapping cones. Although we do not make direct use of the results of these papers, we remark that Bott & Samelson [2] and Petrie [7] gave earlier versions of the arguments we use, however neither paper

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contains the full range of our results; in particular the latter does not deal with questions about multiplicative structure.

1. Thom complexes of spherical fibrations

Let B be space and let $\xi\colon S^{n-1} \longrightarrow S \longrightarrow B$ be a spherical fibration with associated disc bundle $D^n \longrightarrow D \longrightarrow B$. The Thom space $M = M\xi$ is the cofibre of the inclusion $S \longrightarrow D$, *i.e.*, the quotient space D/S. In each fibre this corresponds to the inclusion $S^{n-1} \longrightarrow D^n$ and there is a cofibre sequence of based spaces

$$S_+ \longrightarrow D_+ \longrightarrow M \xrightarrow{\delta} \Sigma S_+. \tag{1.1}$$

Here we implicitly allow for generalizations to include localized spheres as fibres and bundles with structure monoids obtained from the invertible components of $\mathrm{Maps}(S^{n-1}, S^{n-1})$.

We are interested in the based loop space ΩM. There is an obvious unbased map $S \longrightarrow \Omega M$ which sends $v \in S_b$ (the fibre above $b \in B$) to the non-constant loop $[0,1] \longrightarrow M$ given by $t \mapsto [(2t-1)v]$, running through b parallel to v and passing through the base point at times $t = 0, 1$. This extends to a based map $\theta\colon S_+ \longrightarrow \Omega M$. We write $\mathrm{ev}\colon \Sigma\Omega M \longrightarrow M$ for the evaluation map. See [8] for a related construction

Our next result is surely standard but we don't know an explicit reference.

Lemma 1.1. *The composition*

$$M \xrightarrow{\delta} \Sigma S_+ \xrightarrow{\Sigma\theta} \Sigma\Omega M \xrightarrow{\mathrm{ev}} M$$

is a homotopy equivalence.

Proof. This follows by unravelling definitions. Depending on the sign conventions used for the coboundary map of a cofibration, it is homotopic to $\pm\,\mathrm{Id}$. ☐

Corollary 1.2. *Let $h^*(-)$ be a reduced cohomology theory. Then the cohomology suspension map*

$$h^*(M) \xrightarrow{\mathrm{ev}^*} h^*(\Sigma\Omega M) \xrightarrow{\cong} h^{*-1}(\Omega M)$$

is a monomorphism.

These two results are analogues of results for a suspension ΣX in [15, section 2] which depend on the fact that Σ, Ω is an adjoint pair.

The next result is standard, although it seems to be hard to find it stated in this form in the literature, see for example [7, section 1]. To clarify what is involved, we give details. First recall an algebraic notion.

Let $\Bbbk$ be a commutative unital ring; tensor products will be taken over $\Bbbk$ unless otherwise specified. Let A be a commutative unital graded $\Bbbk$-algebra with product $\varphi\colon A \otimes A \longrightarrow A$.

Definition 1.3. A *non-unital A-algebra* is a left A-module M with multiplication

$$A \otimes M \longrightarrow M; \quad a \otimes m \mapsto a \cdot m$$

and a non-unital associative product $\mu\colon M\otimes_A M \longrightarrow M$. Thus the following diagram commutes, where $\mathrm{T}\colon M\otimes A \longrightarrow A\otimes M$ is the switch map with appropriate signs based on gradings.

$$\begin{array}{ccc} A\otimes M\otimes A\otimes M & \xrightarrow{I\otimes \mathrm{T}\otimes I} & A\otimes A\otimes M\otimes M \\ \Big\downarrow{\scriptstyle \cdot\otimes\cdot} & & \Big\downarrow{\scriptstyle \varphi\otimes\mu} \\ M\otimes M & & A\otimes M \\ & {\scriptstyle \mu}\searrow \qquad \swarrow{\scriptstyle \cdot} & \\ & M & \end{array}$$

For homogeneous elements $a_1, a_2 \in A$, $m_1, m_2 \in M$ and $m_1 m_2 = \mu(m_1\otimes m_2)$,

$$(a_1a_2)\cdot(m_1m_2) = (-1)^{|a_2|\,|m_1|}\mu((a_1\cdot m_1)\otimes(a_2m_2)).$$

There is a Thom diagonal map $\widetilde{\Delta}\colon M \longrightarrow B_+\wedge M$ fitting into a strictly commutative diagram

(1.2)
$$\begin{array}{ccc} D_+ & \xrightarrow{\Delta} & D_+\wedge D_+ \\ {\scriptstyle \text{quot.}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{quot.}} \\ M & \xrightarrow{\widetilde{\Delta}} & B_+\wedge M \end{array}$$

whose vertical maps are the evident quotient maps. If $h^*(-)$ is a multiplicative cohomology theory, then $\widetilde{\Delta}$ induces an external product

$$\cdot : h^*(B)\otimes \widetilde{h}^*(M) \longrightarrow \widetilde{h}^*(B_+\wedge M) \xrightarrow{\widetilde{\Delta}^*} \widetilde{h}^*(M); \quad b\otimes m \mapsto b\cdot m,$$

where $\widetilde{h}^*(-)$ denotes the reduced theory.

