

A VARIATIONAL PRINCIPLE FOR IMPULSIVE SEMIFLOWS

JOSÉ F. ALVES, MARIA CARVALHO, AND CARLOS H. VÁSQUEZ

ABSTRACT. We consider impulsive semiflows defined on compact metric spaces and deduce a variational principle. In particular, we generalize the classical notion of topological entropy to our setting of discontinuous semiflows.

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1. INTRODUCTION

An impulsive semiflow is built from a continuous semiflow φ acting on a compact metric space X , which governs the state of the system between impulses, and a compact set $D \subset X$ where the semiflow φ experiences some drift specified by a continuous function $I : D \rightarrow X$. Dynamical systems with impulse perturbations seem to be an efficient mathematical tool

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to describe real world phenomena that exhibit abrupt transitions in their phase space; see the introduction of [2] and references therein for a thorough list of applications of impulsive semiflows, particularly in Life Sciences and Physics.

A major problem in this field is inherent to the dynamics: an impulsive semiflow is discontinuous. So, a first concern has been to understand the behavior on the non-wandering set, which may be non-invariant, and to find out sufficient conditions for the system to preserve a probability measure on the σ -algebra of the Borel sets. This issue has been successfully addressed in [2]. The second main query in this context is the existence of probability measures suitable for specific investigations or applications. A natural way of selecting invariant measures is provided by the variational principle [9, 4], a relation between the topological and the measure-theoretic entropy which aims to find relevant extremal elements of the convex set of invariant probability measures.

However, the classical notion of topological entropy requires continuity of the dynamical system, a request we can no longer fulfill. Accordingly, in Section 1.1 we introduce a generalized concept of entropy, which coincides with the classical one for continuous semiflows and is invariant under flow conjugacy. This new entropy concept turns out to be adequate to the kind of discontinuities under consideration and the right notion to establish a variational principle for impulsive semiflows.

1.1. Topological entropy. Here we recall the usual notion of topological entropy when X is a compact metric space and $\varphi : \mathbb{R}_0^+ \times X \rightarrow X$ is a continuous semiflow and introduce a modified definition adapted to our setting.

1.1.1. *The classical definition.* Given $x \in X$, $T > 0$ and $\epsilon > 0$ we define the *dynamic ball*

$$B(x, \varphi, T, \epsilon) = \{y \in X : \text{dist}(\varphi_t(x), \varphi_t(y)) < \epsilon, \text{ for every } t \in [0, T]\}.$$

The continuity of φ implies that $B(x, \varphi, T, \epsilon)$ is an open set of X since it is the open ball centered at x of radius ϵ for the metric

$$\text{dist}_T^\varphi(x, y) = \max_{0 \leq t \leq T} \{\text{dist}(\varphi_t(x), \varphi_t(y))\}.$$

A set $E \subseteq X$ is said to be (φ, T, ϵ) -*separated* if, for each $x \in E$, inside the ball $B(x, \varphi, T, \epsilon)$ there is no other point of E besides x . As a consequence of the compactness of X and the continuity of φ , any set $E \subseteq X$ which is (φ, T, ϵ) -separated is finite. If we denote by $|E|$ the cardinality of E , then we define the largest number of distinct, up to ϵ , initial T -blocks of orbits of φ by

$$s(\varphi, T, \epsilon) = \max\{|E| : E \text{ is } (\varphi, T, \epsilon)\text{-separated}\},$$

and the growth rate of this number as

$$h(\varphi, \epsilon) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \log s(\varphi, T, \epsilon).$$

The *topological entropy* of φ is then given by

$$h_{\text{top}}(\varphi) = \lim_{\epsilon \rightarrow 0^+} h(\varphi, \epsilon).$$

1.1.2. *A modified definition.* We now change the previous definition of topological entropy. Let X be a metric space and $\psi : \mathbb{R}_0^+ \times X \rightarrow X$ a (not necessarily continuous) semiflow.

Definition 1. Consider a function τ assigning to each $x \in X$ a strictly increasing (possibly finite) sequence of positive real numbers $(\tau_n(x))_{n \in A(x)}$, where either $A(x) = \{1, \dots, \ell\}$ for some $\ell \in \mathbb{N}$ or $A(x) = \mathbb{N}$. We say that τ is *admissible with respect to* $Z \subset X$ if there exists $\eta > 0$ such that $\tau_1(x) \geq \eta$ for all $x \in Z$, and for all $x \in X$:

- (1) $\tau_n(\psi_s(x)) = \tau_n(x) - s$, for all $n \in \mathbb{N}$ and all $s \geq 0$;
- (2) $\tau_{n+1}(x) - \tau_n(x) \geq \eta$, for all $n \in \mathbb{N}$ with $n + 1 \in A(x)$.

For each admissible function τ , $x \in X$, $T > 0$ and $0 < \delta < \eta/2$, we define

$$J_{T,\delta}^\tau(x) = (0, T] \setminus \left(\bigcup_{j=1}^{n_T(x)}]\tau_j(x) - \delta, \tau_j(x) + \delta[\right),$$

where $n_T(x) = \max\{n \geq 1 : \tau_n(x) \leq T\}$. The τ -dynamical ball of radius $\epsilon > 0$ centered at x is the set

$$B^\tau(x, \psi, T, \epsilon, \delta) = \{y \in X : \text{dist}(\psi_t(x), \psi_t(y)) < \epsilon, \forall t \in J_{T,\delta}^\tau(x)\}.$$

Accordingly, a set $E \subseteq X$ is said to be $(\psi, \tau, T, \epsilon, \delta)$ -separated if, for each $x \in E$, we have

$$y \notin B^\tau(x, \psi, T, \epsilon, \delta), \quad \forall y \in E \setminus \{x\}.$$

As before, define

$$s^\tau(\psi, T, \epsilon, \delta) = \sup\{|E| : E \text{ is a finite } (\psi, \tau, T, \epsilon, \delta)\text{-separated set}\},$$

and the growth rate

$$h^\tau(\psi, \epsilon, \delta) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \log s^\tau(\psi, T, \epsilon, \delta),$$

where $\log \infty = \infty$. As the function $\epsilon \mapsto h^\tau(\psi, \epsilon, \delta)$ is decreasing, the following limit exists

$$h^\tau(\psi, \delta) = \lim_{\epsilon \rightarrow 0^+} h^\tau(\psi, \epsilon, \delta).$$

Finally, as the function $\delta \mapsto h^\tau(\psi, \delta)$ is also decreasing, we define the τ -topological entropy of ψ

$$h_{\text{top}}^\tau(\psi) = \lim_{\delta \rightarrow 0^+} h^\tau(\psi, \delta).$$

Theorem A. *Let $\varphi : \mathbb{R}_0^+ \times X \rightarrow X$ be a continuous semiflow on a compact metric space X and τ an admissible function on X . Then $h_{\text{top}}^\tau(\varphi) = h_{\text{top}}(\varphi)$.*

1.2. **Impulsive semiflows.** Consider a compact metric space X , a continuous semiflow $\varphi : \mathbb{R}_0^+ \times X \rightarrow X$, a nonempty compact set $D \subset X$ and a continuous map $I : D \rightarrow X$. Given $\xi > 0$, let

$$D_\xi = \bigcup_{x \in D} \{\varphi_t(x) : 0 < t < \xi\}. \tag{1.1}$$

Definition 2. We say that D satisfies a *half-tube condition* if there is $\xi_0 > 0$ such that:

- (1) D_ξ is an open set for each $0 < \xi \leq \xi_0$;
- (2) If $\varphi_t(x) \in D_{\xi_0}$ for some $x \in X$ and $t > 0$, then there is $0 \leq t' < t$ with $\varphi_{t'}(x) \in D$;
- (3) $\{\varphi_t(x_1) : 0 < t < \xi_0\} \cap \{\varphi_t(x_2) : 0 < t < \xi_0\} = \emptyset$ for all $x_1, x_2 \in D$ with $x_1 \neq x_2$.

The first visit of each φ -trajectory to D will be registered by the function $\tau_1 : X \rightarrow [0, +\infty]$ defined by

$$\tau_1(x) = \begin{cases} \inf \{t > 0 : \varphi_t(x) \in D\}, & \text{if } \varphi_t(x) \in D \text{ for some } t > 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

It is known that the function τ_1 is lower semicontinuous on the set $X \setminus D$; see [5, Theorem 2.7]. Additionally, the *tube condition* proposed in [5] ensures that the restriction of τ_1 to $X \setminus D$ is also upper semicontinuous.

