

GROMOV-WITTEN INVARIANTS OF STABLE MAPS WITH FIELDS

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ABSTRACT. We construct the Gromov-Witten invariants of moduli of stable morphisms to \mathbb{P}^4 with fields. This is the all genus mathematical theory of the Guffin-Sharp-Witten model, and is a modified twisted Gromov-Witten invariants of \mathbb{P}^4 . These invariants are constructed using the cosection localization of Kiem-Li, an algebro-geometric analogue of Witten's perturbed equations in Landau-Ginzburg theory. We prove that these invariants coincide, up to sign, with the Gromov-Witten invariants of quintics.

1. INTRODUCTION

The Candelas-dela Ossa-Green-Parkes' genus zero generating function [Ca] of the Gromov-Witten invariants of quintic Calabi-Yau threefolds was proved by Givental [Gi] and Lian-Liu-Yau [LLY]; the genus one generating function of Bershadsky-Cecotti-Ooguri-Vafa's [BCOV] was proved by Zinger [Zi]. Both proofs rely on the "hyperplane property" of the Gromov-Witten invariants of quintics, which expresses the invariants in terms of "Euler class of bundles" over the moduli of stable morphisms to \mathbb{P}^4 . The hyperplane property for genus zero was derived by Kontsevich [Ko]; the case of genus one was proved by Li-Zinger [LZ]. This paper is our first step to build such a theory for all genus Gromov-Witten invariants of quintics, and beyond.

In this paper, we introduce a new class of moduli spaces: the moduli of stable morphisms to \mathbb{P}^4 with fields. These moduli spaces are cones over the usual moduli of stable morphisms to \mathbb{P}^4 ; they are not proper for positive genus. We use Kiem-Li's cosection localized virtual cycle to construct their localized virtual cycles, thus their Gromov-Witten invariants. Applying degeneration, we prove that these invariants coincide (up to signs) with the Gromov-Witten invariants of the quintics.

We briefly outline our construction and the main theorem. Given non-negative integers g and d , we form the moduli $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ of genus g degree d stable morphisms to \mathbb{P}^4 with p -fields:

$$\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p = \{[u, C, p] \mid [u, C] \in \overline{\mathcal{M}}_g(\mathbb{P}^4, d), p \in \Gamma(C, u^* \mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_C)\} / \sim.$$

Here $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$ is the moduli of degree d genus g stable morphisms to \mathbb{P}^4 .

It is a Deligne-Mumford stack; forgetting the fields, the induced morphism

$$\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p \rightarrow \overline{\mathcal{M}}_g(\mathbb{P}^4, d)$$

has fiber $H^0(u^* \mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_C)$ over $[u, C] \in \overline{\mathcal{M}}_g(\mathbb{P}^4, d)$. When g is positive, it is not proper.

The moduli space $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ has a perfect obstruction theory, thus has a virtual class. To overcome its non-properness in order to define its Gromov-Witten invariant, we construct a cosection (homomorphism) of its obstruction sheaf. The choice

of the cosection depends on the choice of a degree five homogeneous polynomial, like $\mathbf{w} = x_1^5 + \dots + x_5^5$. The non-surjective loci (called the degeneracy loci) of the cosection associated to \mathbf{w}

$$\sigma : \mathcal{O}_{b\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p} \longrightarrow \mathcal{O}_{\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p}$$

is

$$\overline{\mathcal{M}}_g(Q, d) \subset \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p, \quad Q = (x_1^5 + \dots + x_5^5 = 0) \subset \mathbb{P}^4,$$

which is proper. Applying Kiem-Li cosection localized virtual class construction, we obtain a localized virtual cycle

$$[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]_{\sigma}^{\text{vir}} \in A_0 \overline{\mathcal{M}}_g(Q, d).$$

We define the Gromov-Witten invariant of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ be

$$N_g(d)_{\mathbb{P}^4}^p = \deg[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]_{\sigma}^{\text{vir}}.$$

(We also call them the Gromov-Witten invariants of the space $(K_{\mathbb{P}^4}, \mathbf{w})$.)

It relates to the Gromov-Witten invariants the quintic Q :

Theorem 1.1. *For $g \geq 0$ and $d > 0$, the Gromov-Witten invariant of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ (or $(K_{\mathbb{P}^4}, \mathbf{w})$) coincides with the Gromov-Witten invariant $N_g(d)_Q$ of the quintic Q up to a sign:*

$$N_g(d)_{\mathbb{P}^4}^p = (-1)^{5d+1-g} N_g(d)_Q.$$

When $g = 0$, this is derived in Guffin-Sharpe [GS] using path-integral. This identity also is the Kontsevich's formula on $g = 0$ Gromov-Witten invariants of quintics. If one views the localized virtual cycle of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$ as ‘‘Euler class of bundles’’, this theorem is a substitute of the ‘‘hyperplane property’’ of the Gromov-Witten invariants of quintics in high genus.

We believe this construction will lead to a mathematical approach to Witten's Gauged-Linear-Sigma model for all genus. In [Wi1], Witten constructed a (gauged) topological field theory (for $g = 0$) whose target is the stacky quotient $[\mathbb{C}^6/\mathbb{C}^*]$ (of weights $(1, 1, 1, 1, 1, -5)$) with a superpotential, say \mathbf{w} . This theory has two GIT quotients: one is $(K_{\mathbb{P}^4}, \mathbf{w})$, called the massive theory; the other is $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$, called the linear Landau-Ginzberg model.¹ Witten proposed to A-twist both models: the A-twist of $(K_{\mathbb{P}^4}, \mathbf{w})$ likely is a theory of moduli of stable quotients, and the resulting theory is of Landau-Ginzburg type. The A-twist of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$ is related to the generalized Witten conjecture [Wi2] for $A_4 = (\mathbb{C}, x^5)$.

The program proposed in [Wi1] provides a possible road map towards an all genus mathematical theory linking the Gromov-Witten theory of quintic to the Landau-Ginzburg model of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$. A bolder speculation is that there is a geometric mirror construction identifying the A-twisted topological string theory of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$ with the B-side invariants of its Landau-Ginzburg Mirror.

In [FJRW], Fan, Jarvis and Ruan constructed the virtual cycle of the A-twisted topological string theories of the linear Landau-Ginzburg model of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$; their construction is via analytic perturbation of Witten's equation. Later, Ruan and Chiodo proved [CR] the genus zero mirror symmetry for $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$ and its mirror.

For massive theory of $(K_{\mathbb{P}^4}, \mathbf{w})$, Marian, Oprea and Pandharipande constructed the moduli of stable quotients [MOP], which is believed to be an example of massive

¹Linear Landau-Ginzburg model means the space is the orbifold quotient of an affine space.

instantons. It is interesting to see how the invariants of A -twisting the construction in [MOP] relate to the invariants of the massive instantons in $(K_{\mathbb{P}^4}, \mathbf{w})$ in Witten's program.

Using Super-String theories, Guffin and Sharpe constructed a special type of genus zero Landau-Ginzberg model for $(K_{\mathbb{P}^4}, \mathbf{w})$, and equated it with the genus zero Gromov-Witten invariants of the quintic Q [GS]. The notion of p -fields was introduced in this work. Using non-perturbative localization of path-integral, they reduced this theory to the genus zero Gromov-Witten invariants of quintics. Since this follows Witten's Gauged-Linear-Sigma-Model program, we call this construction the Guffin-Sharpe-Witten model.

Our work is an algebro-geometric construction of Guffin-Sharpe-Witten model for all genus. The moduli of stable morphisms with p -fields is the algebro-geometric substitute of the phase space of all smooth maps with smooth fields. The cosection localized virtual cycle is the analogue of Witten's perturbed equation. Theorem 1.1 shows that the Gromov-Witten invariants of the algebro-geometric Guffin-Sharpe-Witten model of all genus coincide up to signs with the Gromov-Witten invariants of quintic threefolds.

Our construction applies to global complete intersection Calabi-Yau threefolds of toric varieties. In the subsequent papers, we will apply the techniques developed to the moduli of stable quotients (cf. [MOP]) to obtain all genus invariants of massive theory of $(K_{\mathbb{P}^4}, \mathbf{w})$ [CL]; we will also apply it to the linear Landau-Ginzberg model to obtain an alternative algebro-geometric construction of Fan-Jarvis-Ruan-Witten invariants [CLL]. In the later case, the resulting invariants are equal to those defined using perturbed the Witten equations [FJRW].

We believe the new invariants and their equivalence with the Gromov-Witten invariants of quintics provide the first step toward building a geometric bridge establishing the conjectural equivalence of Gromov-Witten invariants of quintics and the Fan-Jarvis-Ruan-Witten invariants of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w})$. Constructing such bridge will be the long term goal of this project.

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Conventions. In this paper, the primary focus is on moduli of stable morphisms with fields to \mathbb{P}^4 , to a smooth quintic Calabi-Yau $Q \subset \mathbb{P}^4$ defined by $\sum x_i^5 = 0$, and a deformation of \mathbb{P}^4 to the normal cone to $Q \subset \mathbb{P}^4$.

Throughout the paper, we fix a homogeneous coordinates $[x_1, \dots, x_5]$ of \mathbb{P}^4 , with $x_i \in H^0(\mathbb{P}^4, \mathcal{O}(1))$ and $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^4}(1)$. We denote by N the normal bundle to Q in \mathbb{P}^4 . Using the defining section $\sum x_i^5 = 0$, we obtain a canonical isomorphism $N \cong \mathcal{O}_Q(5)$.

In this paper, we will fix positive integers g and d throughout. We will use (f, \mathcal{C}) with subscripts to denote the universal families of various moduli spaces. For instance, after abbreviating $\mathcal{P} = \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$, the universal curve and map of \mathcal{P} is denoted by

$$(f_{\mathcal{P}}, \pi_{\mathcal{P}}) : \mathcal{C}_{\mathcal{P}} \longrightarrow \mathbb{P}^4 \times \mathcal{P}.$$

For any locally free sheaf \mathcal{L} on \mathcal{C} , we denote by $\mathrm{Vb}(\mathcal{L})$ the underlying vector bundle of \mathcal{L} ; namely, the sheaf of sections of $\mathrm{Vb}(\mathcal{L})$ is \mathcal{L} .

In this paper, we will use fonts \mathbb{E} , etc. to denote derived objects (of complexes). We reserve $\mathbb{L}_{X/Y}$ to denote the cotangent complex of $X \rightarrow Y$; we denote by $\mathbb{T}_{X/Y}$ its derived dual $\mathbb{T}_{X/Y} = \mathbb{L}_{X/Y}^\vee$, called the tangent complex of $X \rightarrow Y$. We use $\phi_{X/Y} : \mathbb{T}_{X/Y} \rightarrow \mathbb{E}_{X/Y}$ to denote a relative obstruction theory of $X \rightarrow Y$, following Behrend-Fantechi [BF].

Without causing confusion, all pull back of derived objects (resp. sheaves) are derived pull back (resp. sheaves pull back) unless otherwise stated.

2. DIRECT IMAGE CONES AND MODULI OF SECTIONS

In this section, to a locally free sheaf \mathcal{L} over a family of nodal curves $\pi : \mathcal{C} \rightarrow \mathfrak{A}$ over an Artin stack \mathfrak{A} , we will construct its direct image cone $C(\pi_*\mathcal{L})$, and its relative obstruction theory.

2.1. Direct image cones. Let \mathfrak{A} be an Artin stack, $\pi : \mathcal{C} \rightarrow \mathfrak{A}$ be a flat family of connected, nodal, arithmetic genus g curves, and \mathcal{L} a locally free sheaf on \mathcal{C} .

Definition 2.1. For any scheme S , we define $C(\pi_*\mathcal{L})(S)$ be the collection of (ρ, p) so that $\rho : S \rightarrow \mathfrak{A}$ is a morphism and $p \in H^0(\mathcal{C}_S, \rho^*\mathcal{L})$, where $\mathcal{C}_S = S \times_{\mathfrak{A}} \mathcal{C}$ and $\rho^*\mathcal{L} = \mathcal{L} \times_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}_{\mathcal{C}_S}$.

An arrow from (ρ, p) to (ρ', p') in $C(\pi_*\mathcal{L})(S)$ consists of an arrow $\tau : \mathcal{C}_S \rightarrow \mathcal{C}_{S'}$ in $\mathfrak{A}(S)$ such that under the induced isomorphism $\tau^*\rho'^*\mathcal{L} \cong \rho^*\mathcal{L}$, $p = \tau^*p'$. Given $S \rightarrow S'$, we define $C(\pi_*\mathcal{L})(S') \rightarrow C(\pi_*\mathcal{L})(S)$ by pull back.

We show that $C(\pi_*\mathcal{L})$ is a stack over \mathfrak{A} . Given a module \mathcal{F} , we denote by $\mathrm{Sym}\mathcal{F}$ the algebra of symmetric product of \mathcal{F} .

Proposition 2.2. Let the notation be as in Definition 2.1. We have canonical \mathfrak{A} -isomorphism

$$C(\pi_*\mathcal{L}) \cong \mathrm{Spec}_{\mathfrak{A}} \mathrm{Sym} R^1 \pi_*(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}/\mathfrak{A}}).$$

Proof. For any scheme S and a morphism $\rho : S \rightarrow \mathfrak{A}$, we let

$$C(\pi_*\mathcal{L})(\rho) = \{(\rho, p) \mid p \in H^0(\mathcal{C}_S, \rho^*\mathcal{L})\} \cong \Gamma(\mathcal{C}_S, \rho^*\mathcal{L}).$$

We define a transformation

$$(2.1) \quad C(\pi_*\mathcal{L})(\rho) \longrightarrow \mathrm{Hom}_S(S, \mathrm{Spec}_{\mathfrak{A}} \mathrm{Sym} R^1 \pi_*(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}/\mathfrak{A}}) \times_{\mathfrak{A}} S)$$

as follows. We let $\mathcal{F} = R^1 \pi_*(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}/\mathfrak{A}})$. Given a $\rho : S \rightarrow \mathfrak{A}$, an S -morphism $S \rightarrow \mathrm{Spec}_{\mathfrak{A}} \mathrm{Sym}\mathcal{F} \times_{\mathfrak{A}} S$ is given by a morphism of sheaves of $\mathcal{O}_{\mathfrak{A}}$ -algebra

$$\mathrm{Sym}\mathcal{F} \longrightarrow \mathcal{O}_S,$$

which is equivalent to a morphism of sheaves of $\mathcal{O}_{\mathfrak{A}}$ -modules

$$R^1 \pi_{S*}(\mathcal{L}_S^\vee \otimes \omega_{\mathcal{C}_S/S}) = \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{A}}} \mathcal{O}_S \longrightarrow \mathcal{O}_S.$$

Here we have used the base change property of $R^1 \pi_*$.

Applying Serre duality [Co] to the complete intersection morphism $\pi_S : \mathcal{C}_S \rightarrow S$, we obtain

$$\mathrm{Hom}_S(R^1 \pi_{S*}(\mathcal{L}_S^\vee \otimes \omega_{\mathcal{C}_S/S}), \mathcal{O}_S) = \Gamma(\mathcal{C}_S, \mathcal{L}_S).$$

This defines the transformation (2.1). It is direct to check that this is an isomorphism, and satisfies base change property. This proves the Proposition. \square

2.2. Moduli of sections. One can also construct the direct image cone via the moduli of sections. Let $\mathcal{C} \rightarrow \mathfrak{A}$ be as in Definition 2.1; let $\mathcal{Z} \rightarrow \mathcal{C}$ be an Artin stack such that the arrow $\mathcal{Z} \rightarrow \mathcal{C}$ is representable and quasi-projective. We define a groupoid \mathfrak{S} (with dependence on \mathcal{Z} implicitly understood) as follows.

For any scheme $S \rightarrow \mathfrak{A}$, we denote $\mathcal{C}_S = \mathcal{C} \times_{\mathfrak{A}} S$ and $\mathcal{Z}_S = \mathcal{Z} \times_{\mathcal{C}} \mathcal{C}_S$; we view \mathcal{Z}_S as a scheme over \mathcal{C}_S via the projection $\pi_S : \mathcal{Z}_S \rightarrow \mathcal{C}_S$. We define

$$\mathfrak{S}(S) = \{s : \mathcal{C}_S \rightarrow \mathcal{Z}_S \mid s \text{ are } \mathcal{C}_S\text{-morphisms}\}.$$

The arrows are defined by pull backs.

Proposition 2.3. *The groupoid \mathfrak{S} is an Artin stack with a natural projection to \mathfrak{A} . The morphism $\mathfrak{S} \rightarrow \mathfrak{A}$ is representable and quasi-projective.*

Proof. This follows from the functorial construction of Hilbert scheme and that $\mathcal{Z} \rightarrow \mathcal{C}$ is representable and quasi-projective. \square

Corollary 2.4. *Let $\pi : \mathcal{C} \rightarrow \mathfrak{A}$ be as in Definition 2.1, and let $\mathcal{Z} = \text{Vb}(\mathcal{L})$, which is the underlying vector bundle of the locally free sheaf \mathcal{L} . Then canonically $\mathcal{C}(\pi_*\mathcal{L}) \cong \mathfrak{S}$ as stacks over \mathfrak{A} .*

2.3. The obstruction theory. We give the perfect obstruction theory of \mathfrak{S} . Let $\mathcal{Z} \rightarrow \mathcal{C} \rightarrow \mathfrak{A}$ be as in Proposition 2.3. Let $\pi_{\mathfrak{S}} : \mathcal{C}_{\mathfrak{S}} \rightarrow \mathfrak{S}$ be the universal family of \mathfrak{S} and let $\epsilon : \mathcal{C}_{\mathfrak{S}} \rightarrow \mathcal{Z}$ be the tautological evaluation map. Namely, $(\pi_{\mathfrak{S}}, \epsilon) : \mathcal{C}_{\mathfrak{S}} \rightarrow \mathfrak{S} \times \mathcal{Z}$ is the universal family of \mathfrak{S} .

As mentioned at the end of the introduction, we let $\mathbb{T}_{\mathfrak{S}/\mathfrak{A}}$ be the tangent complex of $\mathfrak{S} \rightarrow \mathfrak{A}$, which is the dual of the cotangent complex $\mathbb{L}_{\mathfrak{S}/\mathfrak{A}}$.