Theorem 1.4. *Suppose that $h^*(-)$ is a commutative multiplicative cohomology theory. Then the external product induced from $\widetilde{\Delta}$ makes $\widetilde{h}^*(M)$ into a left $h^*(B)$-module enjoying the following properties.*
(a) *If M has an orientation $u\in \widetilde{h}^n(M)$ then the associated Thom isomorphism*

$$h^*(B) \xrightarrow{\cong} \widetilde{h}^*(M); \quad x \leftrightarrow x\cdot u$$

makes $\widetilde{h}^*(M)$ into a free $h^*(B)$-module of rank 1.
(b) *The cup product on $\widetilde{h}^*(M)$ makes it a commutative non-unital $h^*(B)$-algebra.*
(c) *When $h^*(-) = H^*(-;\mathbb{F}_p)$ for a prime p, the mod p Steenrod algebra acts compatibly so that the Cartan formula holds for products of the form $t\cdot w$ with $t\in H^*(B;\mathbb{F}_p)$ and $w\in \widetilde{H}^*(M;\mathbb{F}_p)$.*

Proof. The main point is to verify that the following diagram commutes, where Δ always denotes an internal based diagonal map $X \longrightarrow X\wedge X$.

(1.3)
$$\begin{array}{ccccc} & & M & & \\ & {\scriptstyle \Delta}\swarrow & & \searrow{\scriptstyle \widetilde{\Delta}} & \\ M\wedge M & & & & B_+\wedge M \\ {\scriptstyle \widetilde{\Delta}\wedge\widetilde{\Delta}}\searrow & & & & \swarrow{\scriptstyle \Delta\wedge\Delta} \\ B_+\wedge M\wedge B_+\wedge M & & \xrightarrow[\text{switch}]{} & & B_+\wedge B_+\wedge M\wedge M \end{array}$$

Making use of the commutative diagram (1.2), this follows from properties of the diagonal $\Delta\colon D_+ \longrightarrow D_+ \wedge D_+$ which is (strictly) coassociative, cocommutative and counital (the counit is the projection $D_+ \longrightarrow S^0$). The diagram

$$\begin{array}{ccccc} & & D_+ & & \\ & \swarrow^{\Delta} & & \searrow^{\Delta} & \\ D_+ \wedge D_+ & & & & D_+ \wedge D_+ \\ & \searrow^{\Delta\wedge\Delta} & & \swarrow^{\Delta\wedge\Delta} & \\ & D_+ \wedge D_+ \wedge D_+ \wedge D_+ & \xrightarrow[\text{switch}]{} & D_+ \wedge D_+ \wedge D_+ \wedge D_+ & \end{array}$$

commutes, so by passing to the diagram of quotients we obtain commutativity of (1.3).

Applying $h^*(-)$ and $\widetilde{h}^*(-)$ now give the algebraic properties asserted. Of course $h^*(M)$ is also a commutative unital h^*-algebra.

The statement about the Steenrod action follows from the Cartan formula for external smash products and naturality. □

Corollary 1.5. *If the orientation u satisfies $u^2 = 0$, then the product in $\widetilde{h}^*(M)$ is trivial.*

Notice that the condition $u^2 = 0$ for one orientation implies that the same is true for any orientation.

We end with another result involving the external diagonal.

Lemma 1.6. *The following diagram commutes.*

$$\begin{array}{ccccccc} & & \Sigma S_+ & \xrightarrow{\Sigma\theta} & \Sigma\Omega M & \xrightarrow{\text{ev}} & M \\ & \swarrow^{\Sigma\Delta} & & & & & \downarrow{\scriptstyle\widetilde{\Delta}} \\ \Sigma S_+ \wedge S_+ & & & & & & B_+ \wedge M \\ \downarrow & & & & & & \uparrow{\scriptstyle \mathrm{Id}\wedge \mathrm{ev}} \\ \Sigma S_+ \wedge B_+ & \xrightarrow{\cong} & B_+ \wedge \Sigma S_+ & & \xrightarrow{\mathrm{Id}\wedge\Sigma\theta} & & B_+ \wedge \Sigma\Omega M \end{array}$$

Hence if $h^*(-)$ is a multiplicative cohomology theory, then $(\mathrm{ev}\circ\Sigma\theta)^*\colon \widetilde{h}^*(M) \longrightarrow h^*(S)$ is a homomorphism of $h^*(B)$-modules.

2. Recollections on the Eilenberg-Moore spectral sequence

There is of course an extensive literature on Eilenberg-Moore spectral sequence, but for our purposes most of what we need can be found in Smith's excellent survey article [15], together with Rector and Smith's papers on Steenrod operations [9, 14]. For the homological algebra background and construction, see [11]. Other useful sources are [3, 10, 12, 13].

In the following we will assume that $\Bbbk$ is a field, and $H^*(-) = H^*(-;\Bbbk)$. We will also assume that our Thom space M from Section 1 has an orientation in $H^*(-)$, M is simply connected, and $H^*(B)$ has finite type; these conditions are needed for convergence of the Eilenberg-Moore spectral sequence we will use.