Assuming $\tau_1(x) > 0$ for all $x \in X$, we define the *impulsive trajectory* $\gamma_x : [0, T(x)[\rightarrow X$ and the subsequent *impulsive times* of $x \in X$ according to the following rules:

- (1) If $0 \leq t < \tau_1(x)$, then we set $\gamma_x(t) = \varphi_t(x)$.
- (2) If $\tau_1(x) < \infty$, then we proceed inductively:
 - (a) Firstly we set

$$\gamma_x(\tau_1(x)) = I(\varphi_{\tau_1(x)}(x)).$$

Defining the second impulsive time of x as

$$\tau_2(x) = \tau_1(x) + \tau_1(\gamma_x(\tau_1(x))),$$

we set

$$\gamma_x(t) = \varphi_{t-\tau_1(x)}(\gamma_x(\tau_1(x))), \quad \text{for } \tau_1(x) < t < \tau_2(x).$$

- (b) Assuming that $\gamma_x(t)$ is defined for $t < \tau_n(x)$, for some $n \geq 2$, we set

$$\gamma_x(\tau_n(x)) = I(\varphi_{\tau_n(x)-\tau_{n-1}(x)}(\gamma_x(\tau_{n-1}(x)))).$$

Defining the $(n+1)^{\text{th}}$ impulsive time of x as

$$\tau_{n+1}(x) = \tau_n(x) + \tau_1(\gamma_x(\tau_n(x))),$$

we set

$$\gamma_x(t) = \varphi_{t-\tau_n(x)}(\gamma_x(\tau_n(x))), \quad \text{for } \tau_n(x) < t < \tau_{n+1}(x).$$

Finally, we define the time length of the trajectory of x as

$$T(x) = \sup_{n \geq 1} \{\tau_n(x)\}.$$

We say that (X, φ, D, I) is an *impulsive dynamical system* if

$$\tau_1(x) > 0 \quad \text{and} \quad T(x) = +\infty, \quad \text{for all } x \in X.$$

As observed in [2, Remark 1.1], under the condition $I(D) \cap (D) = \emptyset$ we have $T(x) = \infty$ for all $x \in X$ and $\tau = \{\tau_n(x)\}_{n \geq 1}$ is an admissible function with respect to D . The *impulsive semiflow* ψ of an impulsive dynamical system (X, φ, D, I) is defined by

$$\begin{aligned} \psi : \mathbb{R}_0^+ \times X &\longrightarrow X \\ (t, x) &\longmapsto \gamma_x(t), \end{aligned}$$

where $\gamma_x(t)$ is the impulsive trajectory of x determined by (X, φ, D, I) . It has been proved in [3, Proposition 2.1] that ψ is indeed a semiflow, though not necessarily continuous.

For small enough $\xi > 0$ we define

$$X_\xi = X \setminus (D_\xi \cup D).$$

Observe that, as we are assuming that D satisfies a half-tube condition (see item (2) of Definition 2), then X_ξ is forward invariant under ψ (that is, $\psi_t(X_\xi) \subseteq X_\xi$ for all $t \geq 0$).

To control the moments a φ -trajectory visits D , we introduce the function

$$\tau^* : X_\xi \cup D \rightarrow [0, +\infty]$$

defined by

$$\tau^*(x) = \begin{cases} \tau_1(x), & \text{if } x \in X_\xi; \\ 0, & \text{if } x \in D \end{cases}$$

and, in what follows, we will assume that τ^* is a continuous map.

Definition 3. We say that $I(D)$ is *transverse* if there are $s_0 > 0$ and $\xi_0 > 0$ such that

- (1) $\varphi_t(x) \in I(D) \Rightarrow \varphi_{t+s}(x) \notin I(D), \forall 0 < s < s_0;$
- (2) $\{\varphi_t(x_1) : 0 < t < \xi_0\} \cap \{\varphi_t(x_2) : 0 < t < \xi_0\} = \emptyset$ for all $x_1, x_2 \in I(D)$ with $x_1 \neq x_2$.

This property holds, for instance, when φ is a C^1 semiflow and $I(D)$ is transversal to the flow direction.

The map I is said to be *1-Lipschitz* if for all $x, y \in D$ we have

$$\text{dist}(I(x), I(y)) \leq \text{dist}(x, y).$$

Theorem B. *Let ψ be the semiflow of an impulsive dynamical system (X, φ, D, I) such that I is 1-Lipschitz, $I(D) \cap D = \emptyset$, D satisfies a half-tube condition, $I(D)$ is transverse and τ^* is continuous. Then there exist a compact metric space \tilde{X} , a continuous semiflow $\tilde{\psi}$ in \tilde{X} and a continuous invertible bimeasurable map $h : X_\xi \rightarrow \tilde{X}$ such that $\tilde{\psi}_t \circ h = h \circ \psi_t$ for all $t \geq 0$ and*

$$h_{\text{top}}^\tau(\psi) = h_{\text{top}}^\tau(\tilde{\psi}).$$

We are left to relate the topological entropy of $\tilde{\psi}$ with the metric entropies of the time-one map ψ_1 induced by the impulsive semiflow ψ . In the sequel, $\mathcal{M}_\psi(X)$ will stand for the set of probability measures defined on the σ -algebra of the Borel subsets of X and invariant by the impulsive semiflow associated to the impulsive dynamical system (X, φ, D, I) .

Theorem C. *Let ψ be the semiflow of an impulsive dynamical system (X, φ, D, I) satisfying the assumptions of Theorem B and such that $\mathcal{M}_\psi(X) \neq \emptyset$. Then*

$$h_{\text{top}}^\tau(\psi) = \sup \{h_\mu(\psi_1) : \mu \in \mathcal{M}_\psi(X)\}.$$

Regarding the additional demand in the statement of the previous theorem, we recall that [2, Theorem A] shows that conditions $I(D) \cap D = \emptyset$ and $I(\Omega_\psi \cap D) \subset \Omega_\psi \setminus D$ together are sufficient for $\mathcal{M}_\psi(X)$ to be nonempty, where Ω_ψ denotes the non-wandering set of ψ .

In the last section, we will present a simple example satisfying the assumptions of our theorems. These results also apply, for instance, to the discontinuous local semiflows for Kurzweil equations studied in [1].

2. TOPOLOGICAL ENTROPY: CLASSICAL AND NEW

In this section we will verify that the modified definition of topological entropy coincides with the classical one for continuous semiflows defined on compact metric spaces. We start proving that the trajectory of any point is uniformly continuous.

Lemma 2.1. *Let $\varphi : \mathbb{R}_0^+ \times X \rightarrow X$ be a continuous semiflow on a compact metric space X . For each $\alpha > 0$ there exists $\beta > 0$ such that, for all $x \in X$ and all $t, u \geq 0$ with $|t - u| < \beta$, we have $\text{dist}(\varphi_t(x), \varphi_u(x)) < \alpha$.*

Proof. First notice that, as X is a compact metric space, then $\varphi : [0, 1] \times X \rightarrow X$ is uniformly continuous. In particular, given $\alpha > 0$, there exist $\beta_0 > 0$ such that for all $y \in X$ and all $t_0, u_0 \in [0, 1]$ with $|t_0 - u_0| < \beta_0$ we have

$$\text{dist}(\varphi_{t_0}(y), \varphi_{u_0}(y)) < \alpha.$$

Take $\beta = \min\{\beta_0, 1/2\} > 0$. For every $t, u > 0$ with $|t - u| < \beta$, there exist an integer $n_0 \geq 0$ and $t_0, u_0 \in [0, 1]$ such that $t = n_0 + t_0$, $u = n_0 + u_0$ and $|t_0 - u_0| = |t - u| < \beta \leq \beta_0$. Taking $y = \varphi_{n_0}(x)$, then

$$\begin{aligned} \text{dist}(\varphi_t(x), \varphi_u(x)) &= \text{dist}(\varphi_{n_0+t_0}(x), \varphi_{n_0+u_0}(x)) \\ &= \text{dist}(\varphi_{t_0}(\varphi_{n_0}(x)), \varphi_{u_0}(\varphi_{n_0}(x))) \\ &= \text{dist}(\varphi_{t_0}(y), \varphi_{u_0}(y)) \\ &< \alpha. \end{aligned}$$