Proposition 2.5. *Let the situation be as stated. Suppose $\mathcal{Z} \rightarrow \mathcal{C}$ is smooth, then $\mathfrak{S} \rightarrow \mathfrak{A}$ has a perfect relative obstruction theory*

$$\phi_{\mathfrak{S}/\mathfrak{A}} : \mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \longrightarrow \mathbb{E}_{\mathfrak{S}/\mathfrak{A}} := R^{\bullet}\pi_{\mathfrak{S}*}\epsilon^*\Omega_{\mathcal{Z}/\mathcal{C}}^{\vee}.$$

Proof. By our construction, we have the commutative diagrams

$$(2.2) \quad \begin{array}{ccccc} \mathfrak{S} & \longleftarrow & \mathcal{C}_{\mathfrak{S}} & \xrightarrow{\epsilon} & \mathcal{Z} \\ & & \downarrow & & \downarrow \\ \mathfrak{A} & \longleftarrow & \mathcal{C} & \xrightarrow{=} & \mathcal{C}, \end{array}$$

where the left one is Cartesian. Applying the projection formula to

$$(2.3) \quad \pi_{\mathfrak{S}}^*\mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \cong \mathbb{T}_{\mathcal{C}_{\mathfrak{S}}/\mathcal{C}} \longrightarrow \epsilon^*\mathbb{T}_{\mathcal{Z}/\mathcal{C}} = \epsilon^*\Omega_{\mathcal{Z}/\mathcal{C}}^{\vee},$$

and using

$$\mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \longrightarrow R^{\bullet}\pi_{\mathfrak{S}*}\pi_{\mathfrak{S}}^*\mathbb{T}_{\mathfrak{S}/\mathfrak{A}},$$

we obtain

$$(2.4) \quad \phi_{\mathfrak{S}/\mathfrak{A}} : \mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \longrightarrow \mathbb{E}_{\mathfrak{S}/\mathfrak{A}} := R^{\bullet}\pi_{\mathfrak{S}*}\epsilon^*\Omega_{\mathcal{Z}/\mathcal{C}}^{\vee}.$$

We claim that $\phi_{\mathfrak{S}/\mathfrak{A}}$ is a perfect obstruction theory.

We prove this by applying the criterion in [BF, Thm 4.5]. Given an extension $T \subset T'$ by ideal J with $J^2 = 0$, and a commutative diagram

$$(2.5) \quad \begin{array}{ccc} T & \xrightarrow{m} & \mathfrak{S} \\ \downarrow & & \downarrow \\ T' & \xrightarrow{n} & \mathfrak{A}, \end{array}$$

we say that \mathbf{m} lifts to an $\mathbf{m}' : T' \rightarrow \mathfrak{S}$ if \mathbf{m}' fits into (2.5) to form two commuting triangles.

By standard deformation theory, the diagram (2.5) provides a morphism

$$\mathbf{m}^* \mathbb{L}_{\mathfrak{S}/\mathfrak{A}} \longrightarrow \mathbb{L}_{T/T'} \longrightarrow \mathbb{L}_{T/T'}^{\geq -1} = J[1],$$

which gives an element

$$(2.6) \quad \varpi(\mathbf{m}) \in \text{Ext}_T^1(\mathbf{m}^* \mathbb{L}_{\mathfrak{S}/\mathfrak{A}}, J) = H^1(T, \mathbf{m}^* \mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathcal{O}_T} J).$$

Using the morphism $\phi_{\mathfrak{S}/\mathfrak{A}}$ in (2.4), we obtain the homomorphism

$$\phi' : H^1(T, \mathbf{m}^* \mathbb{T}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathcal{O}_T} J) \longrightarrow H^1(T, \mathbf{m}^* \mathbb{E}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathcal{O}_T} J).$$

We define

$$\text{ob}(T, T', \mathbf{m}) := \phi'(\varpi(\mathbf{m})) \in H^1(T, \mathbf{m}^* \mathbb{E}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathcal{O}_T} J).$$

To prove that $\phi_{\mathfrak{S}/\mathfrak{A}}$ is a perfect relative obstruction theory, by the criterion in [BF, Thm 4.5 (3)], we need to show

- (1) $\text{ob}(T, T', \mathbf{m}) = 0$ if and only if \mathbf{m} in (2.5) can be lifted to $\mathbf{m}' : T' \rightarrow \mathfrak{C}$;
- (2) when $\text{ob}(T, T', \mathbf{m}) = 0$, the set of liftings $\mathbf{m}' : T' \rightarrow \mathfrak{C}$ form a torsor under $H^0(T, \mathbf{m}^* \mathbb{E}_{\mathfrak{C}/\mathfrak{A}} \otimes_{\mathcal{O}_T} J)$.

We now verify (1) and (2). Pulling back \mathcal{C} to T and T' via \mathbf{m} and \mathbf{n} , we obtain two families $\pi_T : \mathcal{C}_T \rightarrow T$ and $\pi_{T'} : \mathcal{C}_{T'} \rightarrow T'$; pulling back \mathfrak{e} to T , we have evaluation map $\mathfrak{e}_T : \mathcal{C}_T \rightarrow \mathcal{Z}$. Let

$$\kappa : H^1(T, \mathbf{m}^* \mathbb{E}_{\mathfrak{S}/\mathfrak{A}} \otimes_{\mathcal{O}_T} J) \xrightarrow{\cong} H^1(T, R^\bullet \pi_{T*}(\mathfrak{e}_T^* \Omega_{\mathcal{Z}/\mathcal{C}}^\vee \otimes \pi_T^* J))$$

be the canonical isomorphism defined by the definition of $\mathbb{E}_{\mathfrak{S}/\mathfrak{A}}$ (cf. (2.4)).

Using the standard property of cotangent complex, the commuting square

$$(2.7) \quad \begin{array}{ccc} \mathcal{C}_T & \xrightarrow{\mathfrak{e}_T} & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{C}_{T'} & \xrightarrow{\tilde{\mathbf{n}}} & \mathcal{C}, \end{array}$$

where $\tilde{\mathbf{n}}$ is the lift of \mathbf{n} in (2.5), induces homomorphisms

$$\mathfrak{e}_T^* \Omega_{\mathcal{Z}/\mathcal{C}} \cong \mathfrak{e}_T^* \mathbb{L}_{\mathcal{Z}/\mathcal{C}} \longrightarrow \mathbb{L}_{\mathcal{C}_T/\mathcal{C}_{T'}} = \pi_T^* \mathbb{L}_{T/T'} \longrightarrow \mathbb{L}_{\mathcal{C}_T/\mathcal{C}_{T'}}^{\geq -1} = \pi_T^* J[1].$$

Their composite associates to an element

$$\varpi(\mathfrak{e}_T, \mathcal{Z}, \mathcal{C}) \in H^1(\mathcal{C}_T, \mathfrak{e}_T^* \Omega_{\mathcal{Z}/\mathcal{C}}^\vee \otimes \pi_T^* J) \cong H^1(T, R^\bullet \pi_{T*}(\mathfrak{e}_T^* \Omega_{\mathcal{Z}/\mathcal{C}}^\vee \otimes \pi_T^* J)).$$

By Lemma 6.5, $\varpi(\mathfrak{e}_T, \mathcal{Z}, \mathcal{C}) = 0$ if and only if (2.7) admits a lifting $\mathcal{C}_{T'} \rightarrow \mathcal{Z}$.

As (2.7) is the composition of (2.5) with (2.2), $\varpi(\mathfrak{e}_T, \mathcal{Z}, \mathcal{C}) = \kappa(\phi'(\varpi(\mathbf{m})))$. Thus $\text{ob}(T, T', \mathbf{m}) = 0$ if and only if (2.7) has a lifting, which is equivalent to that \mathbf{m} lifts to an $\mathbf{m}' : T' \rightarrow \mathfrak{S}$ in (2.5). This verifies criterion (1).

Finally, when $\text{ob}(T, T', \mathbf{m}) = 0$, any two liftings $\mathcal{C}_{T'} \rightarrow \mathcal{Z}$ differ by a section in $H^0(\mathcal{C}_T, \mathfrak{e}_T^* \Omega_{\mathcal{Z}/\mathcal{C}}^\vee \otimes \pi_T^* J)$, and vice versa [Il, Thm 2.1.7]. This proves the criterion (2). These complete the proof of the Proposition. \square

2.4. Moduli of stable morphisms. Using the stack \mathfrak{D}_g of curves with line bundles, this construction provides a different perspective of the moduli of stable morphisms to a projective scheme.

Definition 2.6. *We define \mathfrak{D}_g be the groupoid associating to each scheme S the set $\mathfrak{D}_g(S)$ of pairs $(\mathcal{C}_S, \mathcal{L}_S)$, where $\mathcal{C}_S \rightarrow S$ is a flat family of connected nodal curves and \mathcal{L}_S is a line bundle on \mathcal{C}_S of degree d along fibers of \mathcal{C}_S/S . An arrow from $(\mathcal{C}_S, \mathcal{L}_S)$ to $(\mathcal{C}'_S, \mathcal{L}'_S)$ consists of a pair (ρ, τ) , where $\rho : \mathcal{C}_S \rightarrow \mathcal{C}'_S$ and $\tau : \rho^* \mathcal{L}'_S \rightarrow \mathcal{L}_S$ are S isomorphisms.*

It is easy to show that \mathfrak{D}_g is a smooth Artin stack. By forgetting the line bundles one obtains an induced morphism $\mathfrak{D}_g \rightarrow \mathfrak{M}_g$, where \mathfrak{M}_g is the Artin stack of all connected genus g nodal curves. For any $\xi = (C, L) \in \mathfrak{D}_g$, the automorphism group of ξ relative to \mathfrak{M}_g , (i.e. automorphisms of L that fix C), is \mathbb{C}^* . We denote by $(\mathcal{C}_{\mathfrak{D}_g}, \mathcal{L}_{\mathfrak{D}_g})$, with $\pi_{\mathfrak{D}_g} : \mathcal{C}_{\mathfrak{D}_g} \rightarrow \mathfrak{D}_g$ implicitly understood, the universal family of \mathfrak{D}_g .

We now let $X \subset \mathbb{P}^n$ be a projective scheme. For the integer d given, (the integer d will be fixed throughout this paper,) we have the moduli of genus g and degree d stable morphisms to X : $\overline{\mathcal{M}}_g(X, d)$. We now present it as a moduli of sections. We keep the homogeneous coordinates $[x_1, \dots, x_{n+1}]$ of \mathbb{P}^n mentioned in the introduction. The choice of $[x_i]$ provides a presentation

$$(2.8) \quad \mathbb{P}^n = \mathbb{A}^{n+1*} / \mathbb{C}^*, \quad \mathbb{A}^{n+1*} := \mathbb{A}^{n+1} - 0.$$

We form the bundle

$$\mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus(n+1)*}) = \mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus(n+1)}) - 0_{\mathcal{C}_{\mathfrak{D}_g}},$$

where $0_{\mathcal{C}_{\mathfrak{D}_g}}$ is the zero section. Using the \mathbb{C}^* -equivariance of the projection $\mathbb{A}^{n+1*} \rightarrow \mathbb{P}^n$ induced by (2.8), we obtain a canonical morphism

$$\Psi : \mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus(n+1)*}) \longrightarrow \mathbb{P}^n.$$

We let

$$\mathcal{Z}_X = \mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus(n+1)*}) \times_{\mathbb{P}^n} X \subset \mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus(n+1)*}).$$

We let \mathfrak{S}_X be the stack of sections constructed in Subsection 2.2 with \mathcal{Z} replaced by \mathcal{Z}_X .

Proposition 2.7. *There is a canonical open immersion of stacks $\overline{\mathcal{M}}_g(X, d) \rightarrow \mathfrak{S}_X$, as stacks over \mathfrak{M}_g .*

Proof. For notational simplicity, in the remainder of this Section, we abbreviate $Y = \overline{\mathcal{M}}_g(X, d)$, and denote by $(f_Y, \pi_Y) : \mathcal{C}_Y \rightarrow X \times Y$ the universal family. Pulling back $\mathcal{O}(1)$, we obtain $\mathcal{L}_Y = f_Y^* \mathcal{O}(1)$; pulling back the homogeneous coordinates x_i , (viewing $x_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$), we obtain $u_i = f_Y^* x_i$. Since f_Y has degree d along fibers of \mathcal{C}_Y/Y , $(\mathcal{C}_Y, \mathcal{L}_Y)$ defines a morphism

$$(2.9) \quad \lambda : Y = \overline{\mathcal{M}}_g(X, d) \longrightarrow \mathfrak{D}_g;$$

since $f_Y(\mathcal{C}_Y) \subset X$, (u_1, \dots, u_{n+1}) defines a section $Y \rightarrow \mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus(n+1)}) \times_{\mathfrak{D}_g} Y$ (of Y) that factors through a section

$$\xi : Y \rightarrow \mathcal{Z}_X \times_{\mathfrak{D}_g} Y.$$

This defines a morphism $Y \rightarrow \mathfrak{S}_X$.

It is direct to check that this is an open immersion, and is a morphism over \mathfrak{M}_g . This proves the Proposition. \square

It is worth comparing the relative obstruction theory ϕ_{Y/\mathfrak{D}_g} of $\overline{\mathcal{M}}_g(X, d) \rightarrow \mathfrak{D}_g$ constructed using Subsection 2.3 with the relative obstruction theory ϕ_{Y/\mathfrak{M}_g} of $\overline{\mathcal{M}}_g(X, d) \rightarrow \mathfrak{M}_g$ given in [BF].

Following the notation before Proposition 2.5, we have an evaluation map $\epsilon_Y : \mathcal{C}_Y \rightarrow \mathcal{Z}_X$. The induced morphism $\pi_Y^* \mathbb{T}_{Y/\mathfrak{D}_g} \cong \mathbb{T}_{\mathcal{C}_Y/\mathcal{C}_{\mathfrak{D}_g}} \rightarrow \epsilon_Y^* \mathbb{T}_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g}}$ induces

$$\phi_{Y/\mathfrak{D}_g} : \mathbb{T}_{Y/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{Y/\mathfrak{D}_g} := R^* \pi_* \epsilon_Y^* \mathbb{T}_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g}}.$$

Applying Proposition 2.5, ϕ_{Y/\mathfrak{D}_g} is a perfect relative obstruction theory of $Y \rightarrow \mathfrak{D}_g$.

Lemma 2.8. *Suppose X is smooth. The relative obstruction theories ϕ_{Y/\mathfrak{D}_g} and ϕ_{Y/\mathfrak{M}_g} are related by a morphism of distinguished triangles*

$$\begin{array}{ccccc} R^* \pi_{Y*} \mathcal{O}_{\mathcal{C}_Y} & \longrightarrow & \mathbb{E}_{Y/\mathfrak{D}_g} & \longrightarrow & \mathbb{E}_{Y/\mathfrak{M}_g} \xrightarrow{+1} \longrightarrow \\ \uparrow \parallel & & \uparrow \phi_{Y/\mathfrak{M}_g} & & \uparrow \phi_{Y/\mathfrak{D}_g} \\ \lambda^* \mathbb{T}_{\mathfrak{D}_g/\mathfrak{M}_g}[-1] & \longrightarrow & \mathbb{T}_{Y/\mathfrak{D}_g} & \longrightarrow & \mathbb{T}_{Y/\mathfrak{M}_g} \xrightarrow{+1} \longrightarrow . \end{array}$$

Proof. Let $\mathcal{C}_{\mathfrak{M}_g}$ be the universal curve on \mathfrak{M}_g ; let

$$\chi_M : \mathcal{Z}_X \longrightarrow \mathcal{C}_{\mathfrak{M}_g} \times X$$

be the morphism so that its first factor is the composite $\mathcal{Z}_X \rightarrow \mathcal{C}_{\mathfrak{D}_g} \rightarrow \mathcal{C}_{\mathfrak{M}_g}$, and the second factor is the natural projection. Let

$$f : \mathcal{C}_Y \longrightarrow \mathcal{C}_{\mathfrak{M}_g} \times X$$

be the composite of $\epsilon_Y : \mathcal{C}_Y \rightarrow \mathcal{Z}_X$ with $\chi_M : \mathcal{Z}_X \rightarrow \mathcal{C}_{\mathfrak{M}_g} \times X$. Note that the first factor of f is the canonical projection induced by $Y \rightarrow \mathfrak{D}_g \rightarrow \mathfrak{M}_g$; its the second factor is f_Y .

Taking the tangent complex relative to \mathfrak{M}_g , we obtain

$$\pi_Y^* \mathbb{T}_{Y/\mathfrak{M}_g} \cong \mathbb{T}_{\mathcal{C}_Y/\mathcal{C}_{\mathfrak{M}_g}} \longrightarrow f^* \mathbb{T}_{\mathcal{C}_{\mathfrak{M}_g} \times X/\mathcal{C}_{\mathfrak{M}_g}} \cong f_Y^* \mathbb{T}_X.$$

This induces

$$\phi_{Y/\mathfrak{M}_g} : \mathbb{T}_{Y/\mathfrak{M}_g} \longrightarrow \mathbb{E}_{Y/\mathfrak{M}_g} := R^* \pi_* f_Y^* \mathbb{T}_X,$$

which is the perfect relative obstruction theory of $Y \rightarrow \mathfrak{M}_g$ defined in [BF].