Theorem 2.1. *There is a second quadrant Eilenberg-Moore spectral sequence of $\Bbbk$-Hopf algebras $(\mathrm{E}_r^{*,*}, d_r)$ with differentials*

$$d_r\colon \mathrm{E}_r^{s,t} \longrightarrow \mathrm{E}_r^{s+r,t-r+1}$$

and
$$\mathrm{E}_2^{s,t} = \mathrm{Tor}^{s,t}_{H^*(M)}(\Bbbk,\Bbbk) \Longrightarrow H^{s+t}(\Omega M).$$

The grading conventions here give
$$\mathrm{Tor}^{s,*}_{H^*(M)} = \mathrm{Tor}^{H^*(M)}_{-s,*}$$
in the standard homological grading.

When $\Bbbk = \mathbb{F}_p$ for a prime p, this spectral sequence admits Steenrod operations; see [9, 10, 12–14]. We denote the mod p Steenrod algebra by $\mathcal{A}(p)^*$ or $\mathcal{A}^*$ when the prime p is clear.

Theorem 2.2. *If $H^*(-) = H^*(-;\mathbb{F}_p)$ for a prime p, the Eilenberg-Moore spectral sequence is a spectral sequence of $\mathcal{A}^*$-Hopf algebras.*

We will need explicit formulae for the Steenrod action. The main result is the following.

Proposition 2.3. *Suppose that X is a based space. Then in the Eilenberg-Moore spectral sequence*
$$\mathrm{E}_2^{*,*} = \mathrm{Tor}^{*,*}_{H^*(X;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) \Longrightarrow H^*(\Omega X;\mathbb{F}_p)$$
the action of the Steenrod operations on the E_2*-term is given in terms of the cobar construction by*
$$\begin{aligned}
\mathrm{Sq}^s[x_1|\cdots|x_n] &= \sum_{s_1+\cdots+s_n=s} [\mathrm{Sq}^{s_1}x_1|\cdots|\mathrm{Sq}^{s_n}x_n] && \textit{if } p=2,\\
\mathcal{P}^s[x_1|\cdots|x_n] &= \sum_{s_1+\cdots+s_n=s} [\mathcal{P}^{s_1}x_1|\cdots|\mathcal{P}^{s_n}x_n] && \textit{if } p \textit{ is odd.}
\end{aligned}$$

Sketch of Proof. There is a construction of the Eilenberg-Moore spectral sequence for the pullback of a fibration q along a map f.
$$\begin{array}{ccc}
E' & \longrightarrow & E \\
{\scriptstyle q'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle q} \\
B' & \xrightarrow[f]{} & B
\end{array}$$
For details see [3, 14]. This approach involves the cosimplicial space $C^\bullet$ with
$$C^s = E \times B^{\times s} \times B'$$
and structure maps $h_t\colon C^s \longrightarrow C^{s+1}$ $(0 \leqslant t \leqslant s+1)$,
$$h_t(e,b_1,\ldots,b_s,b') = \begin{cases} (e,h(e),b_1,\ldots,b_s,b') & \text{if } t=0,\\ (e,b_1,\ldots,b_{t-1},b_t,b_t,b_{t+1},\ldots,b_s,b') & \text{if } 1 \leqslant t \leqslant s,\\ (e,b_1,\ldots,b_s,q(b'),b') & \text{if } t=s+1. \end{cases}$$
The geometric realisation $|C^\bullet|$ admits a map $E' \longrightarrow |C^\bullet|$, and on applying $H^*(-;\mathbb{F}_p)$ to the coskeletal filtration of $|C^\bullet|$ we obtain the Eilenberg-Moore spectral sequence for $H^*(E';\mathbb{F}_p)$. Then the E_1-term can be identified with bar construction on $H^*(B;\mathbb{F}_p)$ and comes from the cohomology of the filtration quotients which are suspensions of the spaces $E \wedge B^{(s)} \wedge B'$. The action of Steenrod operations on $\widetilde{H}^*(E \wedge B^{(s)} \wedge B';\mathbb{F}_p)$ is determined using the Cartan formula, and gives the claimed formulae in the E_2-term. □

Now we come to a special situation that is our main concern.

Theorem 2.4. *Suppose that the orientation* $u \in H^n(M) = H^n(M;\Bbbk)$ *satisfies* $u^2 = 0$. *Then there is an isomorphism of Hopf algebras*

$$\mathrm{Tor}^{*,*}_{H^*(M)}(\Bbbk,\Bbbk) = \mathrm{B}^*(H^*(M)),$$

where $\mathrm{B}^*(H^*(M))$ *denotes the bar construction with*

$$\mathrm{B}^{-s}(H^*(M)) = (\widetilde{H}^*(M))^{\otimes s}$$

for $s \geqslant 0$. *The coproduct*

$$\psi\colon \mathrm{B}^{-s}(H^*(M)) \longrightarrow \bigoplus_{i=0}^{s} \mathrm{B}^{-i}(H^*(M)) \otimes \mathrm{B}^{i-s}(H^*(M))$$

is the usual one with

$$\psi([u_1|\cdots|u_s]) = \sum_{i=0}^{s} [u_1|\cdots|u_i] \otimes [u_{i+1}|\cdots|u_s],$$

where we use the traditional bar notation $[w_1|\cdots|w_r] = w_1 \otimes \cdots \otimes w_r$.