□

2.1. Proof of Theorem A. Fix $0 < \delta < \eta/2$, $\epsilon > 0$ and $T > 0$. Notice that for every $x \in X$ we have

$$B(x, \varphi, T, \epsilon) \subseteq B^\tau(x, \varphi, T, \epsilon, \delta),$$

and

$$s^\tau(\varphi, T, \epsilon, \delta) \leq s(\varphi, T, \epsilon),$$

so

$$h_{\text{top}}^\tau(\varphi) \leq h_{\text{top}}(\varphi).$$

Let us now prove the other inequality. Fix $T \geq 0$ and $\alpha > 0$. By Lemma 2.1 there exists $\beta > 0$ such that, for all $z \in X$ and all $t, u > 0$ with $|t - u| < \beta$, we have

$$\text{dist}(\varphi_t(z), \varphi_u(z)) < \alpha/4. \tag{2.1}$$

Hence, if $x, y \in X$ and $t \in [0, T]$ satisfy

$$\text{dist}(\varphi_t(x), \varphi_t(y)) > \alpha, \tag{2.2}$$

then, for every $u \in (t - \beta, t + \beta)$, we get

$$\text{dist}(\varphi_t(x), \varphi_t(y)) \leq \text{dist}(\varphi_t(x), \varphi_u(x)) + \text{dist}(\varphi_u(x), \varphi_u(y)) + \text{dist}(\varphi_u(y), \varphi_t(y))$$

which, together with (2.1) and (2.2), implies

$$\text{dist}(\varphi_u(x), \varphi_u(y)) > \alpha/2. \quad (2.3)$$

Consider now $E \subseteq X$ being (φ, T, α) -separated. As φ is continuous and each dynamical ball is contained in the corresponding τ -dynamical ball, the set E is finite. By definition, for every $x, y \in E$, $x \neq y$, there exists $t \in [0, T]$ such that

$$\text{dist}(\varphi_t(x), \varphi_t(y)) \geq \alpha.$$

Choose $0 < \delta < \min\{\eta, \beta/2\}$ and $0 < \epsilon < \alpha/2$. By (2.3), if $u \in (t - 2\delta, t + 2\delta)$, then

$$\text{dist}(\varphi_u(x), \varphi_u(y)) > \alpha/2 > \epsilon.$$

If $t \in J_{T,\delta}^\tau(x)$, then $y \notin B^\tau(x, \varphi, T, \epsilon, \delta)$. Otherwise, $J_{T,\delta}^\tau(x) \cap (t - 2\delta, t + 2\delta) \neq \emptyset$, and then $y \notin B^\tau(x, \varphi, T, \epsilon, \delta)$. So, E is $(\varphi, \tau, T, \epsilon, \delta)$ -separated. Consequently, for every $0 < \delta < \min\{\eta, \beta/2\}$, $0 < \epsilon < \alpha/2$ and $T > 0$,

$$s(\varphi, T, \alpha) \leq s^\tau(\varphi, T, \epsilon, \delta),$$

and so

$$\frac{1}{T} \log s(\varphi, T, \alpha) \leq \frac{1}{T} \log s^\tau(\varphi, T, \epsilon, \delta).$$

Taking the upper limit as $T \rightarrow +\infty$, we get

$$h(\varphi, \alpha) \leq h^\tau(\varphi, \epsilon, \delta).$$

Now, if $\epsilon \rightarrow 0^+$,

$$h(\varphi, \alpha) \leq h^\tau(\varphi, \delta).$$

Noticing that $\beta = \beta(\alpha)$ and $\delta = \delta(\alpha)$, we deduce that, when $\alpha \rightarrow 0^+$, we have $\delta \rightarrow 0^+$, and therefore

$$h_{\text{top}}(\varphi) \leq h_{\text{top}}^\tau(\varphi).$$

2.2. Monotonicity of h_{top}^τ . Given τ and τ' two admissible functions in X , we say that τ' refines τ , and write $\tau' \succ \tau$, if for all $x \in X$ and $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\tau_n(x) = \tau'_m(x)$. Next lemma proves that the new concept of topological entropy is monotone with respect to the refinement of admissible functions.

Lemma 2.2. *For any semiflow $\psi : \mathbb{R}_0^+ \times X \rightarrow X$, if $\tau' \succ \tau$ then $h_{\text{top}}^\tau(\psi) \geq h_{\text{top}}^{\tau'}(\psi)$.*

Proof. Given $\epsilon > 0$, $T > 0$, $0 < \delta < \eta/2$ and a finite $(\psi, \tau', T, \epsilon, \delta)$ -separated subset E , as $\tau' \succ \tau$, the set E is a $(\psi, \tau, T, \epsilon, \delta)$ -separated as well. Therefore

$$s^{\tau'}(\psi, T, \epsilon, \delta) \leq s^\tau(\psi, T, \epsilon, \delta).$$

□

Given two semiflows $\psi : \mathbb{R}_0^+ \times X \rightarrow X$ and $\tilde{\psi} : \mathbb{R}_0^+ \times \tilde{X} \rightarrow \tilde{X}$, acting on metric spaces (X, d) and (\tilde{X}, \tilde{d}) , and two admissible functions τ and $\tilde{\tau}$ defined on X and \tilde{X} , respectively, we say that a uniformly continuous surjective map $h : X \rightarrow \tilde{X}$ is a $(\tau, \tilde{\tau})$ -semiconjugacy between ψ and $\tilde{\psi}$ if

- (1) $\tilde{\psi}_t \circ h = h \circ \psi_t$, for all $t \geq 0$;

(2) $\tilde{\tau}(h(x)) = \tau(x)$, for all $x \in X$.

Lemma 2.3. *Let $h : X \rightarrow \tilde{X}$ be a finite-to-one $(\tau, \tilde{\tau})$ -semiconjugacy between the semiflows ψ and $\tilde{\psi}$ on X and \tilde{X} , with admissible functions τ and $\tilde{\tau}$, respectively. Then $h_{\text{top}}^{\tau}(\psi) \geq h_{\text{top}}^{\tilde{\tau}}(\tilde{\psi})$.*

Proof. Let $\psi : \mathbb{R}_0^+ \times X \rightarrow X$ and $\tilde{\psi} : \mathbb{R}_0^+ \times \tilde{X} \rightarrow \tilde{X}$ be two semiconjugate semiflows and h be such a semiconjugacy. As h is uniformly continuous, given $\epsilon > 0$ there exists $\gamma > 0$ such that

$$d(a, b) < \gamma \quad \Rightarrow \quad \tilde{d}(h(a), h(b)) < \epsilon \quad \forall a, b \in X.$$

Fix $T > 0$ and $0 < \delta < \eta/2$, and consider a finite $(\tilde{\psi}, \tilde{\tau}, T, \epsilon, \delta)$ -separated set $B \subseteq \tilde{X}$. Then $A = h^{-1}(B)$ is finite, although it may have a cardinal bigger or equal than the one of B . Moreover, A is a $(\psi, \tau, T, \gamma, \delta)$ -separated set of X . Indeed, for all $a, b \in A$, there are $t_n \in J_{T, \delta}^{\tilde{\tau}}(h(a))$ and $s_n \in J_{T, \delta}^{\tilde{\tau}}(h(b))$ such that

$$\tilde{d}(\tilde{\psi}_{t_n}(h(a)), \tilde{\psi}_{t_n}(h(b))) \geq \epsilon \quad \text{and} \quad \tilde{d}(\tilde{\psi}_{s_n}(h(a)), \tilde{\psi}_{s_n}(h(b))) \geq \epsilon$$

that is,

$$\tilde{d}(h \circ \psi_{t_n}(a), h \circ \psi_{t_n}(b)) \geq \epsilon \quad \text{and} \quad \tilde{d}(h \circ \psi_{s_n}(a), h \circ \psi_{s_n}(b)) \geq \epsilon.$$

Therefore,

$$d(\psi_{t_n}(a), \psi_{t_n}(b)) \geq \gamma \quad \text{and} \quad d(\psi_{s_n}(a), \psi_{s_n}(b)) \geq \gamma.$$

Taking into account that, by definition of semiconjugacy, $t_n \in J_{T, \delta}^{\tau}(a)$ and $s_n \in J_{T, \delta}^{\tau}(b)$, we deduce that

$$s^{\tau}(\psi, T, \gamma, \delta) \geq s^{\tilde{\tau}}(\tilde{\psi}, T, \epsilon, \delta).$$

When $\epsilon \rightarrow 0$, we have $\gamma = \gamma(\epsilon) \rightarrow 0$, and so we finally conclude that

$$h_{\text{top}}^{\tau}(\psi) \geq h_{\text{top}}^{\tilde{\tau}}(\tilde{\psi}).$$

□

3. TIME AND SPACE RESTRICTIONS

Consider a compact metric space X , a continuous semiflow $\varphi : \mathbb{R}_0^+ \times X \rightarrow X$, a compact set $D \subset X$ and a continuous map $I : D \rightarrow X$ under the assumptions of Theorem B. Let τ be the admissible function with respect to D of the impulsive times associated to the impulsive semiflow (X, φ, D, I) .