We let $\chi_D : \mathcal{Z}_X \rightarrow \mathcal{C}_{\mathfrak{D}_g} \times X$ be defined similar to χ_M , and let $g : \mathcal{C}_{\mathfrak{D}_g} \times X \rightarrow \mathcal{C}_{\mathfrak{M}_g} \times X$ be the projection. Note that $g \circ \chi_D = \chi_M$. By the construction, we have the commutative diagrams

$$(2.10) \quad \begin{array}{ccccc} \mathcal{Z}_X & \xrightarrow{\chi_D} & \mathcal{C}_{\mathfrak{D}_g} \times X & \xrightarrow{g} & \mathcal{C}_{\mathfrak{M}_g} \times X \\ \downarrow \rho_0 & & \downarrow \pi_1 & & \downarrow \\ \mathcal{C}_{\mathfrak{D}_g} & \xlongequal{\quad} & \mathcal{C}_{\mathfrak{D}_g} & \longrightarrow & \mathcal{C}_{\mathfrak{M}_g}. \end{array}$$

It induces an exact sequence of locally free sheaves

$$0 \longrightarrow T_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g} \times X} \longrightarrow T_{\mathcal{Z}_X/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow \chi_D^* T_{\mathcal{C}_{\mathfrak{D}_g} \times X/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow 0.$$

Since χ_D is a \mathbb{C}^* -principal bundle, $\mathcal{O}_{\mathcal{Z}_X} \cong T_{\mathcal{Z}_X/\mathcal{C}_{\mathcal{D}_g} \times X}$. Also we have canonical isomorphism $\chi_D^* T_{\mathcal{C}_{\mathcal{D}_g} \times X/\mathcal{C}_{\mathcal{D}_g}} \cong \chi_M^* T_{\mathcal{C}_{\mathfrak{M}_g} \times X/\mathcal{C}_{\mathfrak{M}_g}}$. Let $\lambda_C : \mathcal{C}_Y \rightarrow \mathcal{C}_{\mathcal{D}_g}$ be induced by λ . The above sequence fits into a morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathfrak{e}_Y^* T_{\mathcal{Z}_X/\mathcal{C}_{\mathcal{D}_g} \times X} & \longrightarrow & \mathfrak{e}_Y^* T_{\mathcal{Z}_X/\mathcal{C}_{\mathcal{D}_g}} & \longrightarrow & \mathfrak{e}_Y^* \chi_M^* T_{\mathcal{C}_{\mathfrak{M}_g} \times X/\mathcal{C}_{\mathfrak{M}_g}} \cong f_X^* T_X & \xrightarrow{+1} & \longrightarrow \\ \uparrow & & \uparrow & & \uparrow & & \\ \lambda_C^* \mathbb{T}_{\mathcal{C}_{\mathcal{D}_g}/\mathcal{C}_{\mathfrak{M}_g}}[-1] & \longrightarrow & \mathbb{T}_{\mathcal{C}_Y/\mathcal{C}_{\mathcal{D}_g}} & \longrightarrow & \mathbb{T}_{\mathcal{C}_Y/\mathcal{C}_{\mathfrak{M}_g}} & \xrightarrow{+1} & \longrightarrow \end{array}$$

where the left vertical arrow is the composition

$$\lambda^* \mathbb{T}_{\mathcal{C}_{\mathcal{D}_g}/\mathcal{C}_{\mathfrak{M}_g}} \cong \mathfrak{e}_Y^* \chi_D^* \pi_1^* T_{\mathcal{C}_{\mathcal{D}_g}/\mathcal{C}_{\mathfrak{M}_g}} \cong \mathfrak{e}_Y^* \chi_D^* T_{\mathcal{C}_{\mathcal{D}_g} \times X/\mathcal{C}_{\mathfrak{M}_g} \times X} \longrightarrow \mathfrak{e}_Y^* T_{\mathcal{Z}_X/\mathcal{C}_{\mathcal{D}_g} \times X}[1],$$

where the last arrow is given by the distinguished triangle of cotangent complexes associated to the top row of (2.10). Here the commutativity of squares in the above diagram can be checked by diagram chasing using (2.10).

Therefore we have a homomorphism of distinguished triangles

$$\begin{array}{ccccccc} R^* \pi_{Y*} \mathcal{O}_{\mathcal{C}_Y} & \longrightarrow & \mathbb{E}_{Y/\mathcal{D}_g} & \longrightarrow & \mathbb{E}_{Y/\mathfrak{M}_g} & \xrightarrow{+1} & \longrightarrow \\ \uparrow & & \uparrow \phi_{Y/\mathcal{D}_g} & & \uparrow \phi_{Y/\mathfrak{M}_g} & & \\ \lambda^* \mathbb{T}_{\mathcal{D}_g/\mathfrak{M}_g}[-1] & \longrightarrow & \mathbb{T}_{Y/\mathcal{D}_g} & \longrightarrow & \mathbb{T}_{Y/\mathfrak{M}_g} & \xrightarrow{+1} & \longrightarrow \end{array}$$

By the property of cotangent complex of Picard stacks the left vertical arrow of the above diagram is an isomorphism. \square

Let $[Y/\mathcal{D}_g]^{\text{vir}}$ and $[Y/\mathfrak{M}_g]^{\text{vir}} \in A_* Y$ be the virtual cycles using the respective perfect relative obstruction theories.

Corollary 2.9. *We have identity*

$$[Y/\mathcal{D}_g]^{\text{vir}} = [Y/\mathfrak{M}_g]^{\text{vir}} \in A_* Y.$$

Proof. Applying [BF, prop 2.7] to Lemma 2.8, we obtain a diagram of cone stacks

$$\begin{array}{ccccc} h^1/h^0(R^* \pi_{Y*} \mathcal{O}_{\mathcal{C}_Y}) & \longrightarrow & h^1/h^0(\mathbb{E}_{Y/\mathcal{D}_g}) & \xrightarrow{\theta} & h^1/h^0(\mathbb{E}_{Y/\mathfrak{M}_g}) \\ \parallel & & \uparrow (\phi_{Y/\mathcal{D}_g})^* & & \uparrow (\phi_{Y/\mathfrak{M}_g})^* \\ h^1/h^0(\lambda^* \mathbb{T}_{\mathcal{D}_g/\mathfrak{M}_g}[-1]) & \longrightarrow & h^1/h^0(\mathbb{T}_{Y/\mathcal{D}_g}) & \xrightarrow{\theta_{\text{int}}} & h^1/h^0(\mathbb{T}_{Y/\mathfrak{M}_g}) \end{array}$$

of which the two rows are exact sequence of abelian cone stacks. Applying argument analogous to the second line in the proof of [KKP, Prop 3], one checks $(\theta_{\text{int}})^*(C_{Y/\mathfrak{M}_g}) = C_{\mathcal{D}_g/\mathfrak{M}_g}$. Hence θ is a quotient of bundle stacks such that $\theta^*(C_{Y/\mathfrak{M}_g}) = C_{Y/\mathcal{D}_g}$. By projection formula

$$[Y/\mathcal{D}_g]^{\text{vir}} = [Y/\mathfrak{M}_g]^{\text{vir}} \in A_* Y.$$

This proves the Corollary. \square

3. GROMOV-WITTEN INVARIANT OF THE GSW MODEL

In this section, we will construct the moduli of stable morphisms to \mathbb{P}^4 coupled with p -fields. We will construct its localized virtual cycle, using Kiem-Li's cosection localized virtual cycles. We define its degree be the virtual counting of stable maps to \mathbb{P}^4 with p -field. This class of invariants is a generalization of genus zero Guffin-Sharpe-Witten model $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$ [GS].

3.1. Moduli of stable maps with p -fields. Let $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$ be the moduli of genus g degree d stable maps to \mathbb{P}^4 . For the moment, we denote by $(f_M, \mathcal{C}_M, \pi_M)$ be the universal family of $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$, and $\mathcal{L}_M = f_M^* \mathcal{O}(1)$ the tautological line bundle. We form

$$\mathcal{P}_M := \mathcal{L}_M^{-\otimes 5} \otimes \omega_{\mathcal{C}_M/M},$$

and call it the auxiliary invertible sheaf on $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$.

We define the moduli of genus g degree d stable morphisms with p -fields be the direct image cone:

$$(3.1) \quad \mathcal{P} := \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p := C(\pi_{M*} \mathcal{P}_M).$$

(We abbreviate it to \mathcal{P} , as indicated above.)

Like before, we can embed \mathcal{P} into the moduli of sections for a choice of $\mathcal{Z} \rightarrow \mathfrak{D}_g$. Let $[x_1, \dots, x_5]$ be the homogeneous coordinates of \mathbb{P}^4 specified in the Introduction. Let

$$(f_{\mathcal{P}}, \pi_{\mathcal{P}}) : \mathcal{C}_{\mathcal{P}} \rightarrow \mathbb{P}^4 \times \mathcal{P}$$

be the universal map of \mathcal{P} . We let $\mathcal{L}_{\mathcal{P}} = f_{\mathcal{P}}^* \mathcal{O}(1)$ the tautological invertible sheaf; let $\mathcal{P}_{\mathcal{P}} = \mathcal{L}_{\mathcal{P}}^{-\otimes 5} \otimes \omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}}$ be the auxiliary invertible sheaf, and let

$$(3.2) \quad \mathfrak{p} \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathcal{P}_{\mathcal{P}}) \quad \text{and} \quad \mathbf{u}_i = f_{\mathcal{P}}^* x_i \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathcal{L}_{\mathcal{P}})$$

be the universal p -field and the tautological coordinate functions, respectively. Note that $(\mathcal{C}_{\mathcal{P}}, \mathcal{L}_{\mathcal{P}})$ induces a morphism $\mathcal{P} \rightarrow \mathfrak{D}_g$ so that $(\mathcal{C}_{\mathcal{P}}, \mathcal{L}_{\mathcal{P}})$ is isomorphic to the pull back of $(\mathcal{C}_{\mathfrak{D}_g}, \mathcal{L}_{\mathfrak{D}_g})$.

Using the line bundle $\mathcal{L}_{\mathfrak{D}_g}$ on $\mathcal{C}_{\mathfrak{D}_g}$ and its auxiliary invertible sheaf

$$\mathcal{P}_{\mathfrak{D}_g} = \mathcal{L}_{\mathfrak{D}_g}^{-\otimes 5} \otimes \omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g},$$

we form the bundle

$$\mathcal{Z} := \text{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g})$$

over $\mathcal{C}_{\mathfrak{D}_g}$. Then the section $((\mathbf{u}_i)_{i=1}^5, \mathfrak{p})$ defines a section of

$$\mathcal{Z} \times_{\mathcal{C}_{\mathfrak{D}_g}} \mathcal{C}_{\mathcal{P}} \longrightarrow \mathcal{C}_{\mathcal{P}}.$$

This section induces a $\mathcal{C}_{\mathfrak{D}_g}$ -morphism $\mathcal{C}_{\mathcal{P}} \rightarrow \mathcal{Z} \times_{\mathcal{C}_{\mathfrak{D}_g}} \mathcal{C}_{\mathcal{P}}$. Composed with the projection $\mathcal{Z} \times_{\mathfrak{D}_g} \mathcal{P} \rightarrow \mathcal{Z}$, we obtain the evaluation morphism over $\mathcal{C}_{\mathfrak{D}_g}$:

$$(3.3) \quad \tilde{\epsilon} : \mathcal{C}_{\mathcal{P}} \longrightarrow \mathcal{Z}.$$

Proposition 3.1. *The pair $\mathcal{P} \rightarrow \mathfrak{D}_g$ admits a perfect relative obstruction theory*

$$\phi_{\mathcal{P}/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} := R^* \pi_{\mathcal{P}*} (\mathcal{L}_{\mathcal{P}}^{\oplus 5} \oplus \mathcal{P}_{\mathcal{P}}).$$

Proof. The proof follows from Proposition 2.5 applied to the (evaluation) morphism $\tilde{\epsilon}$, using that $\Omega_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}}^{\vee} = \mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g}$. \square

3.2. Constructing a cosection. We define a multi-linear bundle morphism

$$(3.4) \quad h_1 : \text{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g}) \longrightarrow \text{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g}), \quad h_1(z, p) = p \cdot \sum_{i=1}^5 z_i^5,$$

where $(z, p) = ((z_i)_{i=1}^5, p) \in \text{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g})$. This map is based on the dual-pairing $\mathcal{L}_{\mathfrak{D}_g}^{\otimes 5} \otimes \mathcal{P}_{\mathfrak{D}_g} \rightarrow \omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g}$.

The morphism h_1 induces a homomorphism of tangent complexes

$$dh_1 : \mathbb{T}_{\mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow h_1^* \mathbb{T}_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g/\mathfrak{D}_g}})/\mathcal{C}_{\mathfrak{D}_g}} = h_1^* \Omega_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g/\mathfrak{D}_g}})/\mathcal{C}_{\mathfrak{D}_g}}^\vee.$$

In explicit form, for any closed $\xi \in \mathcal{C}_{\mathcal{P}}$ and $(z, p) \in \mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g})|_{\xi}$, $dh_1|_{(z,p)}$ sends

$$((\dot{z}_i), \dot{p}) \in \Omega_{\mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}^\vee|_{(z,p)} = (\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g}) \otimes_{\mathcal{O}_{\mathcal{C}_{\mathfrak{D}_g}}} \mathbf{k}(\xi)$$

to

$$(3.5) \quad dh_1|_{(z,p)}(\dot{z}, \dot{p}) = \left(\sum_{i=1}^5 z_i^5 \right) \cdot \dot{p} + p \cdot \sum_{i=1}^5 5z_i^4 \cdot \dot{z}_i.$$

On the other hand, by pulling back dh_1 to $\mathcal{C}_{\mathcal{P}}$ via the evaluation morphism $\tilde{\mathbf{e}}$ (cf. (3.3)) one has (homomorphism and canonical isomorphisms)

$$\tilde{\mathbf{e}}^*(dh_1) : \tilde{\mathbf{e}}^* \Omega_{\mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}^\vee \longrightarrow \tilde{\mathbf{e}}^* h_1^* \Omega_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g/\mathfrak{D}_g}})/\mathcal{C}_{\mathfrak{D}_g}}^\vee.$$

Because the right hand side is canonically isomorphic to $\omega_{\mathcal{C}_{\mathcal{P}/\mathcal{P}}}$, applying $R^\bullet \pi_{\mathcal{P}*}$, we obtain

$$(3.6) \quad \sigma_1^\bullet : \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} \longrightarrow R^\bullet \pi_{\mathcal{P}*}(\mathbf{e}^* h_1^* \Omega_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g/\mathfrak{D}_g}})/\mathcal{C}_{\mathfrak{D}_g}}^\vee) \cong R^\bullet \pi_{\mathcal{P}*}(\omega_{\mathcal{C}_{\mathcal{P}/\mathcal{P}}}).$$

We define

$$(3.7) \quad \sigma_1 := H^1(\sigma_1^\bullet) : \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g} = H^1(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow R^1 \pi_{\mathcal{P}*}(\omega_{\mathcal{C}_{\mathcal{P}/\mathcal{P}}}) \cong \mathcal{O}_{\mathcal{P}}.$$

By Proposition 3.1, σ_1 is in the form (of homomorphism of sheaves)

$$\sigma_1 : \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g} = R^1 \pi_{\mathcal{P}*} \mathcal{L}_{\mathcal{P}}^{\oplus 5} \oplus R^1 \pi_{\mathcal{P}*} \mathcal{P}_{\mathcal{P}} \longrightarrow \mathcal{O}_{\mathcal{P}}.$$

3.3. Degeneracy loci of the cosection. We give a coordinate expression of the cosection σ_1 . We denoting by $u_i = f_{\mathcal{P}}^* x_i$ and $\mathbf{p} \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathcal{P}_{\mathcal{P}})$ be the tautological section of \mathcal{P} . Take any étale chart $T \rightarrow \mathcal{P}$, and let $\mathcal{C}_T = \mathcal{C}_{\mathcal{P}} \times_{\mathcal{P}} T$. For

$$\dot{p} \in H^1(\mathcal{C}_T, \mathcal{P}_{\mathcal{P}}) \quad \text{and} \quad \dot{u} = (\dot{u}_i)_{i=1}^5 \in H^1(\mathcal{C}_T, \mathcal{L}_{\mathcal{P}}^{\oplus 5}),$$

we define

$$(3.8) \quad \zeta(\dot{p}, \dot{u}) := 5p \cdot \sum_{i=1}^5 u_i^4 \cdot \dot{u}_i + \left(\sum_{i=1}^5 u_i^5 \right) \cdot \dot{p},$$

where p and u_i are the pull back of \mathbf{p} and u_i to \mathcal{C}_T , respectively. The expression (3.8) is an element in $R^1 \pi_{\mathcal{P}*}(\omega_{\mathcal{C}_{\mathcal{P}/\mathcal{P}}}) \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_T \cong \mathcal{O}_T$.

One checks that this defines a homomorphism

$$\zeta : R^1 \pi_{\mathcal{P}*} \mathcal{L}_{\mathcal{P}}^{\oplus 5} \oplus R^1 \pi_{\mathcal{P}*} \mathcal{P}_{\mathcal{P}} \longrightarrow \mathcal{O}_{\mathcal{P}}.$$

Lemma 3.2. *The two homomorphisms ζ and σ_1 coincide.*

Proof. This follows from the explicit expression of dh_1 in affine coordinate generalizing the expression (3.5). It is straightforward. \square

Definition 3.3. *We define the degeneracy loci of σ_1 be*

$$D(\sigma_1) = \left\{ \xi \in \mathcal{P} \mid \sigma_1|_{\xi} : \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g} \otimes_{\mathcal{O}_{\mathcal{P}}} \mathbf{k}(\xi) \longrightarrow \mathbf{k}(\xi) \text{ vanishes} \right\}.$$

Following our convention, we denote by $Q \subset \mathbb{P}^4$ the quintic threefold defined by $\sum x_i^5 = 0$. We let $\overline{\mathcal{M}}_g(Q, d)$ be the moduli of genus g degree d stable morphisms to Q . Using $\overline{\mathcal{M}}_g(Q, d) \subset \overline{\mathcal{M}}_g(\mathbb{P}^4, d)$, we obtain embedding

$$\overline{\mathcal{M}}_g(Q, d) \subset \overline{\mathcal{M}}_g(\mathbb{P}^4, d) \subset \mathcal{P},$$

where the second inclusion is by assigning zero p -fields.

Proposition 3.4. *The degeneracy loci of σ_1 is $\overline{\mathcal{M}}_g(Q, d) \subset \mathcal{P}$; it is proper.*

Proof. Let $\xi = (C, L, \phi, p) \in \mathcal{P}$, where $\phi = (\phi_i)_{i=1}^5 \in H^0(C, L^{\oplus 5})$. The restriction of $\sigma_1 = \zeta$ to ξ takes the form $\sigma_1|_\xi(\mathring{p}, \mathring{\phi}) = 5p \sum \phi_i^4 \mathring{\phi}_i + \sum \phi_i^5 \mathring{p}$.