Proof. The proof is identical to that for the case of ΣX in [15, section 2, example 4], and uses the fact that $\widetilde{H}^*(N)$ has only trivial products by Corollary 1.5. □

Remark 2.5. The product in the E_2-term is the shuffle product,

$$[u_1|\cdots|u_r] \sqcup [v_1|\cdots|v_s] = \sum_{(r,s)\text{ shuffles }\sigma} (-1)^{\mathrm{Sgn}(\sigma)} [w_{\sigma(1)}|w_{\sigma(2)}|\cdots|w_{\sigma(r+s)}],$$

where $\sigma \in \Sigma_{r+s}$ is an (r,s)-*shuffle* if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(r), \quad \sigma(r+1) < \sigma(r+2) < \cdots < \sigma(r+s),$$

$$w_{\sigma(i)} = \begin{cases} u_{\sigma(i)} & \text{if } 1 \leqslant \sigma(i) \leqslant r, \\ v_{\sigma(i)-r} & \text{if } r+1 \leqslant \sigma(i) \leqslant r+s, \end{cases}$$

and

$$\mathrm{Sgn}(\sigma) = \sum_{(i,j)} (\deg w_i + 1)(\deg w_{r+j} + 1))$$

where the summation is over pairs (i,j) for which $\sigma(i) > \sigma(r+j)$.

In the situation of this Theorem we have

Corollary 2.6. *The Eilenberg-Moore spectral sequence of* Theorem 2.1 *collapses at the* E_2-*term.*

The proof is similar to that of [15, section 2, example 4], and depends on two observations on this spectral sequence for $H^*(\Omega M)$ under the conditions of Theorem 2.1.

Lemma 2.7. *The edge homomorphism* $e\colon \mathrm{E}_2^{-1,*+1} \longrightarrow H^*(\Omega M)$ *can be identified with the composition*

$$H^{*+1}(M) \xrightarrow{\mathrm{ev}^*} H^{*+1}(\Sigma\Omega M) \xrightarrow{\cong} H^*(\Omega M)$$

using the canonical isomorphism $\mathrm{E}_2^{-1,*+1} \xrightarrow{\cong} H^{*+1}(M)$.

Corollary 2.8. *The edge homomorphism* $e\colon \mathrm{E}_2^{-1,*+1} \longrightarrow H^*(\Omega M)$ *is a monomorphism.*

Proof. This follows from Lemma 1.1 since $(\Sigma\theta \circ \delta)^*$ provides a left inverse for e. □

3. On the cohomology of sphere bundles

In this section we recall some results of Massey [5, part II]. We continue to use the notation and general set-up of Section 1.

We assume that our spherical fibration ξ is orientable in $H^*(-) = H^*(-;\Bbbk)$. Choosing an orientation class $u \in H^n(M)$, we also suppose that $u^2 = 0$. Then (1.1) induces an exact sequence

$$0 \to H^*(B) \longrightarrow H^*(S) \xrightarrow{\delta^*} \widetilde{H}^{*+1}(M) \to 0$$

in which δ^* is a an $H^*(B)$-module homomorphism with respect to the obvious module structure on $H^*(S)$ and the Thom module structure on $\widetilde{H}^*(M)$. Since the left hand map is a monomorphism we regard $H^*(B)$ as a subring of $H^*(S)$.

Now choose $v \in H^{n-1}(S)$ so that $\delta^*(v) = u$. Then by [5, (8.1)] there is a relation of the form

$$v^2 = s + tv, \tag{3.1}$$

where $s \in H^{2n-2}(B)$ and $t \in H^{n-1}(B)$. If we make a different choice $v' \in H^{n-1}(S)$ with $\delta^*(v') = u$, then $w = v' - v \in H^{n-1}(B)$ and we find that

$$(v')^2 = s' + t'v',$$

where

$$s' = s - wt - w^2,$$
$$t' = \begin{cases} t & \text{if } n \text{ is even,} \\ t + 2w & \text{if } n \text{ is odd.} \end{cases}$$

Massey also shows that when n is odd and $\Bbbk = \mathbb{F}_2$,

$$t = w_{n-1}(\xi). \tag{3.2}$$

Here we define the Stiefel-Whitney class through the Wu formula in $H^*(M)$,

$$w_{n-1}(\xi) \cdot u = \mathrm{Sq}^{n-1} u.$$

Of course this makes sense for any spherical fibration, not just those associated with vector bundles.

Here are two examples that we will discuss again later.

