

The Harnack estimate for the Yamabe flow on CR manifolds of dimension 3

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Abstract

We deform the contact form by the amount of the Tanaka-Webster curvature on a closed spherical CR three-manifold. We show that if a contact form evolves with free torsion and positive Tanaka-Webster curvature as initial data, then a certain Harnack inequality for the Tanaka-Webster curvature holds.

Key Words: contact form, Tanaka-Webster curvature, spherical CR manifold, Harnack inequality.

1 Introduction

Let M denote a closed (i.e., compact without boundary) CR manifold. The Yamabe problem is to find a contact form on M with constant Tanaka-Webster curvature. In serial papers, Jerison and Lee initiated the study of this problem. ([JL1],[JL2],[JL3]) In this paper, we study an evolution equation of contact form so that the solution is expected to converge to a solution of the Yamabe problem.

Let $\theta_{(t)}$ denote a family of contact forms on M . We can associate to it the so called Tanaka-Webster curvature ([Ta],[We];see also section 2), denoted $W_{(t)}$. Our evolution equation, the so called (unnormalized) Yamabe flow, reads as follows:

$$\partial_t \theta_{(t)} = -2W_{(t)}\theta_{(t)}. \quad (1.1)$$

Write $\theta_{(t)} = e^{2\lambda_{(t)}}\hat{\theta}$ with respect to a fixed contact form $\hat{\theta}$. Then we can express the equation (1.1) in $\lambda_{(t)}$:

$$\partial_t \lambda_{(t)} = -W_{(t)}. \quad (1.2)$$

Since the linearization of $-W$ with respect to λ is a second-order subelliptic operator, the short time solution and the uniqueness of (1.2) follows from a standard argument. (we will discuss this and the long time solution elsewhere) In this paper we will do the Harnack

estimate for W . The first step is to obtain a geometric quantity, usually called the Harnack quantity. The Harnack quantity is a candidate quantity for us to do the estimate. Let $\nabla_b, \Delta_b, \langle, \rangle_{J,\theta}$ denote the subgradient, sublaplacian, and the Levi form, respectively. (see section 2 for the definitions) Following the idea of Hamilton in [H1], we can "derive" the following Harnack quantity:

$$Z(\theta, \eta) \equiv 2\Delta_b W + W^2 + \frac{W}{t} + \langle \nabla_b W, \eta \rangle_{J,\theta} + \frac{1}{8}W|\eta|_{J,\theta}^2 \quad (1.3)$$

in which η is a Legendrian vector field. (see section 3 for the definition and more details) In section 4, we prove the following theorem:

Theorem A: Let (M, ξ, J) be a closed spherical CR 3-manifold. Suppose there is a contact form $\hat{\theta}$ (together with J defining a positive pseudohermitian structure) with vanishing torsion and positive Tanaka-Webster curvature. Then under the Yamabe flow (1.1),

$$Z(\theta, \eta) \geq 0 \quad (1.4)$$

for any Legendrian vector field η .

Integrating (1.4) from time t_1 to time t_2 , we obtain the following Harnack inequality. (see section 4 for more details)

Theorem B: Suppose we have the same assumptions as in Theorem A. Then, under the Yamabe flow (1.1), we have, for all points x_1, x_2 in M and times $t_1 < t_2$,

$$\frac{W(x_2, t_2)}{W(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-2} \exp\left(-\frac{1}{16}L\right) \quad (1.5)$$

where

$$L = \inf_{\gamma} \int_{t_1}^{t_2} |\dot{\gamma}|_{J,\theta(t)}^2 dt$$

and the infimum is taken over all Legendrian paths γ with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$.

In section 5, we show by examples that there are contact forms on the standard CR 3-sphere with vanishing torsion and nonconstant positive Tanaka-Webster curvature. We remark that the study here was motivated by the beautiful work of Richard Hamilton [H2] on the Ricci flow for surfaces and the work of Ben Chow about the Yamabe flow on locally conformally flat manifolds. ([Ch]) In [H2], Hamilton proved, among other results, that on the 2-sphere, if the initial metric has positive curvature, then the solution metric of the normalized Ricci flow converges to the limiting metric of constant curvature. One of the important ingredients in Hamilton's proof is the Harnack inequality for the evolved curvatures.