Suppose $\sum \phi_i^5 \neq 0$, then by Serre duality, we can find $\mathring{p} \in H^1(C, L^{-\otimes 5} \otimes \omega_C)$ so that $\mathring{p} \cdot \sum \phi_i^5 \neq 0 \in H^1(C, \omega_C)$. Letting $\mathring{\phi}_i = 0$, we obtain $\sigma_1|_\xi \neq 0$.

Suppose $\sum \phi_i^5 = 0$ and $p \neq 0$. Then since ϕ_i have no common vanishing locus, for some k , $p \cdot \phi_k^4 \neq 0$. By Serre duality, we can find a $\mathring{\phi}_k$ so that $p \cdot \phi_k^4 \cdot \mathring{\phi}_k \neq 0 \in H^1(C, \omega_C)$. By choosing other $\mathring{\phi}_i = 0$, we obtain the surjectivity of $\sigma_1|_\xi$. This proves that the degeneracy loci (i.e. the non-surjective loci) of σ_1 is the collection of (C, L, ϕ, p) such that $\sum \phi_i^5 = 0$ and $p = 0$. This set is $\overline{\mathcal{M}}_g(Q, d) \subset \mathcal{P}$. \square

3.4. The cosection factorizes. Let $q : \mathcal{P} \rightarrow \mathfrak{D}_g$ be the tautological morphism. We form the distinguished triangle

$$(3.9) \quad q^* \mathbb{L}_{\mathfrak{D}_g} \longrightarrow \mathbb{L}_{\mathcal{P}} \longrightarrow \mathbb{L}_{\mathcal{P}/\mathfrak{D}_g} \xrightarrow{\delta} q^* \mathbb{L}_{\mathfrak{D}_g}[1].$$

Composing $\phi_{\mathcal{P}/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g} \rightarrow \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}$ with the dual of δ in the above distinguished triangle, we obtain the morphism

$$\phi_{\mathcal{P}/\mathfrak{D}_g} \circ \delta^\vee : q^* \mathbb{T}_{\mathfrak{D}_g} \longrightarrow \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}[1] \longrightarrow \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}[1].$$

Denoting $\eta = H^0(\phi_{\mathcal{P}/\mathfrak{D}_g} \circ \delta^\vee)$, we obtain the composite

$$(3.10) \quad \eta : q^* T_{\mathfrak{D}_g} \longrightarrow H^1(\mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow H^1(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) = \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g}.$$

Following the construction in [KL2, (4.3)], the cokernel of (3.10) is the absolute obstruction sheaf of \mathcal{P} , which we denote by $\mathcal{O}b_{\mathcal{P}}$.

In this subsection, we show

Proposition 3.5. *The cosection $\sigma_1 : \mathcal{O}b_{\mathcal{P}/\mathfrak{D}_g} \rightarrow \mathcal{O}_{\mathcal{P}}$ lifts to a $\bar{\sigma}_1 : \mathcal{O}b_{\mathcal{P}} \rightarrow \mathcal{O}_{\mathcal{P}}$.*

We continue to use the notation developed in the proof of Proposition 3.1.

Lemma 3.6. *The following composition is trivial:*

$$0 = H^1(\sigma_1^* \circ \phi_{\mathcal{P}/\mathfrak{D}_g}) : H^1(\mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow H^1(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow R^1 \pi_{\mathcal{P}*} \omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}}.$$

Proof. Using the universal curve $\pi_{\mathfrak{D}_g} : \mathcal{C}_{\mathfrak{D}_g} \rightarrow \mathfrak{D}_g$ of \mathfrak{D}_g , we introduce the direct image cone $\mathfrak{C}_\omega = C(\pi_* \omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})$; we denote by $\text{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})$ the underlying bundle of $\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g}$. Let $\mathcal{C}_{\mathfrak{C}_\omega} = \mathcal{C}_{\mathfrak{D}_g} \times_{\mathfrak{D}_g} \mathfrak{C}_\omega$ be the universal curve over \mathfrak{C}_ω , and $\pi_{\mathfrak{C}_\omega} : \mathcal{C}_{\mathfrak{C}_\omega} \rightarrow \mathfrak{C}_\omega$ be the projection.

Continue to denote by $(f_{\mathcal{P}}, \mathcal{C}_{\mathcal{P}}, \mathcal{L}_{\mathcal{P}})$ the universal family of \mathcal{P} , and using $u_i = f_{\mathcal{P}}^* x_i \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathcal{L}_{\mathcal{P}})$ and $\mathfrak{p} \in \Gamma(\mathcal{C}_{\mathcal{P}}, \mathcal{P}_{\mathcal{P}})$ the universal coordinate functions and p -field (cf. (3.2)), we form

$$\epsilon := \mathfrak{p} \cdot (u_1^5 + \dots + u_5^5) \in \Gamma(\mathcal{C}_{\mathcal{P}}, \omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}}).$$

It defines a morphism $\Phi_\epsilon : \mathcal{P} \rightarrow \mathfrak{C}_\omega$ so that if we denote by $\tilde{\Phi}_\epsilon : \mathcal{C}_{\mathcal{P}} \rightarrow \mathcal{C}_{\mathfrak{C}_\omega}$ the tautological lift of Φ_ϵ using that both $\mathcal{C}_{\mathcal{P}}$ and $\mathcal{C}_{\mathfrak{C}_\omega}$ are pull backs of $\mathcal{C}_{\mathfrak{D}_g}$, and denote

by ϵ and ϵ' the evaluation morphisms as shown, we have a commutative diagram of morphisms of stacks over $\mathcal{C}_{\mathfrak{D}_g}$:

$$(3.11) \quad \begin{array}{ccc} \mathcal{C}_{\mathcal{P}} & \xrightarrow{\epsilon} & \mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g}) \\ \downarrow \tilde{\Phi}_\epsilon & & \downarrow h_1 \\ \mathcal{C}_{\mathfrak{C}_\omega} & \xrightarrow{\epsilon'} & \mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g}). \end{array}$$

Here h_1 is defined in (3.4). This shows that the square below is commutative

$$(3.12) \quad \begin{array}{ccccc} \pi_{\mathcal{P}}^* \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g} & \xlongequal{\quad} & \mathbb{T}_{\mathcal{C}_{\mathcal{P}}/\mathcal{C}_{\mathfrak{D}_g}} & \longrightarrow & \epsilon^* \Omega_{\mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}^\vee \\ \downarrow & & \downarrow & & \downarrow dh_1 \\ \pi_{\mathcal{P}}^* \tilde{\Phi}_\epsilon^* \mathbb{T}_{\mathfrak{C}_\omega/\mathfrak{D}_g} & \xlongequal{\quad} & \tilde{\Phi}_\epsilon^* \mathbb{T}_{\mathfrak{C}_{\mathfrak{C}_\omega}/\mathcal{C}_{\mathfrak{D}_g}} & \longrightarrow & \tilde{\Phi}_\epsilon^* \epsilon'^* \Omega_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}^\vee \end{array}$$

Applying $R^1\pi_{\mathcal{P}*}$ to the lower horizontal arrow we obtain the obstruction assignment homomorphism

$$(3.13) \quad (0 =) H^1(\tilde{\Phi}_\epsilon^* \phi_{\mathfrak{C}_\omega/\mathfrak{D}_g}) : H^1(\tilde{\Phi}_\epsilon^* \mathbb{T}_{\mathfrak{C}_\omega/\mathfrak{D}_g}) \longrightarrow \Phi_\epsilon^* R^1\pi_{\mathfrak{C}_\omega*} \omega_{\mathfrak{C}_{\mathfrak{C}_\omega}/\mathfrak{C}_\omega},$$

which is trivial since \mathfrak{C}_ω is a vector bundle over \mathfrak{D}_g and $\mathcal{C}_{\mathfrak{C}_\omega} \rightarrow \mathcal{C}_{\mathfrak{D}_g}$ is smooth.

Therefore, using the Cartesian squares

$$(3.14) \quad \begin{array}{ccc} \mathcal{C}_{\mathcal{P}} & \xrightarrow{\tilde{\Phi}_\epsilon} & \mathcal{C}_{\mathfrak{C}_\omega} \\ \downarrow \pi_{\mathcal{P}} & & \downarrow \pi_{\mathfrak{C}_\omega} \\ \mathcal{P} & \xrightarrow{\Phi_\epsilon} & \mathfrak{C}_\omega \end{array}$$

and the commutativity of (3.12), applying $R^1\pi_{\mathcal{P}*}$, we see that the composite

$$H^1(\mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow R^1\pi_{\mathcal{P}*} \epsilon^* \Omega_{\mathrm{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5} \oplus \mathcal{P}_{\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}^\vee \longrightarrow R^1\pi_{\mathcal{P}*} \epsilon^* h_1^* \Omega_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}^\vee$$

coincides with the composite

$$H^1(\mathbb{T}_{\mathcal{P}/\mathfrak{D}_g}) \longrightarrow H^1(\tilde{\Phi}_\epsilon^* \mathbb{T}_{\mathfrak{C}_\omega/\mathfrak{D}_g}) \xrightarrow{0} \Phi_\epsilon^* R^1\pi_{\mathfrak{C}_\omega*} \Omega_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}^\vee.$$

Since the composite in the second line is trivial (cf. (3.13)), the composite in the first line is trivial. Using

$$\epsilon^* h_1^* \Omega_{\mathrm{Vb}(\omega_{\mathcal{C}_{\mathfrak{D}_g}/\mathfrak{D}_g})/\mathcal{C}_{\mathfrak{D}_g}}^\vee \cong \omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}},$$

this is exactly the vanishing desired by the Lemma, \square

Proof of Proposition 3.5. The composition of σ with (3.10) is the H^1 of the composition

$$\mathbb{T}_{\mathfrak{D}_g}[-1] \longrightarrow \mathbb{T}_{\mathcal{P}/\mathfrak{D}_g} \xrightarrow{\phi_{\mathcal{P}/\mathfrak{D}_g}} \mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} \xrightarrow{\sigma_1^\bullet} R^\bullet \pi_{\mathcal{P}*} \omega_{\mathcal{C}_{\mathcal{P}}/\mathcal{P}},$$

where the first arrow is the δ^\vee in (3.9). Lemma 3.6 implies the H^1 of the above composition is trivial. \square

Here we comment the background of this construction in Super-String Theories. Let $K_{\mathbb{P}^4}$ be the total space of the canonical line bundle \mathbb{P}^4 . The quintic polynomial $\sum x_i^5$ defines a regular map $\mathbf{w}_{\mathbb{P}^4} \in \Gamma(\mathcal{O}_{K_{\mathbb{P}^4}})$. Its critical locus is the quintic threefold $Q \subset \mathbb{P}^4$. In physics literature, the pair $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$ is called a Landau-Ginzburg Model (non-linear). In [GS], Guffin and Sharpe constructed a path integral for

genus zero A-twisted theory of the Landau Ginzburg space $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$ [GS]. In this paper, we have constructed a mathematical theory generalizing it to all genus.

3.5. The virtual dimension. We calculate the virtual dimension of \mathcal{P} . Let $\xi = (f, C, L, p) \in \mathcal{P}$ be any closed point. The virtual dimension of $\mathcal{P}/\mathfrak{D}_g$ at ξ is

$$\dim H^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} \otimes_{\mathcal{O}_{\mathcal{P}}} \mathbf{k}(\xi)) - \dim H^1(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g} \otimes_{\mathcal{O}_{\mathcal{P}}} \mathbf{k}(\xi)).$$

By the expression of $\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}$, the above term equals to

$$h^0(L^{\oplus 5}) + h^0(L^{-\otimes 5} \otimes \omega_C) - h^1(L^{\oplus 5}) - h^1(L^{-\otimes 5} \otimes \omega_C) = 4 - 4g.$$

Because

$$\dim \mathfrak{D}_g = \dim \mathfrak{D}_g/\mathfrak{M}_g + \dim \mathfrak{M}_g = (h^0(\mathcal{O}_C) - 1) + 3g - 3 = 4g - 4.$$

The virtual dimension of \mathcal{P} at ξ is zero.

3.6. Localized virtual cycle. We apply the theory developed in [KL2]. We define a subcone-stack

$$h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})_{\sigma_1} \subset h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})$$

as follows. Let $\mathcal{U} \subset \mathcal{P}$ be the locus where σ_1 is surjective; we denote by

$$D(\sigma_1) = \mathcal{P} - \mathcal{U}$$

its complement. Since σ_1 is surjective over \mathcal{U} , it induces a surjective bundle-homomorphism

$$(3.15) \quad \sigma_1|_{\mathcal{U}} : h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) \times_{\mathcal{P}} \mathcal{U} \longrightarrow \mathbb{C}_{\mathcal{U}},$$

where $\mathbb{C}_{\mathcal{U}}$ is the trivial line bundle on \mathcal{U} . We let $\ker(\sigma_1|_{\mathcal{U}})$ be the kernel bundle-stack of (3.15); it is a codimension one subbundle-stack of $h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) \times_{\mathcal{P}} \mathcal{U}$.

We define

$$(3.16) \quad h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})_{\sigma_1} = (h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) \times_{\mathcal{P}} D(\sigma_1)) \cup \ker(\sigma_1|_{\mathcal{U}}).$$

It is closed in $h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})$. We endow it with the reduced structure. (We call (3.16) the kernel of $h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}) \rightarrow \mathbb{C}_{\mathcal{P}}$ induced by σ_1 , where $\mathbb{C}_{\mathcal{P}}$ is the trivial line bundle on \mathcal{P} .)

Proposition 3.7. *The virtual normal cone cycle $[\mathbf{C}_{\mathcal{P}/\mathfrak{D}_g}] \in Z_* h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})$ lies inside $Z_* h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})_{\sigma_1}$.*

Proof. This is Proposition [KL2, Thm 5.1]. □

In [KL2], Kiem and the second named author constructed a localized Gysin map

$$0^!_{\sigma_1, \text{loc}} : A_* h^1/h^0(\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g})_{\sigma_1} \longrightarrow A_{*-n} D(\sigma_1),$$

where $-n$ is the rank of $\mathbb{E}_{\mathcal{P}/\mathfrak{D}_g}$.

Definition-Proposition 3.8. *We define the localized virtual cycle of (\mathcal{P}, σ_1) be*

$$[\mathcal{P}]_{\sigma_1}^{\text{vir}} = [\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]_{\sigma_1}^{\text{vir}} := 0^!_{\sigma_1, \text{loc}}([\mathbf{C}_{\mathcal{P}/\mathfrak{D}_g}]) \in A_0 \overline{\mathcal{M}}_g(Q, d).$$

We define the virtual enumeration $N_g(d)_{\mathbb{P}^4}^p := \deg [\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p]_{\sigma_1}^{\text{vir}}$.

The number $N_g(d)_{\mathbb{P}^4}^p$ is the virtual counting of the Guffin-Sharp-Witten Model (\mathcal{P}, σ_1) . We call it the Gromov-Witten invariants of the moduli of stable morphisms to \mathbb{P}^4 with p -fields, or of the Landau-Ginzburg space $(K_{\mathbb{P}^4}, \mathbf{w}_{\mathbb{P}^4})$.

4. DEGENERATION OF MODULI OF STABLE MORPHISMS WITH p -FIELDS

In the second part, we will use degeneration to prove that $N_g(d)_{\mathbb{P}^4}^p$ coincides up to a sign with the Gromov-Witten invariants $N_g(d)_Q$ of the quintic three-fold Q .

The degeneration we will use is to degenerate the moduli \mathcal{P} to the moduli of stable morphisms to the normal bundle to $Q \subset \mathbb{P}^4$ coupled with p -field. After constructing a cosection of its obstruction sheaf, the degeneration admits a localized virtual cycle that provides the proof of the equivalence of two classes of invariants.

4.1. The degeneration. We let V be the total space of the deformation of \mathbb{P}^4 to the normal bundle of $Q \subset \mathbb{P}^4$; it is the blowing up of $\mathbb{P}^4 \times \mathbb{A}^1$ along $Q \times 0$, after taking out the proper transform of $\mathbb{P}^4 \times 0$. Let

$$(4.1) \quad q_{\mathbb{A}^1} : V \longrightarrow \mathbb{A}^1 \quad \text{and} \quad q_{\mathbb{P}^4} : V \longrightarrow \mathbb{P}^4$$

be the two projections. Then the fiber of V over $c \neq 0$ is the \mathbb{P}^4 , and the central fiber (over $0 \in \mathbb{A}^1$) is the normal bundle N to $Q \subset \mathbb{P}^4$. We define the degree of a morphism $u : C \rightarrow V$ be $\deg u = \deg(\rho \circ u)^* \mathcal{O}(1)$.

We form the moduli of genus g and degree d stable morphisms $\overline{\mathcal{M}}_g(V, d)$. For the moment, we denote by

$$(\tilde{f}, \tilde{\pi}) : \tilde{\mathcal{C}} \longrightarrow V \times \overline{\mathcal{M}}_g(V, d)$$

the universal family of $\overline{\mathcal{M}}_g(V, d)$. Since $q_{\mathbb{A}^1}$ is proper away from the central fiber $N = V \times_{\mathbb{A}^1} 0$, and since \mathbb{A}^1 is affine, the composite $q_{\mathbb{A}^1} \circ \tilde{f} : \tilde{\mathcal{C}} \rightarrow \mathbb{A}^1$ factors through a $\overline{\mathcal{M}}_g(V, d) \rightarrow \mathbb{A}^1$. Its fiber over $c \neq 0 \in \mathbb{A}^1$ are $\overline{\mathcal{M}}_g(\mathbb{P}^4, d)$; its central fiber is $\overline{\mathcal{M}}_g(N, d)$.