Example 3.1. Consider the universal Spin(2) and Spin(3) bundles $\zeta_2 \downarrow B\mathrm{Spin}(2)$ and $\zeta_3 \downarrow B\mathrm{Spin}(3)$ obtained from the canonical representations into SO(2) and SO(3). Of course the bases of these bundles can be taken to be

$$B\mathrm{Spin}(2) = \mathbb{C}P^\infty, \quad B\mathrm{Spin}(3) = \mathbb{H}P^\infty,$$

and $\zeta_2 = \eta^2$, the square of the universal complex line bundle $\eta \downarrow \mathbb{C}P^\infty$. Since there are Spin(3)-equivariant homeomorphisms

$$\mathrm{Spin}(3)/\mathrm{Spin}(2) \cong \mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2,$$

the sphere bundle of ζ_3

$$E\mathrm{Spin}(3)/\mathrm{Spin}(2) \xrightarrow{\doteq} E\mathrm{Spin}(3) \times_{\mathrm{Spin}(3)} \mathrm{Spin}(3)/\mathrm{Spin}(2) \longrightarrow E\mathrm{Spin}(3)/\mathrm{Spin}(3)$$

can be realised as the natural map $\mathbb{C}P^\infty \longrightarrow \mathbb{H}P^\infty$. In cohomology this induces a monomorphism

$$H^*(\mathbb{H}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[y] \longrightarrow H^*(\mathbb{C}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[x]; \quad y \mapsto x^2.$$

It is clear that in $H^*(-;\mathbb{F}_2)$, $w_2(\zeta_2) = 0 = w_2(\zeta_3)$ and also $w_3(\zeta_3) = 0$ since $H^3(\mathbb{H}P^\infty) = 0$.

So we can take $v = x$ and then (3.1) becomes

$$x^2 = y + 0x,$$

since $t = w_2(\zeta_3) = 0$. Similarly, if p is an odd prime, we have $t = 0$ and the analogous relations hold in $H^*(\mathbb{C}P^\infty;\mathbb{F}_p)$ and in $H^*(\mathbb{C}P^\infty;\mathbb{Q})$.

4. Results on cohomology over $\mathbb{F}_2$

Now we can give some general results for the case $\Bbbk = \mathbb{F}_2$. Here $H^*(-) = H^*(-;\mathbb{F}_2)$.

We recall Borel's theorem on the structure of Hopf algebras over perfect fields, see [6, theorem 7.11 and proposition 7.8].

Theorem 4.1. *Suppose that the orientation $u \in H^n(M)$ satisfies $u^2 = 0$, $H^*(B)$ has no nilpotents, and $\mathrm{Sq}^{n-1}u \neq 0$. Then $H^*(\Omega M)$ is a polynomial algebra.*

Proof. Let $0 \neq x \in H^k(B)$ and consider $[x \cdot u] \in \mathrm{E}_2^{-1,k+n}$. Then the Steenrod operation Sq^{n+k-1} satisfies

$$\begin{aligned}\mathrm{Sq}^{n+k-1}[x \cdot u] &= [\mathrm{Sq}^{n+k-1}(x \cdot u)] \\ &= [(\mathrm{Sq}^k x) \cdot \mathrm{Sq}^{n-1}u] \\ &= [x^2 \cdot \mathrm{Sq}^{n-1}u] \neq 0,\end{aligned}$$

since all other terms in the sum $\sum_i \mathrm{Sq}^i x \cdot \mathrm{Sq}^{n+k-1-i}u$ are easily seen to be trivial. It follows that the element of $H^*(\Omega M)$ represented in the spectral sequence by $[x \cdot u]$ has non-trivial square since this is represented by $\mathrm{Sq}^{n+k-1}[x \cdot u] = [x^2 \cdot \mathrm{Sq}^{n-1}u] \neq 0$.

More generally, using the description of the E_2-term in Theorem 2.4, we can similarly see that an element $[x_1 \cdot u|\cdots|x_\ell \cdot u]$ with $x_i \in H^{k_i}(B)$ has

$$\mathrm{Sq}^{k_1+\cdots+k_\ell+n\ell-\ell}[x_1 \cdot u|\cdots|x_\ell \cdot u] = [x_1^2 \cdot \mathrm{Sq}^{n-1}u|\cdots|x_\ell^2 \cdot \mathrm{Sq}^{n-1}u] \neq 0.$$

Thus the algebra generators of $H^*(\Omega M)$ are not nilpotent, so by Borel's theorem we see that $H^*(\Omega M)$ is a polynomial algebra. □

Theorem 4.2. *Suppose that the orientation $u \in H^n(M) = H^n(M;\mathbb{F}_2)$ satisfies $u^2 = 0$ and $\mathrm{Sq}^{n-1}u = 0$. Then $H^*(\Omega M)$ is an exterior algebra.*

Proof. First consider an element of $w \in H^{n+k-1}(\Omega M)$ in filtration 1. We can assume that this is represented in the Eilenberg-Moore spectral sequence by $[x \cdot u]$ for some $x \in H^k(B)$. Then $w^2 = \mathrm{Sq}^{n+k-1}w$ is represented by

$$\mathrm{Sq}^{n+k-1}[x \cdot u] = [(\mathrm{Sq}^k x) \cdot \mathrm{Sq}^{n-1}u] = 0,$$

and is also in filtration 1. Since in positive degrees, filtration 0 is trivial, we have $w^2 = 0$.