It follows from the assumption that $I(D)$ is transverse (see Definition 3) that the function that assigns to each $x \in X$ the sequence of visit times to $I(D)$, say $\theta(x) := (\theta_n(x))_{n \in \mathbb{N}}$, is an admissible function with respect to $I(D)$. Moreover, as $I(D) \cap D = \emptyset$, we may re-index the sequences $\tau(x)$ and $\theta(x)$ in order to assemble them in a unique admissible function τ' , with respect to both D and $I(D)$, where $\tau'_n(x)$ is either $\tau_m(x)$ or $\theta_m(x)$, for some m . This way, we have $\tau' \succ \tau$.

Lemma 3.1. $h_{\text{top}}^{\tau'}(\psi) = h_{\text{top}}^{\tau}(\psi)$.

Proof. As $\tau' \succ \tau$, by Lemma 2.2, we have $h_{\text{top}}^{\tau'}(\psi) \leq h_{\text{top}}^{\tau}(\psi)$.

Concerning the other inequality, we first observe that as the impulsive semiflow ψ is continuous on points out of D , the set $I(D)$ is compact and disjoint from D and τ^* is continuous, and strictly positive in the complement of D , we may find a compact neighborhood of $I(D)$, say

$$V = \{x \in X : \text{dist}(x, I(D)) \leq \Delta\}$$

for some small enough $\Delta > 0$, such that

(a) there exists $\rho > 0$ such that

$$\tau_1(x) \geq \rho \quad \forall x \in V; \quad (3.1)$$

(b) for all $x \in V$, there is $\zeta_x \geq \eta/2$ such that the map $t \in [0, \zeta_x] \rightarrow \psi_t(x)$ is continuous;

(c) given $\alpha > 0$, there exists $0 < \beta < \rho$ such that, if $x \in V$ and $0 \leq u \leq \beta$, then $\text{dist}(\psi_u(x), x) < \alpha$.

Take $T > 0$, $0 < \epsilon < \Delta/2$, $\alpha = \epsilon/8$ and its corresponding β given by (c) above. Let $0 < \delta < \min\{\eta/2, \beta/2\}$. We already know that

$$s^{\tau'}(\psi, T, \epsilon, \delta) \leq s^{\tau}(\psi, T, \epsilon, \delta).$$

If $s^{\tau'}(\psi, T, \epsilon, \delta) < s^{\tau}(\psi, T, \epsilon, \delta)$, then $s^{\tau'}(\psi, T, \epsilon, \delta)$ is finite and we may consider a maximal $(\psi, \tau', T, \epsilon, \delta)$ -separated set E . The set E is also $(\psi, \tau, T, \epsilon, \delta)$ -separated, though not maximal. Therefore, we may find $z \in X$ such that $E \cup \{z\}$ is still $(\psi, \tau, T, \epsilon, \delta)$ -separated, but no longer $(\psi, \tau', T, \epsilon, \delta)$ -separated. This means, in particular, that

(1) for every $x \in E$,

$$\exists v = v_x \in J_{T, \delta}^{\tau}(x) : \quad \text{dist}(\psi_v(z), \psi_v(x)) \geq \epsilon$$

and

$$\exists u = u_z \in J_{T, \delta}^{\tau}(z) : \quad \text{dist}(\psi_u(z), \psi_u(x)) \geq \epsilon;$$

(2) there is $e \in E$ such that either $z \in B^{\tau'}(e, \psi, T, \epsilon, \delta)$, that is,

$$\forall t \in J_{T, \delta}^{\tau'}(e) \quad \text{dist}(\psi_t(z), \psi_t(e)) < \epsilon$$

in which case

$$v_e \in J_{T, \delta}^{\tau}(e) \setminus J_{T, \delta}^{\tau'}(e),$$

which means that there is $\theta_j(e)$ satisfying

$$v_e \in (\theta_j(e) - \delta, \theta_j(e) + \delta);$$

or $e \in B^{\tau'}(z, \psi, T, \epsilon, \delta)$, that is,

$$\forall r \in J_{T, \delta}^{\tau'}(z) \quad \text{dist}(\psi_r(z), \psi_r(e)) < \epsilon$$

and so we must have

$$u_z \in J_{T, \delta}^{\tau}(z) \setminus J_{T, \delta}^{\tau'}(z)$$

or, equivalently, there is $\theta_\ell(z)$ such that

$$u_z \in (\theta_\ell(z) - \delta, \theta_\ell(z) + \delta).$$

Assume that $z \in B^{r'}(e, \psi, T, \epsilon, \delta)$ and consider

$$m = \theta_j(e) - \delta.$$

As $m \in J_{T, \delta}^{r'}(e)$, we know that

$$\text{dist}(\psi_m(z), \psi_m(e)) < \epsilon.$$

Moreover, as $\theta_j(e) - m < \delta < \beta$ and $\psi_{\theta_j(e)}(e) \in I(D) \subset V$, we have

$$\text{dist}(\psi_m(e), \psi_{\theta_j(e)}(e)) < \alpha = \epsilon/8$$

and therefore, as $0 < \max\{v_e - m, |\theta_j(e) - v_e|\} < \beta$, we get

$$\begin{aligned} \text{dist}(\psi_m(e), \psi_{v_e}(e)) &\leq \text{dist}(\psi_m(e), \psi_{\theta_j(e)}(e)) + \text{dist}(\psi_{\theta_j(e)}(e), \psi_{v_e}(e)) \\ &< \alpha + \alpha < \epsilon/4 \end{aligned}$$

and

$$\begin{aligned} \text{dist}(\psi_m(z), I(D)) &\leq \text{dist}(\psi_m(z), \psi_{\theta_j(e)}(e)) \\ &\leq \text{dist}(\psi_m(z), \psi_m(e)) + \text{dist}(\psi_m(e), \psi_{\theta_j(e)}(e)) \\ &< \epsilon + \epsilon/4 < \Delta. \end{aligned}$$

That is, $\psi_m(z) \in V$ and so, as $0 < v_e - m < \beta$,

$$\text{dist}(\psi_m(z), \psi_{v_e}(z)) < \epsilon/4.$$

Analogously, as $v_e - m < \delta < \beta$ and $\psi_{v_e}(e) \in V$, we have

$$\text{dist}(\psi_m(e), \psi_{v_e}(e)) < \epsilon/4.$$

Thus

$$\begin{aligned} \epsilon &\leq \text{dist}(\psi_{v_e}(z), \psi_{v_e}(e)) \\ &\leq \text{dist}(\psi_{v_e}(z), \psi_m(z)) + \text{dist}(\psi_m(z), \psi_m(e)) + \text{dist}(\psi_m(e), \psi_{v_e}(e)) \\ &\leq \epsilon/4 + \text{dist}(\psi_m(z), \psi_m(e)) + \epsilon/4 \end{aligned}$$

hence

$$\text{dist}(\psi_m(z), \psi_m(e)) \geq \epsilon/2.$$

This means that $z \notin B^{r'}(e, \psi, T, \epsilon/2, \delta)$.

In a similar way, we conclude that, if $e \in B^{r'}(z, \psi, T, \epsilon, \delta)$, then $e \notin B^{r'}(z, \psi, T, \epsilon/2, \delta)$. In any case, we deduce that the set $E \cup \{z\}$, which is $s^\tau(\psi, T, \epsilon, \delta)$ -separated, is $s^{r'}(\psi, T, \epsilon/2, \delta)$ -separated as well.