We now couple the stable morphisms with p -field. Let $\tilde{\mathcal{L}} = \tilde{f}^* \mathcal{O}(1)$ and $\tilde{\mathcal{P}} = \tilde{\mathcal{L}}^{-\otimes 5} \otimes \omega_{\tilde{\mathcal{C}}/\overline{\mathcal{M}}_g(V, d)}$ be the tautological and auxiliary invertible sheaves. Like before, we define the moduli of stable morphisms coupled with p -fields be

$$\mathcal{V} := \overline{\mathcal{M}}_g(V, d)^p := C(\tilde{\pi}_* \tilde{\mathcal{P}}),$$

the direct image cone. It is over \mathbb{A}^1 , and its fibers over $c \neq 0 \in \mathbb{A}^1$ and $0 \in \mathbb{A}^1$ are

$$\mathcal{V} \times_{\mathbb{A}^1} c \cong \mathcal{P}, \quad \mathcal{V} \times_{\mathbb{A}^1} 0 := \overline{\mathcal{M}}_g(N, d)^p.$$

Here $\overline{\mathcal{M}}_g(N, d)^p$ is the moduli of stable morphisms to N coupled with p -fields.

Following our convention, we denote by

$$(4.2) \quad (f_{\mathcal{V}}, \pi_{\mathcal{V}}) : \mathcal{C}_{\mathcal{V}} \longrightarrow V \times \mathcal{V}$$

the universal map of \mathcal{V} .

4.2. The cone over V . We construct the tautological cone $C(V)$ over V that will be used to construct the evaluation morphism $\epsilon_{\mathcal{V}}$ of $\mathcal{C}_{\mathcal{V}}$. The evaluation map will be used to construct the obstruction theory of \mathcal{V} .

We let $B = \text{Vb}(\mathcal{O}(5))$ be the underlying line bundle of $\mathcal{O}(5)$ over \mathbb{P}^4 ; let

$$\mathbf{q}_{\mathbb{P}^4} : B \times \mathbb{A}^1 \rightarrow B \rightarrow \mathbb{P}^4 \quad \text{and} \quad \mathbf{q}_{\mathbb{A}^1} : B \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$

be the (composite of) projection(s). We let $t \in \Gamma(\mathcal{O}_{\mathbb{A}^1})$ be the standard coordinate function of \mathbb{A}^1 . We introduce tautological sections over $B \times \mathbb{A}^1$:

$$(4.3) \quad \tilde{x}_i = \mathbf{q}_{\mathbb{P}^4}^* x_i \in \Gamma(\mathbf{q}_{\mathbb{P}^4}^* \mathcal{O}(1)), \quad \tilde{t} = \mathbf{q}_{\mathbb{A}^1}^* t \in \Gamma(\mathcal{O}_{B \times \mathbb{A}^1}), \quad \text{and} \quad \tilde{y} \in \Gamma(\mathbf{q}_{\mathbb{P}^4}^* \mathcal{O}(5)),$$

where \tilde{y} is the section so that the morphism $B \times \mathbb{A}^1 \rightarrow \text{Vb}(\mathcal{O}(5))$ induced by \tilde{y} is the projection $B \times \mathbb{A}^1 \rightarrow B = \text{Vb}(\mathcal{O}(5))$. (I.e. \tilde{y} is the pull back of the identity map $B \rightarrow B$.)

Lemma 4.1. *We have a closed immersion*

$$V \cong (\tilde{s} = 0) \subset B \times \mathbb{A}^1, \quad \tilde{s} = \tilde{x}_1^5 + \dots + \tilde{x}_5^5 - \tilde{t} \cdot \tilde{y}.$$

Proof. We define

$$\Phi : V - V \times_{\mathbb{A}^1} 0 \longrightarrow B \times \mathbb{A}^1$$

via $\Phi^*(\tilde{x}_i) = q_{\mathbb{P}^4}^*(x_i)$, $\Phi^*(\tilde{t}) = q_{\mathbb{A}^1}^*t$, and $\Phi^*\tilde{y} = t^{-1} \cdot (x_1^5 + \dots + x_5^5)$, where $q_{\mathbb{P}^4} : V \rightarrow \mathbb{P}^4$ is the projection, etc. (cf. (4.1)). By definition, the image of Φ lies in $\tilde{s} = 0$ and is an open immersion into $\tilde{s} = 0$. Using that V is the deformation of \mathbb{P}^4 to the normal cone of $Q \subset \mathbb{P}^4$, Φ extends to an isomorphism between V and $\tilde{s} = 0$. This proves the Lemma. \square

In the following, we will view $V \subset B \times \mathbb{A}^1$ using this isomorphism. We next construct the cone $C(V)$ desired. We let $W_5 = \mathbb{C}_{\mathbb{A}^1}$ (resp. $W_1 = \mathbb{C}_{\mathbb{A}^1}^{\oplus 5}$) be the trivial line bundle (resp. rank five trivial vector bundle) over \mathbb{A}^1 . We consider the rank six bundle

$$\text{pr}_{\mathbb{A}^1} : W_1 \times_{\mathbb{A}^1} W_5 \longrightarrow \mathbb{A}^1$$

with the \mathbb{C}^* -action: \mathbb{C}^* acts on the base \mathbb{A}^1 trivially and acts on fibers of W_1 (resp. W_5) of weight one (resp. weight five). Namely, for $z \in W_1$ and $y \in W_5$, $z^\sigma = \sigma z$ and $y^\sigma = \sigma^5 y$.

We let $W_1^* = W_1 - 0_{W_1}$, where 0_{W_1} is the zero section of W_1 . We introduce

$$C(V) = (\epsilon = 0) \subset W_1^* \times_{\mathbb{A}^1} W_5, \quad \epsilon = z_1^5 + \dots + z_5^5 - t \cdot y.$$

It is smooth and is \mathbb{C}^* -invariant.

We claim that $(W_1^* \times_{\mathbb{A}^1} W_5)/\mathbb{C}^*$ is isomorphic to $B = \text{Vb}(\mathcal{O}(5))$, and under this isomorphism we have commuting (horizontal) quotient morphisms

$$(4.4) \quad \begin{array}{ccc} W_1^* \times_{\mathbb{A}^1} W_5 & \xrightarrow{\Psi} & B \\ \uparrow \cup & & \uparrow \cup \\ C(V) & \xrightarrow{/\mathbb{C}^*} & V \end{array}$$

Indeed, the top horizontal quotient morphism follows from that of the weights of the \mathbb{C}^* -action on $W_1 \times_{\mathbb{A}^1} W_5$. To see the full diagram, we construct explicitly the morphism Ψ in (4.4). We let $U_i \subset \mathbb{P}^4$ be the open subset $x_i \neq 0$; we fix trivialization $\mathcal{O}(5)|_{U_i} \cong \mathcal{O}_{U_i}$ so that the transition function $\varphi_{ij} = x_i^5/x_j^5 : \mathcal{O}_{U_i}|_{U_i \cap U_j} \rightarrow \mathcal{O}_{U_j}|_{U_i \cap U_j}$. We define

$$\Psi_i : (W_1^* - \{z_i = 0\}) \times_{\mathbb{A}^1} W_5 \longrightarrow (B \times_{\mathbb{P}^4} U_i) \times \mathbb{A}^1$$

via $((z_i), y), t) \mapsto ((\frac{y}{z_i^5}, [z_1, \dots, z_5]), t)$. This collection $\{\Psi_i\}$ form the morphism Ψ in (4.4).

By construction, Ψ is \mathbb{C}^* -equivariant with \mathbb{C}^* acting trivially on B , and factors to a \mathbb{C}^* -quotient morphism

$$\rho : C(V) \longrightarrow V.$$

For later purpose, we describe the tangent bundles $T_{C(V)/\mathbb{A}^1}$ and $T_{C(V)}$. Using the defining equation of $C(V)$, they fit into the exact sequences

$$(4.5) \quad 0 \longrightarrow T_{C(V)/\mathbb{A}^1} \longrightarrow \mathcal{O}_{C(V)}^{\oplus 5} \oplus \mathcal{O}_{C(V)} \xrightarrow{d'\epsilon} \mathcal{O}_{C(V)} \longrightarrow 0, \quad (d't = 0),$$

where d' is the relative differential and $d'\epsilon|_{((z_i), y, t)}$ sends $((\dot{z}_i), \dot{y})$ to $\sum 5z_i^4 \dot{z}_i - t\dot{y}$;

$$(4.6) \quad 0 \longrightarrow T_{C(V)} \longrightarrow \mathcal{O}_{C(V)}^{\oplus 5} \oplus \mathcal{O}_{C(V)} \oplus \mathcal{O}_{C(V)} \xrightarrow{d\epsilon} \mathcal{O}_{C(V)} \longrightarrow 0,$$

where $d\epsilon|_{((z_i), y, t)}$ sends $((\dot{z}_i), \dot{y}, \dot{t})$ to $\sum 5z_i^4 \dot{z}_i - t\dot{y} - y\dot{t}$.

Together they fit into the exact sequence

$$(4.7) \quad 0 \longrightarrow T_{C(V)/\mathbb{A}^1} \longrightarrow T_{C(V)} \longrightarrow \mathcal{O}_{C(V)} \longrightarrow 0.$$

4.3. The evaluation maps. We now construct the evaluation morphism of $\mathcal{C}_{\mathfrak{D}_g}$. Since V is a family over \mathbb{A}^1 , it is natural to construct the obstruction theory of \mathcal{V} relative to $\mathfrak{D}_g \times \mathbb{A}^1$.

To this purpose, we introduce $\tilde{\mathfrak{D}}_g = \mathfrak{D}_g \times \mathbb{A}^1$, viewed as a stack over \mathbb{A}^1 ; denote by

$$\mathcal{C}_{\tilde{\mathfrak{D}}_g} := \mathcal{C}_{\mathfrak{D}_g} \times \mathbb{A}^1 \longrightarrow \mathfrak{D}_g \times \mathbb{A}^1 = \tilde{\mathfrak{D}}_g$$

the universal curve, and denote by $\mathcal{L}_{\tilde{\mathfrak{D}}_g}$ the pull back of $\mathcal{L}_{\mathfrak{D}_g}$ via $\mathcal{C}_{\tilde{\mathfrak{D}}_g} \rightarrow \mathcal{C}_{\mathfrak{D}_g}$.

We form $\text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5})^* = \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5}) - 0_{\tilde{\mathfrak{D}}_g}$, and consider the bundle over $\mathcal{C}_{\mathfrak{D}_g}$:

$$(4.8) \quad \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\mathfrak{D}_g}} \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5}) \longrightarrow \mathcal{C}_{\tilde{\mathfrak{D}}_g}.$$

Note that for each $\xi \in \mathcal{C}_{\mathfrak{D}_g}$, the fibers of (4.8) over $\xi \times \mathbb{A}^1 \subset \tilde{\mathfrak{D}}_g$ is isomorphic to

$$(L^{\oplus 5} - 0) \times L^{\otimes 5} \times \mathbb{A}^1 \cong W_1^* \times_{\mathbb{A}^1} W_5, \quad L := \mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5} \otimes_{\mathcal{C}_{\mathfrak{D}_g}} \mathbf{k}(\xi),$$

where the isomorphism is uniquely determined by an isomorphism $L \cong \mathbb{C}$, and two different isomorphisms are equivalent under a scaling of $(\mathbb{C}^5 - 0) \times \mathbb{C}$ by a $c \in \mathbb{C}^*$ with weights $(1, \dots, 1, 5)$ on the factors of $(\mathbb{C}^5 - 0) \times \mathbb{C}$.

We let \mathbb{C}^* acts on the bundle (4.8) fiberwise with this weights. We obtain the quotient \mathbb{A}^1 -morphisms (the \mathbb{A}^1 is the base of $W \rightarrow \mathbb{A}^1$ and of $\tilde{\mathfrak{D}}_g = \mathfrak{D}_g \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$)

$$\text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5}) \longrightarrow (\text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5})) / \mathbb{C}^* \longrightarrow (W_1^* \times_{\mathbb{A}^1} W_5) / \mathbb{C}^*.$$

We define

$$(4.9) \quad \mathcal{Z}' = (\text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5})) \times_{(W_1^* \times_{\mathbb{A}^1} W_5) / \mathbb{C}^*} V;$$

it is the preimage of $V \subset (W_1^* \times_{\mathbb{A}^1} W_5) / \mathbb{C}^*$ of the morphism above (4.9). We define

$$(4.10) \quad \mathcal{Z} = \mathcal{Z}' \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \text{Vb}(\mathcal{P}_{\tilde{\mathfrak{D}}_g}).$$

We now construct the evaluation morphism

$$(4.11) \quad \epsilon_{\mathcal{V}} : \mathcal{C}_{\mathcal{V}} \longrightarrow \mathcal{Z}.$$

We let $\mathcal{L}_{\mathcal{V}} = f_{\mathcal{V}}^* \mathcal{O}(1)$, where $(f_{\mathcal{V}}, \mathcal{C}_{\mathcal{V}})$ is the universal family of \mathcal{V} (cf. (4.2)), let $\mathcal{P}_{\mathcal{V}} = \mathcal{L}_{\mathcal{V}}^{-\otimes 5} \otimes \omega_{\mathcal{C}_{\mathcal{V}}/\mathcal{V}}$ be the auxiliary invertible sheaf, and let

$$(4.12) \quad \mathfrak{p} \in \Gamma(\mathcal{C}_{\mathcal{V}}, \mathcal{P}_{\mathcal{V}}), \quad \mathfrak{u}_i = f_{\mathcal{V}}^* \tilde{x}_i \in \Gamma(\mathcal{C}_{\mathcal{V}}, \mathcal{L}_{\mathcal{V}}) \quad \text{and} \quad \mathfrak{\eta} = f_{\mathcal{V}}^* \tilde{y} \in \Gamma(\mathcal{C}_{\mathcal{V}}, \mathcal{L}_{\mathcal{V}}^{\otimes 5})$$

(cf. (4.3)) be the universal \mathbf{p} -field and the tautological coordinate functions. Note that $(\mathcal{C}_{\mathcal{V}}, \mathcal{L}_{\mathcal{V}})$ induces an \mathbb{A}^1 -morphism $\mathcal{V} \rightarrow \widetilde{\mathfrak{D}}_g$ so that $(\mathcal{C}_{\mathcal{V}}, \mathcal{L}_{\mathcal{V}})$ is isomorphic to the pull back of $(\mathcal{C}_{\widetilde{\mathfrak{D}}_g}, \mathcal{L}_{\widetilde{\mathfrak{D}}_g})$.

Then the definition of $V \subset B \times \mathbb{A}^1$ implies that the sections in (4.12) satisfy

$$\mathbf{u}_1^5 + \mathbf{u}_2^5 + \mathbf{u}_3^5 + \mathbf{u}_4^5 + \mathbf{u}_5^5 - t \cdot \boldsymbol{\eta} = 0,$$

where t is the coordinate function of \mathbb{A}^1 mentioned before. Therefore the section $((\mathbf{u}_i)_{i=1}^5, \boldsymbol{\eta}, \mathbf{p})$ defines a section of

$$\mathcal{Z} \times_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}} \mathcal{C}_{\mathcal{V}} \longrightarrow \mathcal{C}_{\mathcal{V}}.$$

This section induces a $\mathcal{C}_{\mathcal{V}}$ -morphism $\mathcal{C}_{\mathcal{V}} \rightarrow \mathcal{Z} \times_{\mathcal{C}_{\widetilde{\mathfrak{D}}_g}} \mathcal{C}_{\mathcal{V}}$. Composed with the projection $\mathcal{Z} \times_{\widetilde{\mathfrak{D}}_g} \mathcal{V} \rightarrow \mathcal{Z}$, we obtain the evaluation morphism over $\mathcal{C}_{\widetilde{\mathfrak{D}}_g}$ in (4.11).

4.4. The obstruction theory of $\mathcal{V}/\widetilde{\mathfrak{D}}_g$. We will build the obstruction theories to carry out the degeneration for virtual cycles. We first construct the relative obstruction theory of $\mathcal{V} \rightarrow \widetilde{\mathfrak{D}}_g$. The restriction of this obstruction theory to fibers over $c \in \mathbb{A}^1$ will give the relative obstruction theories of $\mathcal{V}_c = \mathcal{V} \times_{\mathbb{A}^1} c \rightarrow \mathfrak{D}_g$.

We begin with a description of the tangent bundle $T_{\mathcal{Z}'/\widetilde{\mathfrak{D}}_g}$. Let $\varrho : \mathcal{Z}' \rightarrow \widetilde{\mathfrak{D}}_g$ be the tautological projection. Using the explicit description of $T_{C(V)/\mathbb{A}^1}$ given in (4.5), and the construction of \mathcal{Z}' in (4.9), we see that $\Omega_{\mathcal{Z}'/\widetilde{\mathfrak{D}}_g}^{\vee}$ fits into the exact sequence

$$0 \longrightarrow \Omega_{\mathcal{Z}'/\widetilde{\mathfrak{D}}_g}^{\vee} \longrightarrow \varrho^* \mathcal{L}_{\widetilde{\mathfrak{D}}_g}^{\oplus 5} \oplus \varrho^* \mathcal{L}_{\widetilde{\mathfrak{D}}_g}^{\otimes 5} \xrightarrow{d\mathcal{E}} \varrho^* \mathcal{L}_{\widetilde{\mathfrak{D}}_g}^{\otimes 5} \longrightarrow 0,$$

where $d\mathcal{E}$ restricted to $((z_i), y, t) \in \mathcal{Z}'$ sends $((\dot{z}_i), \dot{y})$ to $\sum 5z_i^4 \dot{z}_i - t\dot{y}$. (cf. (4.5).) Using that $\mathcal{L}_{\mathcal{V}} = f_{\mathcal{V}}^* \mathcal{O}(1)$, we obtain

$$(4.13) \quad \mathbf{e}_{\mathbf{v}}^* \Omega_{\mathcal{Z}'/\mathcal{C}_{\widetilde{\mathfrak{D}}_g}}^{\vee} \cong f_{\mathcal{V}}^* \mathcal{H} \quad \text{and} \quad \mathbf{e}_{\mathbf{v}}^* \Omega_{\mathcal{Z}/\mathcal{C}_{\widetilde{\mathfrak{D}}_g}}^{\vee} \cong f_{\mathcal{V}}^* \mathcal{H} \oplus \mathcal{P}_{\mathcal{V}},$$

where \mathcal{H} on $B \times \mathbb{A}^1$ is defined by the exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow q_{\mathbb{P}^4}^* \mathcal{O}(1)^{\oplus 5} \oplus q_{\mathbb{P}^4}^* \mathcal{O}(5) \xrightarrow{d'\tilde{s}} q_{\mathbb{P}^4}^* \mathcal{O}(5) \longrightarrow 0,$$

where $d'\tilde{s}$ is the differential of \tilde{s} in (4.1), after setting $d't = 0$. (Recall that $V \subset B \times \mathbb{A}^1$ by Lemma 4.1.)