2 Basics derived from the flow

Let us first review some basic material in CR geometry. (e.g.,[We],[L1]) Let M be a closed 3-manifold with an oriented contact structure ξ . There always exists a global contact form θ , obtained by patching together local ones with a partition of unity. The characteristic vector field of θ is the unique vector field T such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$. A CR -structure compatible with ξ is a smooth endomorphism $J : \xi \rightarrow \xi$ such that $J^2 = -identity$. A pseudohermitian structure compatible with ξ is a CR -structure J compatible with ξ together with a global contact form θ .

Given a pseudohermitian structure (J, θ) , we can choose a complex vector field Z_1 , an eigenvector of J with eigenvalue i , and a complex 1-form θ^1 such that $\{\theta, \theta^1, \bar{\theta}^1\}$ is dual to $\{T, Z_1, \bar{Z}_1\}$. ($\theta^1 = (\bar{\theta}^1), Z_1 = (\bar{Z}_1)$) It follows that $d\theta = ih_{1\bar{1}}\theta^1 \wedge \bar{\theta}^1$ for some nonzero real function $h_{1\bar{1}}$. If $h_{1\bar{1}}$ is positive, we call such a pseudohermitian structure (J, θ) positive, and we can choose a Z_1 (hence θ^1) such that $h_{1\bar{1}} = 1$. That is to say

$$d\theta = i\theta^1 \wedge \bar{\theta}^1. \quad (2.1)$$

We'll always assume our pseudohermitian structure (J, θ) is positive and $h_{1\bar{1}} = 1$ throughout the paper. The pseudohermitian connection of (J, θ) is the connection $\nabla^{\psi.h.}$ on $TM \otimes \mathbb{C}$ (and extended to tensors) given by

$$\nabla^{\psi.h.}Z_1 = \omega_1^1 \otimes Z_1, \nabla^{\psi.h.}\bar{Z}_1 = \omega_{\bar{1}}^{\bar{1}} \otimes \bar{Z}_1, \nabla^{\psi.h.}T = 0$$

in which the 1-form ω_1^1 is uniquely determined by the following equation with a normalization condition:

$$\begin{aligned} d\theta^1 &= \theta^1 \wedge \omega_1^1 + A_{\bar{1}}^1 \theta \wedge \bar{\theta}^1 \\ \omega_1^1 + \omega_{\bar{1}}^{\bar{1}} &= 0. \end{aligned} \quad (2.2)$$

The coefficient $A_{\bar{1}}^1$ in (2.2) is called the (pseudohermitian) torsion. Since $h_{1\bar{1}} = 1$, $A_{\bar{1}\bar{1}} = h_{1\bar{1}}A_{\bar{1}}^1 = A_{\bar{1}}^1$. And A_{11} is just the complex conjugate of $A_{\bar{1}\bar{1}}$. Differentiating ω_1^1 gives

$$d\omega_1^1 = W\theta^1 \wedge \bar{\theta}^1 + 2iIm(A_{11, \bar{1}}\theta^1 \wedge \theta) \quad (2.3)$$

where W is the Tanaka-Webster curvature. ([We],[Ta])

We can define the covariant differentiations with respect to the pseudohermitian connection. For instance, $f_{,1} = Z_1 f$, $f_{,1\bar{1}} = Z_{\bar{1}} Z_1 f - \omega_1^1(Z_{\bar{1}})Z_1 f$ for a (smooth) function f . (see, e.g., section 4 in [L1]) We define the subgradient operator ∇_b and the sublaplacian operator Δ_b by

$$\begin{aligned}\nabla_b f &= f_{,\bar{1}}Z_1 + f_{,1}Z_{\bar{1}}, \\ \Delta_b f &= f_{,1\bar{1}} + f_{,\bar{1}1},\end{aligned}$$

respectively. (notice the sign difference for Δ_b in [L1]) We also define the Levi form $\langle, \rangle_{J,\theta}$ by

$$\langle V, U \rangle_{J,\theta} = 2d\theta(V, JU) = v_1 u_{\bar{1}} + v_{\bar{1}} u_1$$

for $V = v_1 Z_{\bar{1}} + v_{\bar{1}} Z_1, U = u_1 Z_{\bar{1}} + u_{\bar{1}} Z_1$ in ξ . (note that the second equality follows from (2.1) and our definition is different from the one in [L1] by a factor 2) The associated norm is defined as usual: $|V|_{J,\theta}^2 = \langle V, V \rangle_{J,\theta}$. Let $\hat{\theta}, \hat{\theta}^1, \hat{\theta}^{\bar{1}}$ satisfy (2.1). Now consider the change of contact form: $\theta = e^{2\lambda} \hat{\theta}$. Choose $\theta^1 = e^\lambda (\hat{\theta}^1 + 2i\lambda_{,\bar{1}} \hat{\theta})$ such that $h_{1\bar{1}} = \hat{h}_{1\bar{1}} (=1$ by assumption). One checks easily that $\theta, \theta^1, \theta^{\bar{1}}$ satisfies (2.1). Then the associated connection form ω_1^1 , torsion A_{11} , and Tanaka-Webster curvature W transform as follows: (cf. section 5 in [L1])