Now we proceed by induction on the filtration r. Suppose that for every positive degree element $z \in H^*(\Omega M)$ of filtration $r \geqslant 1$, we have $z^2 = 0$. Suppose that $w \in H^*(\Omega M)$ has filtration $r+1$. We can assume that w is represented by $[x_1 \cdot u|\cdots|x_{r+1} \cdot u]$ where $x_j \in H^{k_j}(B)$.

Applying the Steenrod operation $\mathrm{Sq}^{k_1+\cdots+k_{r+1}+(r+1)n-1}$ we see that w^2 is also in filtration $r+1$ and is represented by

$$\mathrm{Sq}^{k_1+\cdots+k_{r+1}+(r+1)(n-1)}[x_1\cdot u|\cdots|x_{r+1}\cdot u] = [(\mathrm{Sq}^{k_1}x_1)\cdot \mathrm{Sq}^{n-1}u|\cdots|(\mathrm{Sq}^{k_{r+1}}x_{r+1})\cdot \mathrm{Sq}^{n-1}u] = 0.$$

On the other hand, the coproduct on w is

$$\psi(w) = w\otimes 1 + 1\otimes w + \sum_i w'_i \otimes w''_i$$

where the w'_i, w''_i all have filtration in the range 1 to r. On squaring and using the inductive assumption we find that

$$\psi(w^2) = w^2\otimes 1 + 1\otimes w^2,$$

so w^2 is primitive and decomposable. By [6, proposition 4.21], the kernel of the natural homomorphism $\mathrm{P}H^*(\Omega M) \longrightarrow \mathrm{Q}H^*(\Omega M)$ consists of squares of primitives. Since the primitives must all have filtration 1, all such squares are trivial, hence $w^2 = 0$. This shows that all elements of filtration $r+1$ square to zero, giving the inductive step.

Borel's theorem now implies that $H^*(\Omega M)$ is an exterior algebra. □

5. Results on cohomology over $\mathbb{F}_p$ with p odd

In this we give analogous results for the case $\mathbb{k} = \mathbb{F}_p$ where p is an odd prime. Here $H^*(-) = H^*(-;\mathbb{F}_p)$. We assume that n is odd, say $n = 2m+1$, and that M has an orientation class $u \in H^{2m+1}(M)$. For degree reasons, $u^2 = 0$.

Theorem 5.1. *Suppose that $H^*(B)$ has no nilpotents, and $\mathcal{P}^m u \neq 0$. Then $H^*(\Omega M)$ is a polynomial algebra.*

Of course $\mathcal{P}^m u$ defines a Wu class $W_m(\xi)$ by the formula

$$W_m(\xi)\cdot u = \mathcal{P}^m u,$$

and the condition $\mathcal{P}^m u \neq 0$ amounts to its non-vanishing. The no nilpotents condition implies that $H^*(B)$ is concentrated in even degrees.

Proof. Let $0 \neq x \in H^{2k}(B)$ and consider $[x\cdot u] \in \mathrm{E}_2^{-1,2k+2m+1}$. Then the Steenrod operation $\mathcal{P}^{m+k}$ satisfies

$$\begin{aligned}\mathcal{P}^{m+k}[x\cdot u] &= [\mathcal{P}^{m+k}(x\cdot u)]\\ &= (\mathcal{P}^k x)\cdot \mathcal{P}^m u\\ &= x^p\cdot \mathcal{P}^m u \neq 0,\end{aligned}$$

since all other terms in the sum $\sum_i \mathcal{P}^i x\cdot \mathcal{P}^{m+k-i}u$ are easily seen to be trivial. It follows that the element of $H^*(\Omega M)$ represented in the spectral sequence by $[x\cdot u]$ has non-trivial p-th power since it is represented by

$$\mathcal{P}^{m+k}[x\cdot u] = [x^p\cdot \mathcal{P}^m u] \neq 0.$$

Similarly every element represented by $[x_1\cdot u|\cdots|x_\ell\cdot u]$ with $x_i \in H^{2k_i}(B)$ has non-zero p-th power since

$$\mathcal{P}^{k_1+\cdots+k_\ell+m\ell}[x_1\cdot u|\cdots|x_\ell\cdot u] \neq 0.$$

Thus the algebra generators of $H^*(\Omega M)$ are not nilpotent, so by Borel's theorem we see that $H^*(\Omega M)$ is a polynomial algebra. □

We will say that a connective commutative graded $\mathbb{F}_p$-algebra is *p-truncated* if every positive degree element x satisfies $x^p = 0$. When $p = 2$, being 2-truncated is equivalent to being exterior.

Theorem 5.2. *Suppose that $\mathcal{P}^m u = 0$. Then $H^*(\Omega M)$ is a p-truncated algebra.*

Proof. First consider an element of $w \in H^{2m+2k}(\Omega M)$ in filtration 1. We can assume this is represented in the Eilenberg-Moore spectral sequence by $[x \cdot u] \in \mathrm{E}_2^{-1,2m+2k+1}$ for some $x \in H^{2k}(B)$. Then $w^p = \mathcal{P}^{m+k} w$ is represented by

$$\mathcal{P}^{m+k}[x \cdot u] = [(\mathcal{P}^k x) \cdot \mathcal{P}^m u] = 0,$$

and is also in filtration 1. Since filtration 0 is trivial in positive degrees, we have $w^p = 0$.