We have a similar description

$$(4.14) \quad \mathbf{e}_{\mathbf{v}}^* \Omega_{\mathcal{Z}'/\mathcal{C}_{\mathfrak{D}_g}}^{\vee} \cong f_{\mathcal{V}}^* \mathcal{H} \quad \text{and} \quad \mathbf{e}_{\mathbf{v}}^* \Omega_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}}^{\vee} \cong f_{\mathcal{V}}^* \mathcal{H} \oplus \mathcal{P}_{\mathcal{V}},$$

where \mathcal{H} is defined by the exact sequence

$$(4.15) \quad 0 \longrightarrow \mathcal{H} \xrightarrow{i} q_{\mathbb{P}^4}^* \mathcal{O}(1)^{\oplus 5} \oplus q_{\mathbb{P}^4}^* \mathcal{O}(5) \oplus q_{\mathbb{P}^4}^* \mathcal{O} \xrightarrow{d\tilde{s}} q_{\mathbb{P}^4}^* \mathcal{O}(5) \longrightarrow 0,$$

where $d\tilde{s}$ is the differential of \tilde{s} in Lemma (4.1).

Proposition 4.2. *The pair $\mathcal{V} \rightarrow \widetilde{\mathfrak{D}}_g$ admits a perfect relative obstruction theory*

$$\phi_{\mathcal{V}/\widetilde{\mathfrak{D}}_g} : \mathbb{T}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g} \longrightarrow \mathbb{E}_{\mathcal{V}/\widetilde{\mathfrak{D}}_g} := R^* \pi_{\mathcal{V}*} (f_{\mathcal{V}}^* \mathcal{H} \oplus \mathcal{P}_{\mathcal{V}}).$$

Its specialization at $c \neq 0 \in \mathbb{A}^1$ (resp. $0 \in \mathbb{A}^1$) give the perfect relative obstruction theory of $\phi_{\mathcal{P}/\mathfrak{D}_g}$ (resp. $\phi_{\overline{\mathcal{M}}_g(N,d)\mathfrak{p}/\mathfrak{D}_g}$).

Proof. We fit $\mathbf{e}_v : \mathcal{C}_{\mathcal{V}_g} \rightarrow \mathcal{Z}$ (cf. (4.11)) into the commutative diagrams

$$(4.16) \quad \begin{array}{ccccc} \mathcal{V}_g & \xleftarrow{\pi_{\mathcal{V}}} & \mathcal{C}_{\mathcal{V}} & \xrightarrow{\mathbf{e}_v} & \mathcal{Z} \\ \downarrow & & \downarrow & & \downarrow \text{pr} \\ \tilde{\mathcal{D}}_g & \xleftarrow{\pi_{\tilde{\mathcal{D}}_g}} & \mathcal{C}_{\tilde{\mathcal{D}}_g} & \xlongequal{\quad} & \mathcal{C}_{\tilde{\mathcal{D}}_g}, \end{array}$$

where the left one is Cartesian. Using

$$(4.17) \quad \pi_{\mathcal{V}_g}^* \mathbb{T}_{\mathcal{V}_g/\tilde{\mathcal{D}}_g} \cong \mathbb{T}_{\mathcal{C}_{\mathcal{V}_g}/\mathcal{C}_{\tilde{\mathcal{D}}_g}} \longrightarrow \mathbf{e}_{\mathcal{V}_g}^* \mathbb{T}_{\mathcal{Z}/\mathcal{C}_{\tilde{\mathcal{D}}_g}} = \mathbf{e}_{\mathcal{V}_g}^* T_{\mathcal{Z}/\mathcal{C}_{\tilde{\mathcal{D}}_g}}$$

and applying the projection formula, we obtain

$$(4.18) \quad \phi_{\mathcal{V}_g/\tilde{\mathcal{D}}_g} : \mathbb{T}_{\mathcal{V}_g/\tilde{\mathcal{D}}_g} \longrightarrow R^{\bullet} \pi_{\mathcal{V}_g*} \mathbf{e}_{\mathcal{V}_g}^* T_{\mathcal{Z}/\mathcal{C}_{\tilde{\mathcal{D}}_g}}.$$

Let \mathfrak{S} be the moduli of section of $\mathcal{Z} \rightarrow \tilde{\mathcal{D}}_g$ constructed in Subsection 2.2. Because the evaluation morphism \mathbf{e}_v induces an open immersion $\mathcal{V} \rightarrow \mathfrak{S}$, using Proposition 2.5 implies that $\phi_{\mathcal{V}_g/\tilde{\mathcal{D}}_g}$ is a perfect relative obstruction theory.

Finally, the fiber product of every stack in (4.16) with $c \neq 0 \in \mathbb{A}^1$ gives the diagram used to construct $\phi_{\mathcal{P}/\mathcal{D}_g}$. Using $\iota_c : \mathcal{V} \times_{\mathbb{A}^1} c \rightarrow \mathcal{V}$, the functoriality of the construction ensures that $\phi_{\mathcal{P}/\mathcal{D}_g}$ is the composition of $\mathbb{T}_{\mathcal{P}/\mathcal{D}_g} \rightarrow \iota_c^* \mathbb{T}_{\mathcal{V}/\tilde{\mathcal{D}}_g}$ with

$$\iota_c^*(\phi_{\mathcal{V}/\tilde{\mathcal{D}}_g}) : \iota_c^* \mathbb{T}_{\mathcal{V}/\tilde{\mathcal{D}}_g} \longrightarrow \iota_c^* \mathbb{E}_{\mathcal{V}/\tilde{\mathcal{D}}_g} \cong \mathbb{E}_{\mathcal{P}/\mathcal{D}_g}.$$

In case $c = 0$, we define $\mathbb{E}_{\overline{\mathcal{M}}_g(N,d)p/\mathcal{D}_g} := \iota_0^* \mathbb{E}_{\mathcal{V}/\tilde{\mathcal{D}}_g}$. This proves the Proposition. \square

4.5. The obstruction theory of $\mathcal{V}/\mathcal{D}_g$. To compare the virtual cycle of \mathcal{V}_0 with $\mathcal{V}_{c \neq 0}$, we need the relative obstruction theory of $\mathcal{V} \rightarrow \mathcal{D}_g$.

We using the $\phi_{\mathcal{V}/\tilde{\mathcal{D}}_g}$ just constructed. We let

$$(4.19) \quad \mathcal{K} \longrightarrow q_{\mathbb{P}^4}^* \mathcal{O}_{\mathbb{P}^4} \cong \mathcal{O}_{B \times \mathbb{A}^1}$$

be the composition of i in (4.14) with the projection to the last factor. We form

$$\mu : R^* \pi_{\mathcal{V}*} f_{\mathcal{V}}^* \mathcal{K} \longrightarrow R^* \pi_{\mathcal{V}*} f_{\mathcal{V}}^* \mathcal{O}_{\mathcal{V}} \longrightarrow R^1 \pi_{\mathcal{V}*} \mathcal{O}_{\mathcal{C}_{\mathcal{V}}}[-1],$$

where the first arrow is $R^{\bullet} \pi_{\mathcal{V}*}$ of (4.19), and the second arrow if the tautological homomorphism from a two-term complex to its H^1 .

We let $C(\mu^{\vee})$ be the mapping cone of μ^{\vee} , and let $C(\mu^{\vee})^{\vee}$ be its dual. It fits into the distinguished triangle

$$(4.20) \quad R^1 \pi_{\mathcal{V}*} \mathcal{O}_{\mathcal{C}_{\mathcal{V}}}[-2] \longrightarrow C(\mu^{\vee})^{\vee} \longrightarrow R^* \pi_{\mathcal{V}*} f_{\mathcal{V}}^* \mathcal{K} \xrightarrow{+1} R^1 \pi_{\mathcal{V}*} \mathcal{O}_{\mathcal{C}_{\mathcal{V}}}[-1].$$

We define

$$\mathbb{E}'_{\mathcal{V}/\mathcal{D}_g} := R^* \pi_{\mathcal{V}*} (f_{\mathcal{V}}^* \mathcal{K} \oplus \mathcal{P}_{\mathcal{V}}) \quad \text{and} \quad \mathbb{E}_{\mathcal{V}/\mathcal{D}_g} := C(\mu^{\vee})^{\vee} \oplus R^* \pi_{\mathcal{V}*} \mathcal{P}_{\mathcal{V}}.$$

Then one has

$$(4.21) \quad R^1 \pi_{\mathcal{V}*} \mathcal{O}_{\mathcal{C}_{\mathcal{V}}}[-2] \longrightarrow \mathbb{E}_{\mathcal{V}/\mathcal{D}_g} \xrightarrow{\eta} \mathbb{E}'_{\mathcal{V}/\mathcal{D}_g} \xrightarrow{+\mu} R^1 \pi_{\mathcal{V}*} \mathcal{O}_{\mathcal{C}_{\mathcal{V}}}[-1].$$

By construction, $\mathbb{E}_{\mathcal{V}/\mathcal{D}_g}$ is a derived object representable by a two-term complex of locally free sheaves; its H^1 is

$$H^1(\mathbb{E}_{\mathcal{V}/\mathcal{D}_g}) = \ker\{H^1(\mu) : R^1 \pi_{\mathcal{V}*} (f_{\mathcal{V}}^* \mathcal{K} \oplus \mathcal{P}_{\mathcal{V}}) \longrightarrow R^1 \pi_{\mathcal{V}*} \mathcal{O}_{\mathcal{C}_{\mathcal{V}}}\}.$$

Since (4.19) is surjective, $H^1(\mu)$ is also surjective.

We now derive the perfect relative obstruction theory of $\mathcal{V} \rightarrow \mathfrak{D}_g$. Substituting $\tilde{\mathfrak{D}}_g$ and $\tilde{\mathcal{C}}_{\mathfrak{D}_g}$ in Proposition 4.2 by \mathfrak{D}_g and $\mathcal{C}_{\mathfrak{D}_g}$ respectively, and following the recipe in the proof of Proposition 4.2, we obtain a morphism

$$(4.22) \quad \phi'_{\mathcal{V}/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g} \longrightarrow R^* \pi_{\mathcal{V}*} \epsilon_{\mathfrak{v}}^* \mathbb{T}_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}} \cong R^* \pi_{\mathcal{V}*} (f_{\mathcal{V}}^* \mathcal{K} \oplus \mathcal{P}_{\mathcal{V}}) := \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g}.$$

Since moduli of sections of $\mathcal{Z} \rightarrow \tilde{\mathfrak{D}}_g$ is isomorphic to the moduli of sections of $\mathcal{Z} \rightarrow \mathfrak{D}_g$, where $\mathcal{Z} \rightarrow \mathfrak{D}_g$ is via the composite $\mathcal{Z} \rightarrow \tilde{\mathfrak{D}}_g \rightarrow \mathfrak{D}_g$, both are \mathcal{V} , thus Proposition 2.5 implies that $\phi'_{\mathcal{V}/\mathfrak{D}_g}$ is a perfect relative obstruction theory.

According to Proposition 4.2, the obstruction sheaf of $\phi'_{\mathcal{V}/\mathfrak{D}_g}$ has an extra factor $R^1 \pi_{\mathcal{V}*} \mathcal{O}_{\mathcal{V}}$ compared with that of $\phi_{\mathcal{P}/\mathfrak{D}_g}$ and of $\phi_{\overline{\mathcal{M}}_g(N,d)^p/\mathfrak{D}_g}$. Our solution is to lift it to a new obstruction theory (cf. (4.21))

$$(4.23) \quad \phi_{\mathcal{V}/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}$$

whose obstruction sheaf is parallel to that of $\phi_{\mathcal{P}/\mathfrak{D}_g}$ and of $\phi_{\overline{\mathcal{M}}_g(N,d)^p/\mathfrak{D}_g}$.

We denote by $\tilde{\tau} : \mathcal{C}_{\mathcal{V}} \rightarrow \mathcal{C}_{\tilde{\mathfrak{D}}_g}$ the tautological morphism covering the tautological projection τ shown in the Cartesian square

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{V}} & \xrightarrow{\tilde{\tau}} & \mathcal{C}_{\tilde{\mathfrak{D}}_g} \\ \downarrow \pi_{\mathcal{V}} & & \downarrow \\ \mathcal{V} & \xrightarrow{\tau} & \tilde{\mathfrak{D}}_g. \end{array}$$

Applying $\mathbb{T}_{\cdot/\mathcal{C}_{\mathfrak{D}_g}}$ to the evaluation $\mathcal{C}_{\mathfrak{D}_g}$ -morphism $\epsilon_{\mathfrak{v}} : \mathcal{C}_{\mathcal{V}} \rightarrow \mathcal{Z}$ (in (4.16)), the identity

$$\tilde{\tau} = \text{pr} \circ \epsilon_{\mathfrak{v}} : \mathcal{C}_{\mathcal{V}} \xrightarrow{\epsilon_{\mathfrak{v}}} \mathcal{Z} \xrightarrow{\text{pr}} \mathcal{C}_{\tilde{\mathfrak{D}}_g}$$

provides us a commutative square

$$\begin{array}{ccc} \epsilon_{\mathfrak{v}}^* \Omega_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}}^{\vee} & \longrightarrow & (\text{pr} \circ \epsilon_{\mathfrak{v}})^* \Omega_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}/\mathcal{C}_{\mathfrak{D}_g}}^{\vee} \cong \mathcal{O}_{\mathcal{C}_{\mathcal{V}}} \\ \uparrow & & \uparrow \\ \mathbb{T}_{\mathcal{C}_{\mathcal{V}}/\mathcal{C}_{\mathfrak{D}_g}} \cong \pi_{\mathcal{V}}^* \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g} & \longrightarrow & \tilde{\tau}^* \Omega_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}/\mathcal{C}_{\mathfrak{D}_g}}^{\vee} \cong \pi_{\mathcal{V}}^* \tau^* \Omega_{\tilde{\mathfrak{D}}_g/\mathfrak{D}_g}^{\vee}; \end{array}$$

applying projection formula to both vertical arrows, we further obtain the commutative diagrams

$$(4.24) \quad \begin{array}{ccccc} \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g} & \longrightarrow & R^* \pi_{\mathcal{V}*} \mathcal{O}_{\mathcal{C}_{\mathcal{V}}} & \longrightarrow & R^1 \pi_{\mathcal{V}*} \mathcal{O}_{\mathcal{C}_{\mathcal{V}}}[-1] \\ \uparrow \phi'_{\mathcal{V}/\mathfrak{D}_g} & & \uparrow & & \uparrow \\ \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g} & \longrightarrow & \tau^* \Omega_{\tilde{\mathfrak{D}}_g/\mathfrak{D}_g}^{\vee} = \mathcal{O}_{\mathcal{V}} & \longrightarrow & 0. \end{array}$$

This shows that $\mu \circ \phi'_{\mathcal{V}/\mathfrak{D}_g} = 0$ (cf. μ is in (4.21)). Applying $\text{Hom}(\mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}, \cdot)$ to (4.21), we conclude that the morphism $\phi'_{\mathcal{V}/\mathfrak{D}_g}$ (in (4.21)) lifts (non-uniquely) as stated in (4.23) such that

$$(4.25) \quad \eta \circ \phi_{\mathcal{V}/\mathfrak{D}_g} = \phi'_{\mathcal{V}/\mathfrak{D}_g}.$$

Proposition 4.3. *The homomorphism $\phi_{\mathcal{V}/\mathfrak{D}_g}$ is a perfect relative obstruction theory of $\mathcal{V} \rightarrow \mathfrak{D}_g$.*

Proof. We only need to check the criterion of perfect obstruction theory stated in the proof of Proposition 2.5. Namely, we need to show that to any square zero extension $T \subset T'$ of affine schemes by J , and a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\mathbf{m}} & \mathcal{V} \\ \downarrow & & \downarrow \\ T' & \xrightarrow{\mathbf{n}} & \mathfrak{D}_g, \end{array}$$

the arrow $\phi_{\mathcal{V}/\mathfrak{D}_g}$ assigns an element $\varpi(\mathbf{m}) \in H^1(T, \mathbf{m}^* \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g} \otimes J)$ (cf. (2.6)) such that there is a lifting $\mathbf{m}' : T' \rightarrow \mathcal{V}$ of the square above if and only if $\varpi(\mathbf{m}) = 0$.

Recall that $\phi'_{\mathcal{V}/\mathfrak{D}_g}$ is also a perfect relative obstruction theory. We let $\varpi(\mathbf{m})' \in H^1(T, \mathbf{m}^* \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g} \otimes J)$ be the associated obstruction class. Since $\phi_{\mathcal{V}/\mathfrak{D}_g}$ is a lift of $\phi'_{\mathcal{V}/\mathfrak{D}_g}$, $\varpi(\mathbf{m})'$ is the image of $\varpi(\mathbf{m})$ under the homomorphism

$$H^1(\eta) : H^1(T, \mathbf{m}^* \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g} \otimes J) \longrightarrow H^1(T, \mathbf{m}^* (R^* \pi_{\mathcal{V}*} \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g} \otimes J))$$

induced by the η in (4.21). Because of the distinguished triangle (4.21), $H^1(\eta)$ is injective. This proves that $\varpi(\mathbf{m}) = 0$ if and only if $\varpi(\mathbf{m})' = 0$. Since the later is the obstruction class, the former is too.