$$\omega_1^1 = (\hat{\omega}_1^1) + 3(\lambda_{,1} \hat{\theta}^1 - \lambda_{,\bar{1}} \hat{\theta}^{\bar{1}}) + i(\hat{\Delta}_b \lambda + 4|\hat{\nabla}_b \lambda|_{J,\hat{\theta}}^2) \hat{\theta} \quad (2.4)$$

$$A_{11} = e^{-2\lambda} (\hat{A}_{11} + 2i\lambda_{,11} - 4i(\lambda_{,1})^2) \quad (2.5)$$

$$W = e^{-2\lambda} (-4\hat{\Delta}_b \lambda - 4|\hat{\nabla}_b \lambda|_{J,\hat{\theta}}^2 + \hat{W}). \quad (2.6)$$

Here the operators or quantities with "hat" are with respect to the coframe $(\hat{\theta}, \hat{\theta}^1, \hat{\theta}^{\bar{1}})$, and so are the covariant derivatives of λ . Now consider a family of contact forms $\theta_{(t)} = e^{2\lambda(t)} \hat{\theta}$, a solution to the Yamabe flow (1.1) or (1.2).

Lemma 2.1. Under the Yamabe flow (1.1) ($W = W_{(t)}$ for short), we have

$$\dot{W} = 4\Delta_b W + 2W^2 \quad (2.7)$$

$$(\dot{A}_{11}) = 2W A_{11} - 2iW_{,11} \quad (2.8)$$

in which Δ_b , the torsion, and covariant derivatives are with respect to $\theta_{(t)}$ and induced coframes as shown previously.

Proof: We will omit the t -dependence for simplicity of notation if no confusion occurs. First note that $Z_1 = e^{-\lambda} \hat{Z}_1$ and $\nabla_b f = e^{-2\lambda} \hat{\nabla}_b f$. It follows that

$$\Delta_b f = e^{-2\lambda} (\hat{\Delta}_b f + 2 \langle \hat{\nabla}_b \lambda, \hat{\nabla}_b f \rangle_{J,\hat{\theta}}). \quad (2.9)$$

Differentiating (2.6) with respect to t , we obtain (2.7) by making use of (1.2) and (2.9). From (2.4), it is easy to see that

$$(\hat{\omega}_1^1)(\hat{Z}_1) = e^\lambda(\omega_1^1(Z_1) - 3\lambda_{,1}). \quad (2.10)$$

Substituting (2.10) in the expression of $W_{,\hat{1}\hat{1}}$, we obtain

$$W_{,\hat{1}\hat{1}} = e^{2\lambda}(W_{,11} + 4\lambda_{,1}W_{,1}). \quad (2.11)$$

Now differentiating (2.5) with respect to t and making use of (1.2),(2.11), we finally reach (2.8).

Q.E.D.

Now applying the maximum principle to (2.7), we obtain

Corollary 2.2. Suppose $(M, J, \theta_{(0)})$ is closed with $W \geq c > 0$. Then the inequality $W \geq c > 0$ is preserved under the Yamabe flow (1.1).

The following formula will be used to compute the evolution of $\Delta_b W$.

Lemma 2.3. Under the Yamabe flow (1.1), we have

$$\partial_t(\Delta_b f) = \Delta_b(\dot{f}) + 2W\Delta_b f - 2 \langle \nabla_b W, \nabla_b f \rangle_{J,\theta} \quad (2.12)$$

for a (smooth) real-valued function $f = f(x, t)$ defined on $M \times R$. (note that we have suppressed the t -dependence in the above expression)

Proof: Differentiating $Z_1 = e^{-\lambda}\hat{Z}_1$ and (2.4) with respect to t gives

$$\dot{Z}_1 = WZ_1 \quad (2.13)$$

$$(\dot{\omega}_1^1) = -3W_{,1}\theta^1 + 3W_{,\bar{1}}\theta^{\bar{1}} \pmod{\theta} \quad (2.14)$$

by (1.2). Now our formula (2.12) follows from (2.13), (2.14) by a straightforward computation.

Q.E.D.