Consequently, for every $T > 0$, $0 < \delta < \min\{\eta/2, \beta/2\}$ and $0 < \epsilon < \Delta$, we get

$$s^\tau(\psi, T, \epsilon, \delta) \leq s^{r'}(\psi, T, \epsilon/2, \delta)$$

and therefore

$$h_{\text{top}}^\tau(\psi) \leq h_{\text{top}}^{r'}(\psi).$$

□

Let $a > 0$ be the distance between the compact sets D and $I(D)$, and assume that

$$0 < \xi < \min \{\eta/4, \xi_0/2, a/2\}, \quad (3.2)$$

where $\eta > 0$ and $\xi_0 > 0$ are given in Definition 1 and Definition 2, respectively. The next result shows that the τ and τ' -topological entropies of the semiflows ψ and $\psi|_{X_\xi}$ coincide.

Lemma 3.2. $h_{\text{top}}^{\tau'}(\psi) = h_{\text{top}}^{\tau'}(\psi|_{X_\xi})$.

Proof. As $X_\xi \subset X$, then

$$h_{\text{top}}^{\tau'}(\psi|_{X_\xi}) \leq h_{\text{top}}^{\tau'}(\psi).$$

We are left to prove the other inequality.

Take $\epsilon > 0$, $T > 0$, $0 < \delta < \eta/4$ and a finite $(\psi, \tau', T, \epsilon, \delta)$ -separated set $E \subseteq X$. Let

$$A = E \cap (D \cup D_\xi) \quad \text{and} \quad B = E \cap X_\xi.$$

The set B is $(\psi|_{X_\xi}, \tau', T, \epsilon, \delta)$ -separated in X_ξ , and so its cardinal is smaller than $s^{\tau'}(\psi|_{X_\xi}, T, \epsilon, \delta)$.

We claim that the cardinal of A is also bounded by $s^{\tau'}(\psi|_{X_\xi}, T, \epsilon, \delta)$. Indeed, for each pair of points $a, b \in A$, take $u \in J_{T, \delta}^{\tau'}(a)$ and $v \in J_{T, \delta}^{\tau'}(b)$ such that

$$\text{dist}(\psi_u(a), \psi_u(b)) \geq \epsilon \quad \text{and} \quad \text{dist}(\psi_v(a), \psi_v(b)) \geq \epsilon$$

that is,

$$\text{dist}(\psi_{u-\xi}(\psi_\xi(a)), \psi_{u-\xi}(\psi_\xi(b))) \geq \epsilon \quad \text{and} \quad \text{dist}(\psi_{v-\xi}(\psi_\xi(a)), \psi_{v-\xi}(\psi_\xi(b))) \geq \epsilon.$$

Let

$$S = \min \{t \in J_{T, \delta}^{\tau'}(x) : x \in A\}.$$

By Definition 1 and condition (3.2), we have $S \geq \eta/2 - \delta > \eta/4$ and so $\xi < S$. Hence, as τ' is admissible (check item (1) of Definition 1),

$$u - \xi \in J_{T, \delta}^{\tau'}(\psi_\xi(a)) \quad \text{and} \quad v - \xi \in J_{T, \delta}^{\tau'}(\psi_\xi(b)).$$

Moreover, the points $\psi_\xi(a)$ and $\psi_\xi(b)$ are not in $D_\xi \cup D$. Thus the set $\psi_\xi(A)$ is $(\psi|_{X_\xi}, \tau', T, \epsilon, \delta)$ -separated and so

$$|\psi_\xi(A)| \leq s^{\tau'}(\psi|_{X_\xi}, T, \epsilon, \delta).$$

Finally, by item (3) of Definition 2, $|\psi_\xi(A)| = |A|$. Therefore

$$|E| = |A| + |B| = |\psi_\xi(A)| + |B| \leq 2 s^{\tau'}(\psi|_{X_\xi}, T, \epsilon, \delta)$$

and so

$$s^{\tau'}(\psi, T, \epsilon, \delta) \leq 2 s^{\tau'}(\psi|_{X_\xi}, T, \epsilon, \delta).$$

Letting $T \rightarrow +\infty$, this inequality implies that

$$h^{\tau'}(\psi, \epsilon, \delta) \leq h^{\tau'}(\psi|_{X_\xi}, \epsilon, \delta)$$

and consequently, as ϵ, δ may be chosen arbitrarily small, this last inequality yields

$$h_{\text{top}}^{\tau'}(\psi) \leq h_{\text{top}}^{\tau'}(\psi|_{X_\xi}).$$

□

Remark 3.3. As we are assuming that $I(D)$ is transverse, a similar argument proves that $h_{\text{top}}^{\tau'}(\psi) = h_{\text{top}}^{\tau'}(\psi|_{X_\xi \setminus I(D)})$.

4. A QUOTIENT SPACE

Given an impulsive dynamical system (X, φ, D, I) , consider the quotient space X/\sim endowed with the quotient topology, where \sim is the equivalence relation given by

$$x \sim y \iff x = y, \quad y = I(x), \quad x = I(y) \quad \text{or} \quad I(x) = I(y).$$

Let $\pi : X \rightarrow X/\sim$ be the natural projection.

4.1. The induced metric. If d denotes the metric on X , the metric \tilde{d} in $\pi(X)$ that induces the quotient topology is given by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf \{d(p_1, q_1) + d(p_2, q_2) + \cdots + d(p_n, q_n)\},$$

where $p_1, q_1, \dots, p_n, q_n$ is any chain of points in X such that $p_1 \sim x$, $q_1 \sim p_2$, $q_2 \sim p_3$, ... $q_n \sim y$; see §23 of [10]. In particular, we have

$$\tilde{d}(\tilde{x}, \tilde{y}) \leq d(x, y), \quad \forall x, y \in X.$$

Yet, the length n of the chains needed to evaluate $\tilde{d}(\tilde{x}, \tilde{y})$ may be arbitrarily large, preventing us from comparing $\tilde{d}(\tilde{x}, \tilde{y})$ with $d(p, q)$ for all $p \sim x$ and $q \sim y$. This difficulty is overcome if we are able to uniformly bound the range of n ; this is feasible, for instance, when the map I does not expand distances.

Lemma 4.1. *If I is 1-Lipschitz, then for all $\tilde{x}, \tilde{y} \in \pi(X)$ there exist $p, q \in X$ such that*

$$p \sim x, \quad q \sim y \quad \text{and} \quad d(p, q) \leq 2\tilde{d}(\tilde{x}, \tilde{y}).$$

Proof. We will show that

$$\forall \tilde{x}, \tilde{y} \in \pi(X) \quad \tilde{d}(\tilde{x}, \tilde{y}) = \inf \{d(p, q) : p \sim x, q \sim y\}.$$

Clearly, for all $\tilde{x}, \tilde{y} \in \pi(X)$, we have

$$\tilde{d}(\tilde{x}, \tilde{y}) \leq \inf \{d(p, q) : p \sim x, q \sim y\}.$$

Conversely, take a chain $p_1, q_1, \dots, p_n, q_n \in X$ such that

$$p_1 \sim x, \quad q_1 \sim p_2, \quad q_2 \sim p_3, \quad \dots, \quad q_n \sim y.$$

(1) If $q_1 = p_2$, then

$$d(p_1, q_1) + d(p_2, q_2) = d(p_1, q_1) + d(q_1, q_2) \geq d(p_1, q_2).$$

(2) If $p_2 = I(q_1)$, then

$$d(p_1, q_1) + d(p_2, q_2) \geq d(I(p_1), I(q_1)) + d(p_2, q_2) \geq d(I(p_1), q_2).$$

(3) If $q_1 = I(p_2)$, then

$$d(p_1, q_1) + d(p_2, q_2) \geq d(p_1, q_1) + d(I(p_2), I(q_2)) \geq d(p_1, I(q_2)).$$

(4) If $I(q_1) = I(p_2)$, then

$$d(p_1, q_1) + d(p_2, q_2) \geq d(I(p_1), I(q_1)) + d(I(p_2), I(q_2)) \geq d(I(p_1), I(q_2)).$$

As $I(q_2) \sim q_2$, we may proceed by induction on n , thus concluding that there are $P, Q \in X$ such that $P \sim x$, $Q \sim y$ and

$$d(p_1, q_1) + \cdots + d(p_n, q_n) \geq d(P, Q).$$

Therefore

$$\tilde{d}(\tilde{x}, \tilde{y}) \geq \inf \{d(p, q) : p \sim x, q \sim y\}.$$

Having proved that

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf \{d(p, q) : p \sim x, q \sim y\},$$

we may find $p \sim x$ and $q \sim y$ such that $d(p, q) \leq 2\tilde{d}(\tilde{x}, \tilde{y})$. \square

4.2. An induced semiflow. Assuming that $I(D) \cap D = \emptyset$, then each point in the set $X_\xi = X \setminus (D_\xi \cup D)$ has a representative of the same equivalence class in $X \setminus D_\xi$. This implies that

$$\pi(X_\xi) = \pi(X_\xi \cup D) \tag{4.1}$$

and, by the half-tube condition (see item (1) of Definition 2), this is a compact set. In particular, $\pi(X_\xi)$ with the quotient topology is a compact metric space: indeed, as $X_\xi \cup D$ is a compact metric space and (4.1) holds, then $\pi(X_\xi)$ is a compact pseudometric space; moreover, as D is compact and $I : D \rightarrow X$ is continuous, $\pi(X_\xi)$ is a T_0 space, and so the quotient topology in $\pi(X_\xi)$ is given by a metric; see [2] for more details.