Now as in the proof of Theorem 4.2, we prove by induction on the filtration r that for every positive degree element $z \in H^*(\Omega M)$ of filtration $r \geqslant 1$ has $z^p = 0$. Borel's theorem now implies that every element of $H^*(\Omega M)$ has trivial p-th power. □

6. Rational results

In this section we take $\Bbbk = \mathbb{Q}$. By Borel's Theorem [6, theorem 7.11 and proposition 7.8], we have

Theorem 6.1. *There is an isomorphism of algebras*

$$H^*(\Omega M; \mathbb{Q}) \cong \bigotimes_i \mathbb{Q}[x_i] \otimes \bigotimes_j \mathbb{Q}[y_i]/(y_j^2),$$

where $\deg x_i$ *is even and* $\deg y_i$ *is odd. In particular, if $H^*(M; \mathbb{Q})$ is concentrated in odd degrees then $H^*(\Omega M; \mathbb{Q})$ is a polynomial algebra on even degree generators.*

7. Local to global results

Before giving some examples, we record a variant of the local-global result [1, proposition 2.4]. We follow the convention that a prime p can be 0 or positive, and set $\mathbb{F}_0 = \mathbb{Q}$.

Let $S \subseteq \mathbb{N}$ be the multiplicatively closed set generated by a set of non-zero primes (if this set is empty then $S = \{1\}$). Then

$$\mathbb{Z}[S^{-1}] = \{a/b : a \in \mathbb{Z},\ b \in S\}.$$

In the following, whenever $p \notin S$, $\mathbb{F}_p = \mathbb{Z}[S^{-1}]/(p)$.

Proposition 7.1. *Let H^* be a graded commutative connective $\mathbb{Z}[S^{-1}]$-algebra which is concentrated in even degrees and with each H^{2n} a finitely generated free $\mathbb{Z}[S^{-1}]$-module. Suppose that for each prime $p \notin S$, $H(p)^* = H^* \otimes \mathbb{F}_p$ is a polynomial algebra, then H^* is a polynomial algebra and for every prime p,*

$$\mathrm{rank}_{\mathbb{Z}[S^{-1}]} \mathrm{Q}H^{2n} = \dim_{\mathbb{F}_p} \mathrm{Q}H(p)^{2n}.$$

Proof. The proof of [1, proposition 2.4] can be modified by systematically replacing $\mathbb{Z}$ with the principal ideal domain $\mathbb{Z}[S^{-1}]$ and working only with primes not contained in S (including 0). □

8. Some examples

Our first example is a recasting of the main result of [1].

Example 8.1. Consider the universal line bundle $\eta \downarrow \mathbb{C}P^\infty$, viewed as a real 2-plane bundle. Then the 3-dimensional bundle $\xi = \eta \oplus \mathbb{R}$ has Thom space $M\xi = \Sigma M\mathrm{U}(1) \sim \mathbb{C}P^\infty$. It is straightforward to verify that the conditions of Theorems 4.1 and 5.1 apply. Thus $H^*(\Omega\Sigma\mathbb{C}P^\infty; \mathbb{Z})$ is polynomial.

Example 8.2. Recall Example 3.1.

Here $w_2(\zeta_3) = 0 = w_2(\zeta_2)$, so $H^*(\Omega M\mathrm{Spin}(3); \mathbb{F}_2)$ and $H^*(\Omega\Sigma M\mathrm{Spin}(2); \mathbb{F}_2)$ are exterior algebras.

For an odd prime p, the natural map $\Sigma M\mathrm{Spin}(2) \longrightarrow M\mathrm{Spin}(3)$ induces a monomorphism in $H^*(-; \mathbb{F}_p)$ and in $H^*(M\mathrm{Spin}(2); \mathbb{F}_p) = H^*(\mathbb{C}P^\infty; \mathbb{F}_p)$ we see that for the generator $x \in H^2(\mathbb{C}P^\infty; \mathbb{F}_p)$. $\mathcal{P}^1 x = x^p \neq 0$. Therefore $H^*(\Omega M\mathrm{Spin}(3); \mathbb{F}_p)$ and $H^*(\Omega\Sigma M\mathrm{Spin}(2); \mathbb{F}_p)$ are polynomial algebras.

Combining these results we see that $H^*(\Omega M\mathrm{Spin}(3); \mathbb{Z}[1/2])$ and $H^*(\Omega\Sigma M\mathrm{Spin}(2); \mathbb{Z}[1/2])$ are polynomial algebras.