3 The Harnack quantity

In this section, we apply Hamilton's general method for obtaining a potential quantity for the Harnack estimate. First we need to know what the soliton equation is supposed to be for our flow (1.1). Let ϕ_t be a family of CR automorphisms. Suppose $\phi_t^*\theta_{(t)}$ converges to a fixed contact form θ and differentiating $\phi_t^*\theta_{(t)}$ with respect to t converges to 0. Then θ satisfies the following equation:

$$\mathcal{L}_{X_f}\theta - 2W\theta = 0 \tag{3.1}$$

in which X_f is a CR vector field parametrized by a real-valued function f . We can write (e.g.,[CL])

$$X_f = -fT + if_{,1}Z_{\bar{1}} - if_{,\bar{1}}Z_1. \tag{3.2}$$

We call (3.1) the soliton equation of the flow (1.1). (A solution θ is called a soliton) Substituting (2.1), (3.2) in the formula for the Lie derivative, we can reduce (3.1) to $f_{,0} = -2W$. (recall that $f_{,0} = Tf$ where T is the characteristic vector field of θ) Consider the equation for an expanding soliton:

$$f_{,0} = -W - \frac{1}{t}. \tag{3.3}$$

Substituting the commutation relation $if_{,0} = f_{,1\bar{1}} - f_{,\bar{1}1}$ in (3.3) and differentiating it in $Z_{\bar{1}}$ and Z_1 directions, we get

$$f_{,1\bar{1}\bar{1}} - f_{,\bar{1}1\bar{1}} = -iW_{,1\bar{1}}. \tag{3.4}$$

On the other hand, X_f being a CR vector field means $\mathcal{L}_{X_f}J = 0$, which is equivalent to (e.g.,[CL])

$$f_{,11} + iA_{11}f = 0. \tag{3.5}$$

Differentiating (3.5) in the $Z_{\bar{1}}$ direction and exchanging 1 and $\bar{1}$ using the commutation relation ([L2]), we obtain

$$f_{,1\bar{1}1} + if_{,10} + f_{,1}W + iA_{11,\bar{1}}f + iA_{11}f_{,\bar{1}} = 0. \tag{3.6}$$

Differentiating (3.3) in the Z_1 direction and switching 1 and 0 give $f_{,10} + A_{11}f_{,\bar{1}} = -W_{,1}$. Substituting this in (3.6), we obtain

$$f_{,1\bar{1}\bar{1}} - iW_{,1} + f_{,1}W + iA_{11,\bar{1}}f = 0. \quad (3.7)$$

Also differentiating (3.3) twice in the $Z_{\bar{1}}$ and Z_1 directions and exchanging $\bar{1}1$ and 0 using the commutation relations, we obtain

$$f_{,\bar{1}10} = -W_{,\bar{1}1} - A_{11}f_{,\bar{1}\bar{1}} - A_{11,\bar{1}}f_{,\bar{1}} - A_{\bar{1}\bar{1}}f_{,11} - A_{\bar{1}\bar{1},1}f_{,1}. \quad (3.8)$$

Differentiating the complex conjugate of (3.7) in the Z_1 direction and substituting the result and (3.8) in the commutation relation: $f_{,\bar{1}11\bar{1}} = f_{,\bar{1}\bar{1}11} + if_{,\bar{1}10}$, we obtain an expression for $f_{,\bar{1}11\bar{1}}$. Substituting this for the second term and the result of differentiating (3.7) in the $Z_{\bar{1}}$ direction for the first term in (3.4), we can reduce (3.4) to

$$\begin{aligned} 2i(W_{,\bar{1}\bar{1}} + W_{,\bar{1}1}) - (f_{,\bar{1}\bar{1}} - f_{,\bar{1}1})W + f_{,\bar{1}}W_{,1} - f_{,1}W_{,\bar{1}} \\ + iA_{11}f_{,\bar{1}\bar{1}} + iA_{\bar{1}\bar{1}}f_{,11} - i(A_{11,\bar{1}\bar{1}} + A_{\bar{1}\bar{1},11})f = 0. \end{aligned} \quad (3.9)$$

Substituting $f_{,\bar{1}\bar{1}} - f_{,\bar{1}1} = if_{,0} = -iW - it^{-1}$ (by (3.3)) in (3.9), using (3.5) to replace $f_{,11}$ ($f_{,\bar{1}\bar{1}}$, resp.) by $-iA_{11}f$ ($iA_{\bar{1}\bar{1}}f$, resp.), and noticing the Bianchi identity: $A_{11,\bar{1}\bar{1}} + A_{\bar{1}\bar{1},11} = W_{,0}$ and the definition of the sublaplacian operator Δ_b , we finally obtain

$$2\Delta_b W + \langle \nabla W, X_f \rangle + W^2 + \frac{W}{t} = 0 \quad (3.10)$$

in which $\nabla W = \nabla_b W + W_{,0}T$ and $\langle \nabla W, X_f \rangle = -W_{,0}f - \langle \nabla_b W, J(\nabla_b f) \rangle_{J,\theta}$.