For any $x, y \in X_\xi$ we have $x \sim y$ if and only if $x = y$. This shows that $\pi|_{X_\xi}$ is a continuous bijection (not necessarily a homeomorphism) from X_ξ onto $\pi(X_\xi)$. Then, setting

$$\tilde{\psi}(t, \tilde{x}) = \pi(\psi(t, x)) \tag{4.2}$$

for each $x \in X_\xi$ and $t \geq 0$, we have that

$$\tilde{\psi} : \mathbb{R}^+ \times \pi(X_\xi) \rightarrow \pi(X_\xi)$$

is well defined and obviously satisfies for all $t \geq 0$

$$\tilde{\psi}_t \circ \pi|_{X_\xi} = \pi \circ \psi_t|_{X_\xi}. \tag{4.3}$$

In what follows we will show that $\tilde{\psi}$ is continuous.

Lemma 4.2. *Assume that τ^* is continuous and $I(D) \cap D = \emptyset$. Then $\pi \circ \psi_t|_{X_\xi}$ is continuous for all $t \geq 0$.*

Proof. Given $t > 0$, let us prove the continuity of $\pi \circ \psi_t|_{X_\xi}$ at any point $x \in X_\xi$. By an inductive argument on the impulsive times of x , it is enough to show that, when $y \in X_\xi$ is close to x , then $\pi(\psi_s(y))$ remains close to $\pi(\psi_s(x))$ for all $0 \leq s \leq \tau_1(x)$. Notice that such an inductive argument on the impulsive times can be applied because we are sure that $I(D) \subset X_\xi \setminus D$. The proof follows according to several cases:

Case 1. $\tau_1(x) > t$.

As τ^* is continuous and τ_1 coincides with τ^* in X_ξ , we must have $\tau_1(y) > t$ for any point $y \in X_\xi$ sufficiently close to x . Therefore, the result follows in this case from the continuity of the semiflow φ .

Case 2. $\tau_1(x) \leq t$.

Given $y \in X_\xi$ sufficiently close to x , by the continuity of the semiflow φ the ψ -trajectories of x and y remain close until one of them hits the set D . At this moment the impulsive function acts and, therefore, their ψ -trajectories may not remain close at this first impulsive time. Now we distinguish three possible subcases:

Subcase 2.1. $\tau_1(x) = \tau_1(y)$.

The continuous map I keeps the points $I(\varphi_{\tau_1(x)}(x))$ and $I(\varphi_{\tau_1(x)}(y))$ close, and this implies that $\psi_s(x)$ and $\psi_s(y)$ remain close for all $0 \leq s \leq \tau_1(x)$.

Subcase 2.2. $\tau_1(x) < \tau_1(y)$.

By the continuity of φ we have $\varphi_s(y)$ close to $\varphi_s(x)$ for y sufficiently close to x and $0 \leq s \leq \tau_1(x)$. This in particular implies that $\psi_s(y)$ is close to $\psi_s(x)$ for $0 \leq s < \tau_1(x)$. It remains to check that $\pi(\psi_{\tau_1(x)}(y))$ is close to $\pi(\psi_{\tau_1(x)}(x))$. This is clearly true because $\varphi_{\tau_1(x)}(y)$ is close to $\varphi_{\tau_1(x)}(x)$, and so

$$\pi(\psi_{\tau_1(x)}(y)) = \pi(\varphi_{\tau_1(x)}(y))$$

is close to

$$\pi(\varphi_{\tau_1(x)}(x)) = \pi(I(\varphi_{\tau_1(x)}(x))) = \pi(\psi_{\tau_1(x)}(x)).$$

Subcase 2.3. $\tau_1(x) > \tau_1(y)$.

Again, by the continuity of φ , we have $\psi_s(y)$ is close to $\psi_s(x)$ for $0 \leq s < \tau_1(y)$. We are left to verify that $\pi(\psi_s(y))$ is close to $\pi(\psi_s(x))$ for $\tau_1(y) \leq s \leq \tau_1(x)$.

By the definition of first impulsive time we have $\varphi_{\tau_1(y)}(y) \in D$; so, as we are assuming that $I(D) \cap D = \emptyset$ and have chosen $\xi < a/2$, we know that $\psi_{\tau_1(y)}(y) = I(\varphi_{\tau_1(y)}(y)) \in X_\xi$, which, by (3.1), yields

$$\tau_1(\psi_{\tau_1(y)}(y)) \geq \rho.$$

Using that τ^* is continuous at x , we have $\tau^*(x) - \tau^*(y)$ small for y close to x ; we may ensure, in particular, that

$$\tau^*(x) - \tau^*(y) < \rho.$$

Hence, for $\tau_1(y) \leq s \leq \tau_1(x)$, we have

$$\psi_s(y) = \varphi_{s-\tau_1(y)}(\psi_{\tau_1(y)}(y)) = \varphi_{s-\tau_1(y)}(I(\varphi_{\tau_1(y)}(y))).$$

Observing that $s - \tau_1(y) \leq \tau_1(x) - \tau_1(y)$ is close to 0 for y close to x , we have

$$\varphi_{s-\tau_1(y)}(I(\varphi_{\tau_1(y)}(y))) \quad \text{close to} \quad I(\varphi_{\tau_1(y)}(y)).$$

Hence for $\tau_1(y) \leq s \leq \tau_1(x)$ we have

$$\pi(\psi_s(y)) \quad \text{close to} \quad \pi(I(\varphi_{\tau_1(y)}(y))) = \pi(\varphi_{\tau_1(y)}(y)).$$

Now we just need to notice that, for $\tau_1(y) \leq s \leq \tau_1(x)$, we have $\varphi_{\tau_1(y)}(y)$ close to $\varphi_s(y)$ which is itself close to $\varphi_s(x)$. This way, we get, for $\tau_1(y) \leq s \leq \tau_1(x)$,

$$\pi(\varphi_{\tau_1(y)}(y)) \quad \text{close to} \quad \pi(\varphi_s(x)) = \pi(\psi_s(x)).$$

Lastly, recall that for $s = \tau_1(x)$ we have $\pi(\varphi_{\tau_1(x)}(x)) = \pi(I(\varphi_{\tau_1(x)}(x))) = \pi(\psi_{\tau_1(x)}(x))$. \square

Proposition 4.3. *The semiflow $\tilde{\psi} : \mathbb{R}_0^+ \times \pi(X_\xi) \rightarrow \pi(X_\xi)$ is continuous.*

Proof. Considering for each $\tilde{x} \in \pi(X_\xi)$ the map $\tilde{\psi}^{\tilde{x}} : \mathbb{R}_0^+ \rightarrow \pi(X_\xi)$ defined by

$$\tilde{\psi}^{\tilde{x}}(t) = \tilde{\psi}(t, \tilde{x}),$$

it is enough to prove that $\tilde{\psi}^{\tilde{x}}$ and $\tilde{\psi}_t$ are continuous for all $\tilde{x} \in \pi(X_\xi)$ and all $t \geq 0$.