The other part of the criterion follows from the same reason. This proves the Proposition. \square

4.6. Comparison of obstruction theories. Let $c \in \mathbb{A}^1$ be any closed point. We denote the restrictions to fibers over c by

$$\iota_c : \mathcal{V}_c = \mathcal{V} \times_{\mathbb{A}^1} c \xrightarrow{\subset} \mathcal{V} \quad \text{and} \quad \mathbf{e}_{\mathfrak{v}_c} = \mathbf{e}_{\mathfrak{v}}|_{\mathcal{C}_{\mathfrak{v}_c}} : \mathcal{C}_{\mathfrak{v}_c} = \mathcal{C}_{\mathcal{V}} \times_{\mathcal{V}} \mathcal{V}_c \longrightarrow \mathcal{Z}_c = \mathcal{Z} \times_{\mathbb{A}^1} c.$$

Recall by Proposition 4.2 that composing the tautological $\mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} \rightarrow \iota_c^* \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}$ with $\iota_c^* \phi_{\mathcal{V}/\mathfrak{D}_g}$ gives the perfect relative obstruction theory

$$\phi_{\mathcal{V}_c/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} := \iota_c^* \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}.$$

(Note that for $c \neq 0$, $\mathcal{V}_c = \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$, and this obstruction theory coincide with the one constructed in Subsection 3.1.)

We now compare the obstruction theory $\phi_{\mathcal{V}/\mathfrak{D}_g}$ with $\phi_{\mathcal{V}_c/\mathfrak{D}_g}$. Using the tautological exact sequence

$$(4.26) \quad 0 \longrightarrow T_{\mathcal{Z}_c/\mathcal{C}_{\mathfrak{D}_g}} \longrightarrow T_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}}|_{\mathcal{Z}_c} \longrightarrow \mathcal{O}_{\mathcal{Z}_c} \longrightarrow 0,$$

we obtain a morphism of distinguished triangles (the top line is an exact sequence of sheaves):

$$(4.27) \quad \begin{array}{ccccc} \mathbf{e}_{\mathfrak{v}_c}^* \Omega_{\mathcal{Z}_c/\mathcal{C}_{\mathfrak{D}_g}}^{\vee} & \longrightarrow & \mathbf{e}_{\mathfrak{v}_c}^* (\Omega_{\mathcal{Z}/\mathcal{C}_{\mathfrak{D}_g}}^{\vee}|_{\mathcal{Z}_c}) & \longrightarrow & \mathbf{e}_{\mathfrak{v}_c}^* \mathcal{O}_{\mathcal{Z}_c} \cong \mathcal{O}_{\mathcal{C}_{\mathfrak{v}_c}} \xrightarrow{+1} 0 \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{T}_{\mathcal{C}_{\mathfrak{v}_c}/\mathcal{C}_{\mathfrak{D}_g}} & \longrightarrow & \mathbb{T}_{\mathcal{C}_{\mathfrak{v}}/\mathcal{C}_{\mathfrak{D}_g}}|_{\mathcal{C}_{\mathfrak{v}_c}} & \longrightarrow & \mathbb{T}_{\mathcal{C}_c/\mathcal{C}_{\mathfrak{v}}}[1] \xrightarrow{+1} \end{array}$$

By projection formula, we have a morphism of distinguished triangles

$$(4.28) \quad \begin{array}{ccccc} R^* \pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{V}_c}[-1] & \longrightarrow & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} & \xrightarrow{\beta'} & \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \xrightarrow{+1} \\ \uparrow & & \uparrow \phi_{\mathcal{V}_c/\mathfrak{D}_g} & & \uparrow \phi'_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \\ \mathbb{T}_{\mathcal{V}_c/\mathcal{V}} & \longrightarrow & \mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} & \xrightarrow{\gamma_0} & \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \xrightarrow{+1} \end{array} .$$

Applying the mapping cone construction (4.21) to the top row of (4.28), and using the octahedral axiom, we obtain a compatible diagram of mapping cones

(4.29)

$$\begin{array}{ccccccc}
& & & & R^1 \pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] & \xrightarrow{=} & R^1 \pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] \\
& & & & \uparrow \mu|_{\mathcal{V}_c} & & \uparrow \\
R^* \pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] & \longrightarrow & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} & \xrightarrow{\beta'} & \mathbb{E}'_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} & \longrightarrow & R^* \pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{V}_c} \\
\uparrow & & \uparrow & & \uparrow \eta|_{\mathcal{V}_c} & & \uparrow \\
\pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] & \longrightarrow & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} & \xrightarrow{\beta_0} & \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} & \xrightarrow{\beta_1} & \pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{V}_c}.
\end{array}$$

Restricting the perfect obstruction theory $\phi_{\mathcal{V}/\mathfrak{D}_g}$ (cf. Proposition 4.3) to \mathcal{V}_c , we obtain the following (not necessarily commuting) homomorphisms

$$(4.30) \quad \begin{array}{ccc}
\mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} & \xrightarrow{\beta_0} & \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \\
\uparrow \phi_{\mathcal{V}_c/\mathfrak{D}_g} & & \uparrow \phi_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \\
\mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} & \xrightarrow{\gamma_0} & \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}.
\end{array}$$

We consider

$$\delta = \phi_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \circ \gamma_0 - \beta_0 \circ \phi_{\mathcal{V}_c/\mathfrak{D}_g} : \mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}.$$

Applying the commutative diagrams (4.25), (4.29) and (4.28), we conclude that $\eta|_{\mathcal{V}_c} \circ \delta = \eta|_{\mathcal{V}_c} \circ \phi_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \circ \gamma_0 - \eta|_{\mathcal{V}_c} \circ \beta_0 \circ \phi_{\mathcal{V}_c/\mathfrak{D}_g} = \phi'_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \circ \gamma_0 - \beta' \circ \phi_{\mathcal{V}_c/\mathfrak{D}_g} = 0$.

Therefore, δ factors through $R^1 \pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{V}_c}[-2] \rightarrow \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}$.

Because of this, after applying the truncation functor $\tau_{\leq 1}$ to (4.30), we obtain a commutative square

$$(4.31) \quad \begin{array}{ccc}
\mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} & \xrightarrow{\beta_0} & \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} \\
\uparrow \phi_{\mathcal{V}_c/\mathfrak{D}_g}^{\leq 1} & & \uparrow \phi_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}^{\leq 1} \\
\mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g}^{\leq 1} & \xrightarrow{\gamma_0} & \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}^{\leq 1}
\end{array}$$

On the other hand, applying the truncation functor $\tau_{\leq 1}$ to the left square in (4.28), we obtain another commutative square

$$(4.32) \quad \begin{array}{ccc}
\pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] & \longrightarrow & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} \\
\uparrow & & \uparrow \phi_{\mathcal{V}_c/\mathfrak{D}_g}^{\leq 1} \\
\mathbb{T}_{\mathcal{V}_c/\mathcal{V}}^{\leq 1} & \longrightarrow & \mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g}^{\leq 1}.
\end{array}$$

Combined, we have a commutative diagrams

$$(4.33) \quad \begin{array}{ccccc}
\mathcal{O}_{\mathcal{V}_c}[-1] \cong \pi_{\mathcal{V}_c} \mathcal{O}_{\mathcal{C}_{\mathcal{V}_c}}[-1] & \longrightarrow & \mathbb{E}_{\mathcal{V}_c/\mathfrak{D}_g} & \xrightarrow{\beta_0} & \mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c} & \xrightarrow{+1} \\
\uparrow & & \uparrow \phi_{\mathcal{V}_c/\mathfrak{D}_g}^{\leq 1} & & \uparrow \phi_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}^{\leq 1} & \\
\mathbb{T}_{\mathcal{V}_c/\mathcal{V}}^{\leq 1} & \longrightarrow & \mathbb{T}_{\mathcal{V}_c/\mathfrak{D}_g}^{\leq 1} & \xrightarrow{\gamma_0} & \mathbb{T}_{\mathcal{V}/\mathfrak{D}_g}|_{\mathcal{V}_c}^{\leq 1} &
\end{array}$$

By (4.29) the top row is a distinguished triangle (but not the lower one).

We comment that applying results in [KKP], this diagram implies that the virtual cycles of \mathcal{V}_c is the pull back via $\iota_c : \mathcal{V}_c \rightarrow \mathcal{V}$ of the virtual cycle of \mathcal{V} . In our case, we are using localized virtual cycles via cosections of the obstruction sheaves, thus we need to construct a cosection of the obstruction sheaf

$$Ob_{\mathcal{V}} = \text{coker}\{T_{\mathfrak{D}_g} \otimes_{\mathcal{O}_{\mathfrak{D}_g}} \mathcal{O}_{\mathcal{V}} \longrightarrow H^1(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})\}.$$

4.7. Family cosection. We first construct a cosection of the obstruction sheaf $Ob_{\mathcal{V}/\tilde{\mathfrak{D}}_g}$. The construction is parallel to the case $\mathcal{P} = \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p$.

First, we define a bi-linear morphism of bundles

$$h : \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5} \oplus \mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5} \oplus \mathcal{P}_{\tilde{\mathfrak{D}}_g}) \longrightarrow \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5}) \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \text{Vb}(\mathcal{P}_{\tilde{\mathfrak{D}}_g}) \longrightarrow \text{Vb}(\omega_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}/\tilde{\mathfrak{D}}_g}).$$

Here the first arrow is $(\text{pr}_2, \text{pr}_3)$, where pr_i is the i -th projection; the second arrow is induced by tensoring of sheaves of $\mathcal{O}_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}}$ -modules $\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5} \otimes \mathcal{P}_{\tilde{\mathfrak{D}}_g} \rightarrow \omega_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}/\tilde{\mathfrak{D}}_g}$. Using that the family $\mathcal{Z} \rightarrow \mathcal{C}_{\tilde{\mathfrak{D}}_g}$ in (4.10) is a subfamily

$$\mathcal{Z} \subset \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5})^* \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5}) \times_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}} \text{Vb}(\mathcal{P}_{\tilde{\mathfrak{D}}_g}) \subset \text{Vb}(\mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\oplus 5} \oplus \mathcal{L}_{\tilde{\mathfrak{D}}_g}^{\otimes 5} \oplus \mathcal{P}_{\tilde{\mathfrak{D}}_g}),$$

composing with h , we obtain a $\mathcal{C}_{\tilde{\mathfrak{D}}_g}$ -morphism

$$(4.34) \quad \mathcal{Z} \longrightarrow \text{Vb}(\omega_{\mathcal{C}_{\tilde{\mathfrak{D}}_g}/\tilde{\mathfrak{D}}_g}).$$

Lemma 4.4. *The homomorphism (4.34) induces a homomorphism*

$$\sigma^\bullet : \mathbb{E}_{\mathcal{V}/\tilde{\mathfrak{D}}_g} \longrightarrow R^1\pi_{\mathcal{V}*}\mathcal{O}_{\mathcal{V}}[-1]$$

whose restriction to $\mathcal{V} \times_{\mathbb{A}^1} c \cong \mathcal{P}$, $c \neq 0$, is proportional (by an element in \mathbb{C}^*) to σ_1^\bullet in (3.7).

Proof. The proof is exactly as in Section 3.2. We will omit it here. \square

We denote

$$(4.35) \quad \sigma = H^1(\sigma^\bullet) : Ob_{\mathcal{V}/\tilde{\mathfrak{D}}_g} := H^1(\mathbb{E}_{\mathcal{V}/\tilde{\mathfrak{D}}_g}) \longrightarrow R^1\pi_{\mathcal{V}*}\omega_{\mathcal{V}/\mathcal{V}} \cong \mathcal{O}_{\mathcal{V}}.$$

Let $\tilde{\mathfrak{q}} : \mathcal{V} \rightarrow \tilde{\mathfrak{D}}_g$ be the projection. The distinguished triangle $\tilde{\mathfrak{q}}^*\mathbb{L}_{\tilde{\mathfrak{D}}_g} \rightarrow \mathbb{L}_{\mathcal{V}} \rightarrow \mathbb{L}_{\mathcal{V}/\tilde{\mathfrak{D}}_g} \rightarrow \tilde{\mathfrak{q}}^*\mathbb{T}_{\tilde{\mathfrak{D}}_g}[1]$ gives a morphism $\tilde{\mathfrak{q}}^*\mathbb{T}_{\tilde{\mathfrak{D}}_g} \rightarrow \mathbb{T}_{\mathcal{V}/\tilde{\mathfrak{D}}_g}[1]$, which composed with $\phi_{\mathcal{V}/\tilde{\mathfrak{D}}_g} : \mathbb{T}_{\mathcal{V}/\tilde{\mathfrak{D}}_g} \rightarrow \mathbb{E}_{\mathcal{V}/\tilde{\mathfrak{D}}_g}$ gives

$$\eta : \tilde{\mathfrak{q}}^*\mathbb{T}_{\tilde{\mathfrak{D}}_g} \longrightarrow \mathbb{E}_{\mathcal{V}/\tilde{\mathfrak{D}}_g}[1].$$

Taking the cokernel of the H^0 of this arrow, we obtain the absolute obstruction sheaf

$$(4.36) \quad Ob_{\mathcal{V}} := \text{coker}\{H^0(\eta) : \tilde{\mathfrak{q}}^*\Omega_{\tilde{\mathfrak{D}}_g}^\vee \longrightarrow H^1(\mathbb{E}_{\mathcal{V}/\tilde{\mathfrak{D}}_g})\}.$$

Lemma 4.5. *The following composite vanishes*

$$(4.37) \quad \tilde{\mathfrak{q}}^*\Omega_{\tilde{\mathfrak{D}}_g}^\vee \xrightarrow{H^0(\eta)} H^1(\mathbb{E}_{\mathcal{V}/\tilde{\mathfrak{D}}_g}) \xrightarrow{\sigma} R^1\pi_{\mathcal{V}*}\omega_{\mathcal{V}/\mathcal{V}}.$$

Proof. The proof is exactly the same as the that of Proposition 3.5, and will be omitted. \square

This immediately gives

Corollary 4.6. *The cosection $\sigma : Ob_{\mathcal{V}/\tilde{\mathfrak{D}}_g} \rightarrow \mathcal{O}_{\mathcal{V}}$ lifts to a cosection $\bar{\sigma} : Ob_{\mathcal{V}} \rightarrow \mathcal{O}_{\mathcal{V}}$.*

Lastly, we describe the degeneracy (non-surjective) loci of σ . As before, we say σ is degenerate at $\xi \in \mathcal{V}$ if $\sigma|_\xi$ is not surjective (i.e. is trivial). Let $\xi \in \mathcal{V}$ be any closed point; ξ is represented by

$$((\phi_i), b, p) \in H^0(L^{\oplus 5}) \times H^0(L^{\otimes 5}) \times H^0(L^{-\otimes 5} \otimes \omega_C)$$

for $(C, L) \in \mathfrak{D}_g$ the point under ξ . Then $\sigma|_\xi : \mathcal{O}b_{\mathcal{V}/\tilde{\mathfrak{D}}_g}|_\xi \rightarrow \mathbb{C}$ is identical to the composite of the inclusion

$$\mathcal{O}b_{\mathcal{V}/\tilde{\mathfrak{D}}_g}|_\xi \subset H^1(L^{\oplus 5}) \oplus H^1(L^{\otimes 5}) \oplus H^1(L^{-\otimes 5} \otimes \omega_C)$$

with the pairing

$$H^1(L^{\oplus 5}) \oplus H^1(L^{\otimes 5}) \oplus H^1(L^{-\otimes 5} \otimes \omega_C) \longrightarrow H^1(\omega_C)$$

defined via $((\phi_i), \mathring{b}, \mathring{p}) \mapsto \mathring{b} \cdot p + b \cdot \mathring{p}$. Like the proof of Proposition 3.4, this description shows that the degeneracy loci of σ is $\overline{\mathcal{M}}_g(Q, d) \times \mathbb{A}^1 \subset \mathcal{V}$, where the inclusion is via vanishing p -fields and the inclusion $\overline{\mathcal{M}}_g(Q, d) \times \mathbb{A}^1 \subset \overline{\mathcal{M}}_g(V, d)$ induced by the tautological inclusion $Q \times \mathbb{A}^1 \subset V$.

Lemma 4.7. *The degeneracy loci of the cosection $\bar{\sigma}$ is $\overline{\mathcal{M}}_g(Q, d) \times \mathbb{A}^1 \subset \mathcal{V}$; it is proper over \mathbb{A}^1 .*

Proof. We first need to verify that σ is as given. The proof of this is exactly the same as that of Lemma 3.2. Using this description, we argue that the degeneracy loci of the cosection $\sigma : \mathcal{O}b_{\mathcal{V}/\mathfrak{D}_g} \rightarrow \mathcal{O}_{\mathcal{V}}$ is $\overline{\mathcal{M}}_g(Q, d) \times \mathbb{A}^1 \subset \mathcal{V}$; thus is proper over \mathbb{A}^1 . Since $\bar{\sigma}$ is a lift of σ , the degeneracy loci of $\bar{\sigma}$ coincides with that of σ . This proves the Lemma. \square

4.8. The constancy of the invariants. By direct verification, the virtual dimension of \mathcal{V} is one. Using Lemma 4.7 and Corollary 4.6, following the convention introduced in Subsection 3.6, we denote by

$$h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})_{\bar{\sigma}} \subset h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})$$

the kernel of a cone-stack morphism $h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}) \rightarrow \mathcal{C}_{\mathcal{V}}$ induced by $\bar{\sigma}$ defined as in (3.16).²

Let

$$[\mathbf{C}_{\mathcal{P}/\mathfrak{D}_g}] \in Z_* h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})$$

be the intrinsic normal cone embedded using the obstruction theory $\phi_{\mathcal{V}/\mathfrak{D}_g}$. Because of Lemma 4.7 and Corollary 4.6, applying [KL2, Thm 5.1] we conclude that

$$[\mathbf{C}_{\mathcal{P}/\mathfrak{D}_g}] \in Z_* h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})_{\bar{\sigma}}.$$

We then applying the localized Gysin map [KL2]

$$0_{\bar{\sigma}, \text{loc}}^1 : A_* h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})_{\bar{\sigma}} \longrightarrow A_*(\overline{\mathcal{M}}_g(Q, d) \times \mathbb{A}^1).$$

Definition 4.8. *We define the localized virtual cycle of $(\mathcal{V}, \bar{\sigma})$ be*

$$[\mathcal{V}]_{\bar{\sigma}}^{\text{vir}} := 0_{\bar{\sigma}, \text{loc}}^1([\mathbf{C}_{\mathcal{V}/\mathfrak{D}_g}]) \in A_1(\overline{\mathcal{M}}_g(Q, d) \times \mathbb{A}^1).$$

Now let $c \in \mathbb{A}^1$ be any closed point and let $j_c : c \rightarrow \mathbb{A}^1$ be the closed inclusion. We denote $\mathcal{N} := \mathcal{V} \times_{\mathbb{A}^1} 0$. By the compatibility stated in diagram (4.33) and Corollary (4.6), we apply [KL2, Thm 5.3] to obtain

²It is $h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g})$ along the degeneracy loci and is the kernel of $h^1/h^0(\mathbb{E}_{\mathcal{V}/\mathfrak{D}_g}) \rightarrow \mathcal{O}_{\mathcal{V}}$ induced by σ away from the degeneracy loci.