In the Riemannian case ([Ch]), we can add a certain quadratic term in the involved vector field to get the Harnack quantity. However, in our case, adding a quadratic term like $(constant)W|X_f|^2$ does not seem to work without extra estimates on the torsion. So, as a first try, we assume the torsion vanishes at the initial time. It turns out that the torsion vanishes for all time if our CR structure J is spherical.

Lemma 3.1. Suppose J is spherical and $A_{11} = 0$ for an initial $\theta_{(0)}$. Then, under the Yamabe flow (1.1), A_{11} vanishes for all $\theta_{(t)}$.

Proof: First, recall that the Cartan curvature tensor Q_{11} is related to the Tanaka-Webster curvature W and torsion A_{11} in the following formula: (Lemma 2.2 in [CL])

$$Q_{11} = \frac{1}{6}W_{,11} + \frac{i}{2}WA_{11} - A_{11,0} - \frac{2i}{3}A_{11,\bar{1}\bar{1}}. \quad (3.11)$$

The fundamental theorem of 3-dimensional CR geometry due to Elie Cartan ([Ca]) asserts that J being spherical is equivalent to $Q_{11} = 0$. Now look at the evolution equation (2.8) of A_{11} . Each term in the right side of (2.8) contains the torsion or one of its derivatives in view of (3.11). So obviously $A_{11} = 0$ for all t is a solution to (2.8). Therefore it suffices to show the uniqueness of solutions to (2.8). However, using the commutation relation, we can write the right side of (2.8) as

$$4(A_{11,1\bar{1}} + A_{11,\bar{1}1} - 4iA_{11,0}) - 12WA_{11}.$$

The highest "weight" term is just -4 times the generalized Folland-Stein operator \mathcal{L}_α (defined in [CL]) acting on A_{11} with $\alpha = 4$. Since $\alpha = 4$ is not an odd integer, $-\mathcal{L}_4$ is subelliptic. So the uniqueness follows from the standard theory for subparabolic equations.

Q.E.D.

When the torsion A_{11} vanishes identically, so does W_0 due to the Bianchi identity: ([L2])

$$A_{11,1\bar{1}} + A_{1\bar{1},11} = W_{,0}. \quad (3.12)$$

Thus we can reduce $\langle \nabla W, X_f \rangle$ in (3.10) to $-\langle \nabla_b W, J(\nabla_b f) \rangle_{J,\theta}$. Note that the vector field $\eta = -J(\nabla_b f)$ belongs to ξ , the contact bundle, at each point. We call such a vector field a Legendrian vector field. Releasing the f -dependence, we therefore consider (1.3) for arbitrary Legendrian vector field η as our "Harnack" quantity. (the coefficient $\frac{1}{8}$ of the last term in (1.3) is the minimal value for (1.4) to hold as we'll see in the proof of Theorem A)

4 The Harnack inequality: Proof of Theorems

To apply the maximum principle to $Z(\theta, \eta)$, we compute the evolution equation for $Z(\theta, \eta)$. For convenience, we define $\square = \partial_t - 4\Delta_b$ and $\square\eta = (\square\eta_1)Z_{\bar{1}} + (\square\eta_{\bar{1}})Z_1$ for a Legendrian vector field $\eta = \eta_1 Z_{\bar{1}} + \eta_{\bar{1}} Z_1$, in which $\square\eta_1 = \partial_t \eta_1 - 4(\eta_{1,1\bar{1}} + \eta_{1,\bar{1}1})$. Also we define the modulus of "Legendrian 2-tensor" $\nabla_b \eta$ as follows: $|\nabla_b \eta|_{J,\theta}^2 = 2(\eta_{1,\bar{1}}\eta_{\bar{1},1} + \eta_{1,1}\eta_{\bar{1},\bar{1}})$. (recall that $h_{1\bar{1}} = 1$ and we express all tensors using subindices)