Let us start by proving the continuity of $\tilde{\psi}^{\tilde{x}}$ for $x \in X_\xi$. Take first $t_0 \geq 0$ which is not an impulsive time for x . In this case we have, for t in a sufficiently small neighborhood of t_0 in \mathbb{R}_0^+ ,

$$\tilde{\psi}^{\tilde{x}}(t) = \pi(\varphi(t, x))$$

and, as φ is continuous, this obviously gives the continuity of $\tilde{\psi}^{\tilde{x}}$ at t_0 . On the other hand, if t_0 is an impulsive time for x , then we have

$$\lim_{t \rightarrow t_0^-} \tilde{\psi}^{\tilde{x}}(t) = \lim_{t \rightarrow t_0^-} \pi(\psi(t, x)) = \lim_{t \rightarrow t_0^-} \pi(\varphi(t, x)) = \pi(\varphi(t_0, x)).$$

As $\varphi(t_0, x) \in D$, it follows from the definition of $\psi(t_0, x)$ and the equivalence relation that yields the projection π that

$$\pi(\varphi(t_0, x)) = \pi(I(\varphi(t_0, x))) = \pi(\psi(t_0, x)) = \tilde{\psi}^{\tilde{x}}(t_0).$$

This gives the continuity of $\tilde{\psi}^{\tilde{x}}$ on the left hand side of t_0 . The continuity on the right hand side of t_0 follows easily from the fact that, by definition, the impulsive trajectories are continuous on the right hand side.

Let us now prove the continuity of $\tilde{\psi}_t$ for $t \geq 0$. Notice that as we are considering the quotient topology in $\pi(X_\xi)$, we know that $\tilde{\psi}_t$ is continuous if and only if $\tilde{\psi}_t \circ \pi|_{X_\xi}$ is continuous. The continuity of $\tilde{\psi}_t \circ \pi|_{X_\xi}$ is an immediate consequence of Lemma 4.2 and (4.3). \square

4.3. Proof of Theorem B. We take $\tilde{X} = \pi(X_\xi)$, the semiflow $\tilde{\psi}$ as given in (4.2) and the map $h : X_\xi \rightarrow \tilde{X}$ given by $h(x) = \pi(x)$ for all $x \in X_\xi$. It follows from (4.3) and Proposition 4.3 that $\tilde{\psi}_t \circ h = h \circ \psi_t$ for all $t \geq 0$. Thus, we are left to prove that

$$h_{\text{top}}^\tau(\psi) = h_{\text{top}}(\tilde{\psi}).$$

Let $f : X_\xi \rightarrow \pi(X_\xi)$ be given by $f(x) = \pi(x)$. The map f is a continuous bimeasurable (see [8]) bijection, so, using it, we define admissible functions $\tilde{\tau}$ and $\tilde{\tau}'$ for the semiflow $\tilde{\psi}$ as

$$\tilde{\tau}(f(x)) = \tau(x) \quad \text{and} \quad \tilde{\tau}'(f(x)) = \tau'(x).$$

Notice that f is a $(\tau, \tilde{\tau})$ -semiconjugacy between $\psi|_{X_\xi}$ and $\tilde{\psi}$.

As $I^{-1}(D) = \emptyset$, we have, for all $x \in D$,

$$\pi^{-1}(\{\pi(x)\}) = \{x, I(x)\} \cup I^{-1}(\{I(x)\}).$$

So, we may restrict f to $X_\xi \setminus I(D)$ and define the map

$$g : X_\xi \setminus I(D) \rightarrow \pi(X_\xi) \setminus \pi(D), \quad g(x) = f(x).$$

Observe that, as g is a restriction of π and π is uniformly continuous on X , then g is uniformly continuous as well.

Lemma 4.4. *If I is 1-Lipschitz, then g^{-1} is uniformly continuous.*

Proof. As in $X_\xi \setminus I(D)$ each equivalence class in X has only one member, we deduce from Lemma 4.1 that, for all $\tilde{x}, \tilde{y} \in \pi(X_\xi) \setminus \pi(D)$, we have

$$d(x, y) \leq 2\tilde{d}(\tilde{x}, \tilde{y}).$$

This in turn implies that g^{-1} is uniformly continuous. \square

After Lemma 4.4, we define

$$\tilde{\tau}'(g(x)) = \tau'(x)$$

and this way g is a $(\tau', \tilde{\tau}')$ -semiconjugacy between $\psi|_{X_\xi \setminus I(D)}$ and $\tilde{\psi}|_{\pi(X_\xi) \setminus \pi(D)}$, and g^{-1} is a $(\tilde{\tau}', \tau')$ -semiconjugacy between $\tilde{\psi}|_{\pi(X_\xi) \setminus \pi(D)}$ and $\psi|_{X_\xi \setminus I(D)}$.

Lemma 4.5. $h_{\text{top}}^{\tilde{\tau}'}(\tilde{\psi}) = h_{\text{top}}^{\tau'}(\psi|_{X_\xi})$.

Proof. From Lemma 2.3 applied to the semiconjugacy $f : X_\xi \rightarrow \pi(X_\xi)$ we deduce that

$$h_{\text{top}}^{\tilde{\tau}'}(\tilde{\psi}) \leq h_{\text{top}}^{\tau'}(\psi|_{X_\xi}).$$

Conversely, Lemma 4.4 ensures that the map g^{-1} is a $(\tilde{\tau}', \tau')$ -semiconjugacy between $\tilde{\psi}|_{\pi(X_\xi) \setminus \pi(D)}$ and $\psi|_{X_\xi \setminus I(D)}$, and so we get from Lemma 2.3

$$h_{\text{top}}^{\tilde{\tau}'}(\tilde{\psi}|_{\pi(X_\xi) \setminus \pi(D)}) \geq h_{\text{top}}^{\tau'}(\psi|_{X_\xi \setminus I(D)}).$$

As $h_{\text{top}}^{\tau'}(\psi|_{X_\xi \setminus I(D)}) = h_{\text{top}}^{\tau'}(\psi|_{X_\xi})$ (see Remark 3.3), we conclude that

$$h_{\text{top}}^{\tilde{\tau}'}(\tilde{\psi}) \geq h_{\text{top}}^{\tilde{\tau}'}(\tilde{\psi}|_{\pi(X_\xi) \setminus \pi(D)}) \geq h_{\text{top}}^{\tau'}(\psi|_{X_\xi \setminus I(D)}) = h_{\text{top}}^{\tau'}(\psi|_{X_\xi}).$$

\square

Lemma 4.6. $h_{\text{top}}(\tilde{\psi}) = h_{\text{top}}^{\tau'}(\psi)$.

Proof. Firstly, by Theorem A and Lemma 4.5 we get

$$h_{\text{top}}(\tilde{\psi}) = h_{\text{top}}^{\tilde{\tau}'}(\tilde{\psi}) = h_{\text{top}}^{\tau'}(\psi|_{X_\xi}).$$

Then, by Lemma 3.2 we obtain

$$h_{\text{top}}^{\tau'}(\psi|_{X_\xi}) = h_{\text{top}}^{\tau'}(\psi).$$

\square

To conclude the proof of Theorem B, we have just to notice that from Lemma 3.1 we get

$$h_{\text{top}}^{\tau}(\psi) = h_{\text{top}}^{\tau'}(\psi)$$

and by Lemma 4.6 we deduce that

$$h_{\text{top}}^{\tau'}(\psi) = h_{\text{top}}^{\tau'}(\tilde{\psi}).$$

4.4. Proof of Theorem C. Firstly, Theorem B ensures that

$$h_{\text{top}}^{\tau}(\psi) = h_{\text{top}}^{\tau}(\tilde{\psi}).$$

Additionally, by Proposition 4.3, we may apply the Variational Principle [4, 9] to $\tilde{\psi}$, getting

$$h_{\text{top}}^{\tau}(\tilde{\psi}) = \sup \{h_{\nu}(\tilde{\psi}_1) : \nu \in \mathcal{M}_{\tilde{\psi}}(\pi(X_{\xi}))\}.$$

To conclude the proof of Theorem C, we are due to connect the measure theoretical information of $\tilde{\psi}$ to the corresponding one of ψ , and to ascertain that we may replace X_{ξ} by X in the previous equality. Accordingly, we will start verifying that the space restriction $X_{\xi} = X \setminus (D_{\xi} \cup D)$ is negligible within the measure theoretical context we are dealing with.

Lemma 4.7. *Let μ be a probability measure invariant by the semiflow ψ .*

- (a) *If $I(D) \cap D = \emptyset$, then $\mu(D) = 0$.*
- (b) *If $I(D) \cap D = \emptyset$ and D satisfies a half-tube condition, then $\mu(D_{\xi}) = 0$.*

Proof. Supposing that $\mu(D) > 0$, it follows from Poincaré Recurrence Theorem that for μ almost every $x \in D$ there are infinitely many moments $t > 0$ such that $\psi_t(x) \in D$. Clearly, if $I(D) \cap D = \emptyset$, then the ψ -trajectories do not hit D for $t > 0$, and so we arrive at a contradiction.