9. Homology generators and a stable splitting

The map $\theta \colon S_+ \longrightarrow \Omega M$ introduced in Section 1 allows us to define a *canonical* choice of generator $v \in H^{n-1}(S)$ in the sense of Massey's paper [5], namely

$$v = (\mathrm{ev} \circ \Sigma\theta)^* u.$$

This follows from Lemma 1.1. When $n = 2m + 1$ is odd, in mod p cohomology $H^*(-) = H^*(-; \mathbb{F}_p)$, from (3.1) we obtain

$$v^2 = s + tv,$$

where

$$t = \begin{cases} w_{2m}(\xi) & \text{if } p = 2, \\ W_m(\xi) & \text{if } p \text{ is odd.} \end{cases}$$

and we define these invariants by

$$w_{2m}(\xi) \cdot u = \mathrm{Sq}^{2m} u,$$
$$W_m(\xi) \cdot u = \mathcal{P}^m u.$$

Notice that the multiplicativity given by Lemma 1.6 implies that for $x \in H^*(B)$,

$$(\mathrm{ev} \circ \Sigma\theta)^*(x \cdot u) = xv.$$

Now let $b_i \in H^*(B)$ form an $\mathbb{F}_p$-basis for $H^*(B)$, where we suppose that $b_0 = 1$. Then the elements $b_i v, b_i \in H^*(S)$ form a basis for $H^*(S)$, and the $b_i \cdot u$ form a basis for $\widetilde{H}^*(M)$. Since

$$\delta^*(b_i v) = b_i \cdot u, \quad \delta^*(b_i) = 0,$$

for the dual bases $(b_i \cdot v)^\circ, (b_i)^\circ$ of $H^*(S)$ and $(b_i \cdot u)^\circ$ of $\widetilde{H}^*(M)$ we have

$$\delta_*((b_i \cdot u)^\circ) = (b_i v)^\circ.$$

Furthermore, $(\Sigma\theta \circ \delta)_*((b_i \cdot u)^\circ)$ is dual to the class represented in the Eilenberg-Moore spectral sequence by the primitive $[b_i \cdot u]$, hence the $(\Sigma\theta \circ \delta)_*((b_i \cdot u)^\circ)$ form a basis for the indecomposables

$\mathrm{Q}H_*(\Omega M)$. Using the bar resolution description of the Eilenberg-Moore spectral sequence and the dual cobar resolution for the homology spectral sequence

$$\mathrm{E}^2_{*,*} = \mathrm{Cotor}^{H_*(M)}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow H_*(\Omega M)$$

we obtain

Proposition 9.1. *The homology algebra $H_*(\Omega M; \mathbb{F}_p)$ is the free non-commutative algebra on the elements $(\Sigma\theta \circ \delta)_*((b_i \cdot u)^\circ)$.*

Now we can give an analogue of the James splitting. We need the free S-algebra functor $\mathbb{T}$ of [4, section II.4]. This is defined for an S-module X by

$$\mathbb{T}X = \bigvee_{k\geqslant 0} X^{(k)},$$

where $(-)^{(k)}$ denotes the k-th smash power. The map $\Sigma\theta \circ \delta$ gives rise to a map of spectra

$$\Theta \colon \Sigma^{-1}\Sigma^\infty M \longrightarrow \Sigma^\infty \Omega M$$

and by the freeness property of $\mathbb{T}$, there is an induced morphism of S-algebras

$$\widetilde{\Theta} \colon \mathbb{T}(\Sigma^{-1}\Sigma^\infty M) \longrightarrow \Sigma^\infty(\Omega M)_+,$$

where $\Sigma^\infty(\Omega M)_+$ becomes an S-algebra using the natural A_∞ structure on ΩM.

Theorem 9.2. *Suppose that p is a prime for which* Proposition 9.1 *is true. Then $\widetilde{\Theta}$ is an $H\mathbb{F}_p$-equivalence of S-algebras.*

Proof. Under the map $\widetilde{\Theta}_*$, an exterior product of classes in $H_*(\Sigma^{-k}\Sigma^\infty M^{(k)}; \mathbb{F}_p)$ goes to their internal product in $H_*(\Omega M; \mathbb{F}_p)$. Now Proposition 9.1 shows that $\widetilde{\Theta}$ is an $\mathbb{F}_p$-equivalence for such a prime p. □

Combining our results and using an arithmetic square argument we obtain

Theorem 9.3. *Let $S \subseteq \mathbb{N}$ be the multiplicatively closed set generated by all the primes p for which* Proposition 9.1 *is false. Then $\widetilde{\Theta}$ is an $H\mathbb{Z}[S^{-1}]$-equivalence of S-algebras. Hence there is an $H\mathbb{Z}[S^{-1}]$-equivalence*

$$\bigvee_{k\geqslant 1} \Sigma^{-k}\Sigma^\infty M^{(k)} \longrightarrow \Sigma^\infty \Omega M.$$

Of course, this stable splitting is very different from the James splitting for a connected based space X,

$$\Sigma\Omega\Sigma X \sim \bigvee_{k\geqslant 1} \Sigma X^{(k)}.$$

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School of Mathematics & Statistics, University of Glasgow, Glasgow G12 8QW, Scotland.
E-mail address: `a.baker@maths.gla.ac.uk`
URL: `http://www.maths.gla.ac.uk/∼ajb`