Lemma 4.1. Under the Yamabe flow (1.1), we have the following evolution equations:

$$\square(2\Delta_b W + W^2) = 12W\Delta_b W - 4|\nabla_b W|_{J,\theta}^2 + 4W^3 \quad (4.1)$$

$$\square(W/t) = 2W^2/t - W/t^2 \quad (4.2)$$

$$\square(W|\eta|_{J,\theta}^2) = 2W^2|\eta|_{J,\theta}^2 + 2W \langle \eta, \square\eta \rangle_{J,\theta} - 8W|\nabla_b \eta|_{J,\theta}^2 - 8 \langle \nabla_b W, \nabla_b(|\eta|_{J,\theta}^2) \rangle_{J,\theta} \quad (4.3)$$

Proof: (4.2) follows from (2.7). (4.1) follows from (2.12) with $f = W$, (2.7), and the product formula: $\Delta_b(fg) = (\Delta_b f)g + f\Delta_b g + 2\langle \nabla_b f, \nabla_b g \rangle_{J,\theta}$. Similarly, (4.3) follows from a direct computation using (2.7) and the above product formula. (noting that $|\eta|_{J,\theta}^2 = 2\eta_1\eta_{\bar{1}}$)

Q.E.D.

Let η^1, η^2 be two 2-tensors with components η_{cd}^1, η_{cd}^2 , respectively, in which $c, d = 1$ or $\bar{1}$. We define the Levi form for $\eta^1, \eta^2 : \langle \eta^1, \eta^2 \rangle_{J,\theta} = \sum \eta_{cd}^1 \eta_{c\bar{d}}^2$, in which the sum is taken for all possible (c, d) and $(\bar{1}) = 1$ by convention. We define $\nabla_b \eta$ to be a 2-tensor with components $\eta_{c,d}$ for η_c being components of a Legendrian vector field η . Then the modulus of $\nabla_b \eta$ defined previously is just the square root of the above Levi form for $\eta^1 = \eta^2 = \nabla_b \eta$.

Lemma 4.2. Suppose the torsion A_{11} vanishes identically under the Yamabe flow (1.1). Then we have the following evolution equation:

$$\square(\langle \nabla_b W, \eta \rangle_{J,\theta}) = W \langle \nabla_b W, \eta \rangle_{J,\theta} + \langle \nabla_b W, \square \eta \rangle_{J,\theta} - 8 \langle \nabla_b(\nabla_b W), \nabla_b \eta \rangle_{J,\theta} \quad (4.4)$$

Proof: First observe that

$$\begin{aligned} \square(W_{,1}) &= \partial_t(Z_1 W) - 4(W_{,11\bar{1}} + W_{,1\bar{1}1}) & (4.5) \\ &= 5WW_{,1} + 4(W_{,\bar{1}11} - W_{,1\bar{1}\bar{1}}) \text{ (by (2.7) and } \dot{Z}_1 = WZ_1) \\ &= WW_{,1} - 8iW_{,01} \text{ mod}(A_{11}, A_{11,\bar{1}}) \text{ (by commutation relations)} \\ &= WW_{,1}. \text{ (} A_{11} = 0 \text{ and } W_{,0} = 0 \text{ by (3.12))} \end{aligned}$$

Now (4.4) follows from a direct computation using the product formula and (4.5).

Q.E.D.

By combining Lemmas 4.1 and 4.2, we find that if the torsion vanishes identically, the evolution equation for $Z(\theta, \eta)$ is given by

$$\begin{aligned} \square Z(\theta, \eta) &= 12W\Delta_b W - 4|\nabla_b W|_{J,\theta}^2 + 4W^3 + 2W^2/t - W/t^2 & (4.6) \\ &+ W \langle \nabla_b W, \eta \rangle_{J,\theta} + \frac{1}{4}W^2|\eta|_{J,\theta}^2 + \langle \nabla_b W + \frac{1}{4}W\eta, \square \eta \rangle_{J,\theta} \\ &- 8 \langle \nabla_b(\nabla_b W), \nabla_b \eta \rangle_{J,\theta} - W|\nabla_b \eta|_{J,\theta}^2 - \langle \nabla_b W, \nabla_b(|\eta|_{J,\theta}^2) \rangle_{J,\theta}. \end{aligned}$$

Proof of Theorem A: First observe that by Corollary 2.2, W is always positive, and for all η , $Z(\theta, \eta) \geq Y(\theta)$ where

$$Y(\theta) = 2\Delta_b W + W^2 + \frac{W}{t} - 2W^{-1}|\nabla_b W|_{J,\theta}^2. \quad (4.7)$$

Also there exists a positive constant δ such that $Y(\theta) > 0$ for $t < \delta$, hence $Z(\theta, \eta) > 0$ for $t < \delta$.