Assume that $\mu(D_{\xi}) > 0$. Then, for μ almost every $x \in D_{\xi}$, there are infinitely many times $t > 0$ such that $\psi_t(x) \in D_{\xi}$. Yet, as $I(D) \cap D = \emptyset$ and D satisfies a half-tube condition (see item (2) of Definition 2), no φ -trajectory enters D_{ξ} unless it has previously crossed D ; and, once at D , it is sent, by the impulsive semiflow ψ , to $I(D)$. Thus, no ψ -trajectory comes back to D_{ξ} for $t > 0$; this way we reach a contradiction. \square

We remark that, as D_{ξ} is an open set, the proof of the previous lemma also shows that $\Omega_{\psi} \subset X_{\xi} \cup D$.

Let us now exchange ergodic data between $\tilde{\psi}$ and ψ . Consider the continuous bimeasurable bijection

$$\begin{aligned} f : X_{\xi} &\longrightarrow \pi(X_{\xi}) \\ x &\longmapsto \pi(x) \end{aligned}$$

and the inclusion map $i : X_{\xi} \rightarrow X$. In the next two lemmas we follow the strategies used to prove [2, Lemmas 5.2 & 5.3].

Lemma 4.8. *$(i \circ f^{-1})_* : \mathcal{M}_{\tilde{\psi}}(\pi(X_{\xi})) \longrightarrow \mathcal{M}_{\psi}(X_{\xi})$ is well defined and is a bijection.*

Proof. To see that $(i \circ f^{-1})_*$ is well defined, we need to check that if $\nu \in \mathcal{M}_{\tilde{\psi}}(\pi(X_\xi))$, then one necessarily has $(i \circ f^{-1})_*\nu \in \mathcal{M}_\psi(X_\xi)$. Now, from

$$f \circ \psi_t = \tilde{\psi}_t \circ f, \quad \text{for all } t \geq 0,$$

or equivalently

$$\psi_t \circ f^{-1} = f^{-1} \circ \tilde{\psi}_t, \quad \text{for all } t \geq 0 \quad (4.4)$$

we clearly have that

$$\nu \in \mathcal{M}_{\tilde{\psi}}(\pi(X_\xi)) \Rightarrow (f^{-1})_*\nu \in \mathcal{M}_\psi(X_\xi) \Rightarrow i_*(f^{-1})_*\nu \in \mathcal{M}_\psi(X).$$

Finally, as $i_*(f^{-1})_*\nu = (i \circ f^{-1})_*\nu$, we conclude that $(i \circ f^{-1})_*\nu \in \mathcal{M}_\psi(X)$. This shows that $(i \circ f^{-1})_*$ is well defined.

It remains to check that $(i \circ f^{-1})_*$ is bijective. As $(i \circ f^{-1})_* = i_* \circ f_*^{-1}$ and $(f^{-1})_*$ is invertible, we only need to prove that i_* is invertible. Clearly, being injective, i_* has a left inverse; this implies that i_* has a left inverse. Thus i_* is injective as well.

To prove that i_* is surjective, given $\mu \in \mathcal{M}_\psi(X)$, let ν be the restriction of μ to the Borel subsets of X_ξ . Noticing that the support of μ is contained in Ω_ψ , that $\Omega_\psi \subseteq X_\xi \cup D$ and that $\mu(D) = 0$, we know that $\nu \in \mathcal{M}_\psi(X_\xi)$. Using the ψ -invariance of μ , we also deduce that, for any Borel set $A \subset X_\xi$,

$$\nu(\psi_t^{-1}(A)) = \mu(\psi_t^{-1}(A) \cap (X_\xi)) = \mu(\psi_t^{-1}(A)) = \mu(A) = \mu(A \cap (X_\xi)) = \nu(A).$$

Moreover, for any Borel set $A \subset X$, we have

$$i_*\nu(A) = \nu(i^{-1}(A)) = \nu(A \cap (X_\xi)) = \mu(A \cap (X_\xi)) = \mu(A).$$

Consequently, $\nu \in \mathcal{M}_\psi(X_\xi)$ and $i_*\nu = \mu$. □

So, Lemma 4.8 ensures that

$$h_{\text{top}}(\tilde{\psi}) = \sup \{h_\mu(\psi_1) : \mu \in \mathcal{M}_\psi(X_\xi)\}.$$

Besides, from Lemma 4.7 we get

$$\sup \{h_\mu(\psi_1) : \mu \in \mathcal{M}_\psi(X_\xi)\} = \sup \{h_\mu(\psi_1) : \mu \in \mathcal{M}_\psi(X)\}.$$

Hence,

$$h_{\text{top}}^\tau(\psi) = \sup \{h_\mu(\psi_1) : \mu \in \mathcal{M}_\psi(X)\}.$$

5. AN EXAMPLE

Consider the phase space

$$X = \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2 : 1 \leq r \leq 2, \theta \in [0, 2\pi]\}$$

and define $\varphi : \mathbb{R}_0^+ \times X \rightarrow X$ as the semiflow of the vector field in X given by

$$\begin{cases} r' = 0 \\ \theta' = 1. \end{cases}$$

The trajectories of φ are circles spinning counterclockwise around zero. Take now

$$D = \{(r, 0) \in X : 1 \leq r \leq 2\}$$

and the map

$$I : (r, 0) \in D \mapsto I(r, 0) = \left(-\frac{1}{2} - \frac{1}{2}r, 0\right)$$

whose Lipschitz constant is $1/2$.

The non-wandering set of the semiflow ψ of the impulsive dynamical system (X, φ, D, I) is

$$\Omega_\psi = \{(\cos \theta, \sin \theta) : \pi \leq \theta \leq 2\pi\}$$

and, by [2, Theorem A], ψ has some invariant probability measure. For a suitably small $\xi > 0$,

$$\begin{aligned} D_\xi &= \{(r \cos \theta, r \sin \theta) : 1 \leq r \leq 2, 0 < \theta < 2\pi\xi\} \\ X_\xi &= \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2 : 1 \leq r \leq 2, \theta \in [2\pi\xi, 2\pi)\}. \end{aligned}$$

As regards the requirements of Theorem C, we have, $I(D) \cap D = \emptyset$; the set D satisfies a half-tube condition (with $0 < \xi_0 \leq 1/2$); the function $\tau^* : X_\xi \cup D \rightarrow [0, 2\pi]$ is given by

$$\tau^*(x) = \begin{cases} 2\pi - \theta, & \text{if } x = (r \cos \theta, r \sin \theta) \in X_\xi; \\ 0, & \text{if } x \in D \end{cases}$$

so it is continuous; whenever $\psi_t(x) \in I(D)$, we have $\{\psi_{t+s}(x) : 0 < s < \pi\} \cap I(D) = \emptyset$; and, finally, $I(\Omega_\psi \cap D) = I(\{(-1, 0), (1, 0)\}) = \{(-1, 0)\} \subset \Omega_\psi \setminus D$.

The equality (4.3) and Proposition 4.3 build a conjugacy between $\psi|_{\Omega_\psi \setminus D}$ and the semiflow $\tilde{\psi}$ on the quotient structure where it acts as a circle rotation. Therefore

$$h_{\text{top}}^\tau(\psi) = \sup \{h_\mu(\psi_1) : \mu \in \mathcal{M}_\psi(X)\} = 0.$$

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JOSÉ F. ALVES, CENTRO DE MATEMÁTICA DA UNIVERSIDADE DO PORTO, RUA DO CAMPO ALEGRE 687, 4169-007 PORTO, PORTUGAL

E-mail address: `jfalves@fc.up.pt`

URL: `http://www.fc.up.pt/cmup/jfalves`

MARIA CARVALHO, CENTRO DE MATEMÁTICA DA UNIVERSIDADE DO PORTO, RUA DO CAMPO ALEGRE 687, 4169-007 PORTO, PORTUGAL

E-mail address: `mpcarval@fc.up.pt`

CARLOS VÁSQUEZ, INSTITUTO DE MATEMÁTICA, PONTIFICIA UNIVERSIDAD CATÓLICA DE VALPARAÍSO, VALPARAÍSO, CHILE

E-mail address: `carlos.vasquez@ucv.cl`