Suppose $Z(\theta, \eta) \leq 0$ at some space-time point for some η . Then there exists a first time $\tau > 0$, a point $\zeta \in M$ and a Legendrian tangent vector η at ζ such that at (ζ, τ) ,

$$Z(\theta, \eta) = 0. \quad (4.8)$$

We extend η so that at (ζ, τ) ,

$$\eta_{1,\bar{1}} = W^{-1}(4iW_{,1\bar{1}} - W_{,\bar{1}}\eta_1) \quad (4.9)$$

$$\eta_{1,1} = -W^{-1}W_{,1}\eta_1 \quad (4.10)$$

where $\eta = i\eta_1 Z_{\bar{1}} - i\eta_{\bar{1}} Z_1$. Substituting (4.9), (4.10) in the last three terms of (4.6), involving first derivatives of η , we obtain that at (ζ, τ) ,

$$\begin{aligned} -8 < \nabla_b(\nabla_b W), \nabla_b \eta >_{J,\theta} - W|\nabla_b \eta|_{J,\theta}^2 - < \nabla_b W, \nabla_b(|\eta|_{J,\theta}^2) >_{J,\theta} \\ &= W^{-1}(2|4iW_{,1\bar{1}} - W_{,\bar{1}}\eta_1|^2 + 2|W_{,1}\eta_1|^2). \end{aligned} \quad (4.11)$$

In deriving (4.11), we have used $W_{,11} = 0$ for all time due to Lemma 3.1 and $Q_{11} = 0$ in (3.11). (note that the choice of $\nabla_b \eta$ in (4.9),(4.10) is to maximize the left side of (4.11))

Now if $\nabla_b W + \frac{1}{4}W\eta \neq 0$ at (ζ, τ) , we extend η by choosing the value of $\square\eta$ at (ζ, τ) to kill all terms on the right side of (4.6) except, say, the term $2W^2/t$. Then it follows that at (ζ, τ) , $0 \geq \partial_t Z = 4\Delta_b Z + 2W^2/\tau \geq 2W^2/\tau$, a contradiction. So we assume

$$\nabla_b W + \frac{1}{4}W\eta = 0 \quad (4.12)$$

at (ζ, τ) . By (4.8) and (4.12), we can express $\Delta_b W$ in terms of W and η at (ζ, τ) :

$$2\Delta_b W = \frac{1}{8}W|\eta|_{J,\theta}^2 - W^2 - \frac{W}{t}. \quad (4.13)$$

From Lemma 3.1 and (3.12), $W_{,0}$ vanishes identically for all time. It follows that $W_{,1\bar{1}} = W_{,\bar{1}1}$ by the commutation relation. Therefore $\Delta_b W = 2W_{,1\bar{1}}$, so by (4.13) we can express $W_{,1\bar{1}}$ at (ζ, τ) as follows:

$$W_{,1\bar{1}} = \frac{1}{4}\left(\frac{1}{8}W|\eta|_{J,\theta}^2 - W^2 - \frac{W}{t}\right). \quad (4.14)$$

Now substituting (4.14),(4.12) in (4.11) and making use of (4.13),(4.12), we can reduce (4.6) to an expression in W, η only:

$$\square Z = \frac{W}{t^2} + \frac{1}{2}W^2|\eta|_{J,\theta}^2 + \frac{1}{32}W|\eta|_{J,\theta}^4$$

at (ζ, τ) . Hence the maximum principle implies that at (ζ, τ) ,

$$\begin{aligned} 0 \geq \partial_t Z &= 4\Delta_b Z + \frac{W}{t^2} + \frac{1}{2}W^2|\eta|_{J,\theta}^2 \\ &\quad + \frac{1}{32}W|\eta|_{J,\theta}^4 \geq W\tau^{-2}, \end{aligned}$$

which is a contradiction (to Corollary 2.2). So $Z(\theta, \eta) > 0$ completing the proof of Theorem A.

Q.E.D.

Note that we actually obtain the strict inequality from the proof of Theorem A. Taking $\eta = -4W^{-1}\nabla_b W$ in Theorem A implies that

$$Z(\theta, \eta) = Y(\theta) \geq 0, \quad (4.15)$$

where $Y(\theta)$ is defined in (4.7).

Proof of Theorem B: By (2.7) we rewrite (4.15) as

$$\partial_t W + \frac{2W}{t} - 4W^{-1}|\nabla_b W|_{J,\theta}^2 \geq 0. \quad (4.16)$$

Integrating (4.16) over Legendrian paths connecting x_1, x_2 from t_1 to t_2 and using $az + \bar{a}\bar{z} + b|z|^2 \geq -b^{-1}|a|^2$ for $b > 0$, we obtain (1.5).

Q.E.D.

5 Nontriviality of initial conditions

In this section, we will construct contact forms on the standard CR 3-sphere $(S^3, \hat{\xi}, \hat{J})$ with nonconstant positive Tanaka-Webster curvature and vanishing torsion.

Suppose our S^3 is defined by $|z_1|^2 + |z_2|^2 = 1$ for $(z_1, z_2) \in C^2$. The standard contact form $\hat{\theta} = i(\sigma - \bar{\sigma})$ where $\sigma = z_1 d\bar{z}_1 + z_2 d\bar{z}_2$. Take $\hat{\theta}^1 = \sqrt{2}(z_1 dz_2 - z_2 dz_1)$ such that (2.1) is satisfied for the "hatted" quantities. It is easy to deduce $\hat{Z}_1 = \frac{1}{\sqrt{2}}(\bar{z}_1 \partial_2 - \bar{z}_2 \partial_1)$ where $\partial_j = \frac{\partial}{\partial z_j}$ for $j = 1, 2$. Also $\hat{\omega}_1^1 = -2(\bar{z}_1 dz_1 + \bar{z}_2 dz_2)$ and $\hat{A}_{11} = 0$ by (2.2).

Now let $\theta = e^{2\lambda} \hat{\theta}$. It follows from (2.5) that $A_{11} = 0$ (with respect to θ) if and only if

$$\lambda_{,11} = 2(\lambda_{,1})^2 \quad (5.1)$$

in which $\lambda_{,1} = \hat{Z}_1 \lambda$ and $\lambda_{,11} = (\hat{Z}_1)^2 \lambda$ since $\hat{\omega}_1^1(\hat{Z}_1) = 0$. It is a direct verification that $\lambda = -\ln|az_1 + bz_2 + c|$ (well defined on S^3 for $|c| \gg |a|, |b|$) satisfies (5.1). Next we compute W from the formula (2.6) for $\hat{W} = 1$ and λ given above. The final result is

$$\begin{aligned} W = & -(3|z_1|^2 + 2|z_2|^2)|a|^2 - (3|z_2|^2 + 2|z_1|^2)|b|^2 - (a\bar{b}z_1\bar{z}_2 + \bar{a}bz_1z_2) \\ & - c(\bar{a}\bar{z}_1 + \bar{b}\bar{z}_2) - \bar{c}(az_1 + bz_2) + |c|^2. \end{aligned} \quad (5.2)$$

Now it is easy to see from (5.2) that W is positive on S^3 for $|c| \gg |a|, |b|$, and nonconstant in general.

References

- [Ca] E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexe*, I, Ann. Mat., 11(1932),17-90; II, Ann. Sc. Norm. Sup. Pisa, 1(1932),333-354
- [Ch] B. Chow, *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Commun. Pure and Appl. Math., XLV(1992),1003-1014
- [CL] J.-H. Cheng and J. M. Lee, *The Burns-Epstein invariant and deformation of CR structures*, Duke Math. J., 60(1990),221-254
- [H1] R. S. Hamilton, *The Harnack estimate for the Ricci flow*, J. Diff. Geom., 37(1993),225-243
- [H2] ———, *The Ricci flow on surfaces*, Math. and General Relativity, Contemporary Math. 71,(1988),237-262

- [JL1] D. Jerison and J. M. Lee, *The Yamabe problem on CR manifolds*, J. Diff. Geom., 25(1987),167-197
- [JL2] ———, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc., 1(1988),1-13
- [JL3] ———, *Intrinsic CR normal coordinates and the CR Yamabe problem*, J. Diff. Geom., 29(1989),303-343
- [L1] J. M. Lee, *The Fefferman metric and pseudohermitian invariants*, Trans. Amer. Math. Soc., 296(1986),411-429
- [L2] ———, *Pseudo-Einstein structures on CR manifolds*, Am. J. Math., 110(1988),157-178
- [Ta] N. Tanaka, *A Differential Geometric Study on Strongly Pseudo-Convex Manifolds*, 1975, Kinokuniya Co. Ltd., Tokyo
- [We] S. M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Diff. Geom., 13(1978),25-41

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