Proposition 4.9. *Under the shriek operation of cycles ($c \neq 0$),*

$$j_c^*([\mathcal{V}]_{\bar{\sigma}}^{\text{vir}}) = [\mathcal{P}]_{\bar{\sigma}_1}^{\text{vir}} \in A_0 \overline{\mathcal{M}}_g(Q, d), \quad j_0^*([\mathcal{V}]_{\bar{\sigma}}^{\text{vir}}) = [\mathcal{N}]_{\bar{\sigma}_0}^{\text{vir}} \in A_0 \overline{\mathcal{M}}_g(Q, d).$$

Here $[\mathcal{N}]_{\bar{\sigma}_0}^{\text{vir}}$ is the localized virtual cycle using the obstruction theory of \mathcal{N} induced by the restriction of $\phi_{\mathcal{V}/\tilde{\mathcal{D}}_g}$ (Prop 4.2) and the restriction of the cosection $\bar{\sigma}_0 = \bar{\sigma}|_{\mathcal{N}}$.

5. GROMOV WITTEN INVARIANT OF (K_N, \mathbf{w}_N)

We continue to denote by

$$r : N \longrightarrow Q$$

the normal bundle to Q in \mathbb{P}^4 . Let K_N be the total space of the canonical line bundle of N , which is isomorphic to the underlying line bundle of the pull back $r^* \mathcal{O}(-5)$. The duality pairing $\mathcal{O}_Q(5) \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(-5) \rightarrow \mathcal{O}_Q$ defines a regular function $\mathbf{w}_N \in \Gamma(\mathcal{O}_{K_N})$. The degree $\deg[\mathcal{N}]_{\bar{\sigma}_0}^{\text{vir}}$ are the Gromov-Witten invariants of the Landau-Ginzburg space (K_N, w_N) .

We denote by $\overline{\mathcal{M}}_g(N, d)$ the moduli space of genus g degree d stable morphisms to N , where the degree is measured by their images in \mathbb{P}^4 via $N \rightarrow Q \subset \mathbb{P}^4$. Because $N = V \times_{\mathbb{A}^1} 0$, canonically $\overline{\mathcal{M}}_g(N, d) = \mathcal{V} \times_{\mathbb{A}^1} 0$. The moduli of stable maps coupled with p -fields is identical to \mathcal{N}

$$\mathcal{N} := \mathcal{V} \times_{\mathbb{A}^1} 0 = \overline{\mathcal{M}}_g(V, d)^p \times_{\mathbb{A}^1} 0 \cong \overline{\mathcal{M}}_g(N, d)^p.$$

We let

$$(f_{\mathcal{N}}, \pi_{\mathcal{N}}) : \mathcal{C}_{\mathcal{N}} \longrightarrow N \times \mathcal{N}$$

be the universal map of \mathcal{N} . By definition, it is the restriction of $(f_{\mathcal{V}}, \pi_{\mathcal{V}}, \mathcal{C}_{\mathcal{V}})$ to the fiber over $0 \in \mathbb{A}^1$.

5.1. The invariants and the equivalence. As indicated in the beginning of Subsection 4.6, we have evaluation morphism

$$\mathbf{e}_{\mathcal{N}} : \mathcal{C}_{\mathcal{N}} \longrightarrow \mathcal{Z}_0 = \mathcal{Z} \times_{\mathbb{A}^1} 0.$$

By construction in Proposition 4.2,

$$(5.1) \quad \phi_{\mathcal{N}/\mathcal{D}_g} : \mathbb{T}_{\mathcal{N}/\mathcal{D}_g} \longrightarrow R^* \pi_{\mathcal{N}*} \mathbf{e}_{\mathcal{N}}^* \mathbb{T}_{\mathcal{Z}_0/\mathcal{C}_{\mathcal{D}_g}} := \mathbb{E}_{\mathcal{N}/\mathcal{D}_g}$$

is a perfect relative obstruction theory of $\mathcal{N}/\mathcal{D}_g$, which is identical to the restriction of $\phi_{\mathcal{V}/\tilde{\mathcal{D}}_g}$ to the fiber over $0 \in \mathbb{A}^1$.

We let σ_0 be the restriction of σ to \mathcal{N} :

$$(5.2) \quad \sigma_0 = \sigma|_{\mathcal{N}} : \mathcal{O}b_{\mathcal{N}/\mathcal{D}_g} \longrightarrow \mathcal{O}_{\mathcal{N}}.$$

Proposition 5.1. *The cosection σ_0 lifts to a cosection $\bar{\sigma}_0 : \mathcal{O}b_{\mathcal{N}} \rightarrow \mathcal{O}_{\mathcal{N}}$. The degeneracy (non-surjective) loci of the cosection $\bar{\sigma}_0$ is $\overline{M}_g(Q, d) \subset \overline{M}_g(N, d)^P$; thus is proper.*

Proof. This follows directly from Lemma 4.7. □

Because the virtual dimension of \mathcal{V} is one, the virtual dimension of \mathcal{N} is 0. By Proposition 3.4, applying cosection localization Gysin map [KL2, Theorem 5.1], we obtain

Definition-Proposition 5.2. We define the localized virtual cycle of $\overline{\mathcal{M}}_g(N, d)^P$ be

$$[\overline{\mathcal{M}}_g(N, d)^P]_{\sigma}^{\text{vir}} := 0_{\sigma, \text{loc}}^1([\mathbf{C}_{\overline{\mathcal{M}}_g(N, d)^P/\mathfrak{D}_g}]) \in A_0 \overline{\mathcal{M}}_g(Q, d);$$

we denote $N_g(d)_{K_{N_Q}} = \deg [\overline{\mathcal{M}}_g(N, d)^P]_{\sigma}^{\text{vir}}$.

We call $N_g(d)_{K_{N_Q}}$ the formal Landau-Ginzburg Model.

Theorem 5.3. For any positive d , the invariants coincide: $N_g(d)_{\mathbb{P}^4}^P = N_g(d)_{K_{N_Q}}$.

Proof. It follows directly from Proposition 4.9. \square

5.2. Comparing with the GW invariant of Quintics. We now show that the formal Landau-Ginzburg model gives the same invariants as the Gromov-Witten invariants of Q up to signs.

We first construct a perfect relative obstruction theory of $\mathcal{N} \rightarrow \mathcal{Q}$. In the fiber product over \mathfrak{D}_g

$$(5.3) \quad \begin{array}{ccc} \mathcal{N} & \xrightarrow{\gamma} & \mathfrak{C} := C(\pi_{\mathfrak{D}_g*}(\mathcal{L}_{\mathfrak{D}_g}^{\otimes 5} \oplus \mathcal{P}_{\mathfrak{D}_g})) \\ \downarrow v & & \downarrow \\ \mathcal{Q} := \overline{\mathcal{M}}_g(Q, d) & \longrightarrow & \mathfrak{D}_g, \end{array}$$

where $C(\pi_{\mathfrak{D}_g*}(\mathcal{L}_{\mathfrak{D}_g}^{\otimes 5} \oplus \mathcal{P}_{\mathfrak{D}_g}))$ is the direct image cone constructed in Subsection 2.1, the morphism γ pulls back the relative perfect obstruction theory

$$(5.4) \quad \mathbb{T}_{\mathfrak{C}/\mathfrak{D}_g} \longrightarrow \mathbb{E}_{\mathfrak{C}/\mathfrak{D}_g}$$

to the morphism

$$\phi_{\mathcal{N}/\mathcal{Q}} : \mathbb{T}_{\mathcal{N}/\mathcal{Q}} \longrightarrow \mathbb{E}_{\mathcal{N}/\mathcal{Q}} := \gamma^* \mathbb{E}_{\mathfrak{C}/\mathfrak{D}_g}.$$

By Proposition 2.5, $\phi_{\mathcal{N}/\mathcal{Q}}$ is the perfect relative obstruction theory associated with the direct image cone stack $\mathcal{N} \cong C(\pi_{\mathcal{Q}*}(\mathcal{L}_{\mathcal{Q}}^{\otimes 5} \oplus \mathcal{P}_{\mathcal{Q}}))$ relative to $\mathcal{Q} = \overline{\mathcal{M}}_g(Q, d)$.

We define

$$(5.5) \quad \mathfrak{Q} = \text{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\oplus 5})^* \times_{\mathbb{P}^4} Q.$$

The evaluation maps of \mathcal{N} and \mathcal{Q} fit into the diagram

$$\begin{array}{ccccc} \mathcal{C}_{\mathcal{N}} & \xrightarrow{e_{\mathcal{N}}} & \mathcal{Z}_0 & \longrightarrow & \text{Vb}(\mathcal{L}_{\mathfrak{D}_g}^{\otimes 5}) \times_{\mathfrak{D}_g} \text{Vb}(\mathcal{P}_{\mathfrak{D}_g}) \\ \downarrow v_{\mathcal{C}} & & \downarrow & & \downarrow \\ \mathcal{C}_{\mathcal{Q}} & \xrightarrow{e_{\mathcal{Q}}} & \mathfrak{Q} & \longrightarrow & \mathcal{C}_{\mathfrak{D}_g} \end{array}$$

where the right square is a fiber product of smooth morphisms, and $v_{\mathcal{C}}$ is induced by the vertical arrow v in diagram (5.3).

The diagram associates a morphism between distinguished triangles

$$\begin{array}{ccccccc} e_{\mathcal{N}}^* T_{\mathcal{Z}_0/\mathfrak{Q}} & \longrightarrow & e_{\mathcal{N}}^* T_{\mathcal{Z}_0/\mathcal{C}_{\mathfrak{D}_g}} & \longrightarrow & v_{\mathcal{C}}^* e_{\mathfrak{Q}}^* T_{\mathfrak{Q}/\mathcal{C}_{\mathfrak{D}_g}} & \xrightarrow{+1} & \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathbb{T}_{\mathcal{C}_{\mathcal{N}}/\mathcal{C}_{\mathfrak{Q}}} & \longrightarrow & \mathbb{T}_{\mathcal{C}_{\mathcal{N}}/\mathcal{C}_{\mathfrak{D}_g}} & \longrightarrow & v_{\mathcal{C}}^* \mathbb{T}_{\mathcal{C}_{\mathfrak{Q}}/\mathcal{C}_{\mathfrak{D}_g}} & \xrightarrow{+1} & \end{array}$$

Denoting $\mathbb{E}_{\mathcal{N}/\mathcal{Q}} := \pi_{\mathcal{N}*} \mathbf{e}_{\mathcal{N}}^* T_{\mathcal{Z}_0/\Omega}$, then by the projection formula we have

$$(5.6) \quad \begin{array}{ccccc} \mathbb{E}_{\mathcal{N}/\mathcal{Q}} & \longrightarrow & \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g} & \xrightarrow{h} & v^* \mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g} \xrightarrow{+1} \cdot \\ \uparrow \phi_{\mathcal{N}/\mathcal{Q}} & & \uparrow \phi_{\mathcal{N}/\mathfrak{D}_g} & & \uparrow v^* \phi_{\mathcal{Q}/\mathfrak{D}_g} \\ \mathbb{T}_{\mathcal{N}/\mathcal{Q}} & \longrightarrow & \mathbb{T}_{\mathcal{N}/\mathfrak{D}_g} & \longrightarrow & v^* \mathbb{T}_{\mathcal{Q}/\mathfrak{D}_g} \xrightarrow{+1} \end{array}$$

Composing the cosection $\sigma_0 : \mathcal{O}b_{\mathcal{N}/\mathfrak{D}_g} \rightarrow \mathcal{O}_{\mathcal{N}}$ (cf. (5.2)) with $H^1(\mathbb{E}_{\mathcal{N}/\mathcal{Q}}) \rightarrow H^1(\mathbb{E}_{\mathcal{N}/\mathfrak{D}_g})$, we obtain

$$\sigma'_0 := \mathcal{O}b_{\mathcal{N}/\mathcal{Q}} \longrightarrow \mathcal{O}_{\mathcal{N}}.$$

Arguing similar to Proposition 5.1, one sees that the degeneracy loci of σ'_0 equals $\mathcal{Q} \subset \mathcal{N}$.

Now let $U := \mathcal{N} - \mathcal{Q}$; it is open in \mathcal{N} , and both σ_0 and σ'_0 are surjective on U . By the octahedral axiom, we have a diagram

$$(5.7) \quad \begin{array}{ccccccc} \mathcal{O}_U[-1] & \xrightarrow{=} & \mathcal{O}_U[-1] & & & & \\ \uparrow & & \uparrow & & & & \\ \mathbb{E}_{\mathcal{N}/\mathcal{Q}}|_U & \longrightarrow & \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g}|_U & \longrightarrow & v^* \mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g}|_U & \xrightarrow{+1} \cdot & \\ \uparrow \chi_{\mathcal{Q}} & & \uparrow \chi & & \uparrow \parallel & & \\ \mathbb{E}'_{U/\mathcal{Q}} & \longrightarrow & \mathbb{E}'_{U/\mathfrak{D}_g} & \longrightarrow & v^* \mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g}|_U & \xrightarrow{+1} \end{array}$$

where all rows and columns are distinguished triangles, and the two vertical rows to $\mathcal{O}_U[-1]$ are induced by σ_0 and σ'_0 , respectively.

Lemma 5.4. *There are perfect relative obstruction theories $\phi'_{U/\mathcal{Q}}$ of U/\mathcal{Q} and ϕ'_{U/\mathfrak{D}_g} of U/\mathfrak{D}_g that fit into a compatible diagram*

$$(5.8) \quad \begin{array}{ccccccc} \mathbb{E}'_{U/\mathcal{Q}} & \xrightarrow{\theta_E} & \mathbb{E}'_{U/\mathfrak{D}_g} & \xrightarrow{h \circ \chi} & v^* \mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g} & \xrightarrow{+1} \cdot & \\ \uparrow \phi'_{U/\mathcal{Q}} & & \uparrow \phi'_{U/\mathfrak{D}_g} & & \uparrow v^* \phi_{\mathcal{Q}/\mathfrak{D}_g}^{\leq 1}|_U & & \\ \mathbb{T}_{U/\mathcal{Q}}^{\leq 1} & \xrightarrow{\theta} & \mathbb{T}_{U/\mathfrak{D}_g}^{\leq 1} & \longrightarrow & (v^* \mathbb{T}_{\mathcal{Q}/\mathfrak{D}_g})|_U^{\leq 1} & \xrightarrow{+1} \end{array}$$

Proof. Applying the truncation functor $\tau_{\leq 1}$ to $\phi_{\mathcal{N}/\mathfrak{D}_g}|_U$, we obtain

$$\phi_{\mathcal{N}/\mathfrak{D}_g}^{\leq 1}|_U : \mathbb{T}_{\mathcal{N}/\mathfrak{D}_g}^{\leq 1}|_U \longrightarrow \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g}|_U.$$

Then the commutative diagram

$$\begin{array}{ccc} \mathbb{T}_{\mathcal{N}/\mathfrak{D}_g}^{\leq 1}|_U & \xrightarrow{\phi_{\mathcal{N}/\mathfrak{D}_g}^{\leq 1}|_U} & \mathbb{E}_{\mathcal{N}/\mathfrak{D}_g}|_U \\ \downarrow & & \downarrow \\ H^1(\mathbb{T}_{\mathcal{N}/\mathfrak{D}_g}|_U) & \longrightarrow & \mathcal{O}b_{\mathcal{N}/\mathfrak{D}_g}|_U \end{array}$$

implies that the composition of $\phi_{\mathcal{N}/\mathfrak{D}_g}^{\leq 1}|_U$ with $\mathbb{E}_{\mathcal{N}/\mathfrak{D}_g}|_U \rightarrow \mathcal{O}_U[-1]$ vanishes. Hence $\phi_{\mathcal{N}/\mathfrak{D}_g}^{\leq 1}|_U = \chi \circ \phi'_{U/\mathfrak{D}_g}$ for some

$$\phi_{U/\mathfrak{D}_g} : \mathbb{T}_{U/\mathfrak{D}_g}^{\leq 1} \longrightarrow v^* \mathbb{E}_{\mathcal{Q}/\mathfrak{D}_g}.$$

It is direct to check ϕ_{U/\mathfrak{D}_g} is a perfect obstruction theory, and the middle square of the diagram (5.6) commutes. By similar reason the $\tau_{\leq 1}$ truncation of $\phi_{\mathcal{N}/\mathcal{Q}}|_U$,

$$\phi_{\mathcal{N}/\mathcal{Q}}|_U : \mathbb{T}_{U/\mathcal{Q}}^{\leq 1} \longrightarrow \mathbb{E}_{\mathcal{N}/\mathcal{Q}}|_U,$$

has its composition with $\mathbb{E}_{\mathcal{N}/\mathcal{Q}}|_U \rightarrow \mathcal{O}_U[-1]$ vanishes and lifts to a

$$\phi'_{U/\mathcal{Q}} : \mathbb{T}_{U/\mathcal{Q}}^{\leq 1} \longrightarrow \mathbb{E}_{U/\mathfrak{D}_g}$$

such that $\phi_{\mathcal{N}/\mathcal{Q}}|_U = \chi_Q \circ \phi'_{U/\mathcal{Q}}$. The map $\Delta := \theta_E \circ \phi'_{U/\mathcal{Q}} - \phi'_{U/\mathfrak{D}_g} \circ \theta$ in (5.8) thus satisfies $\chi \circ \Delta = 0$, hence Δ factors through a morphism $\mathcal{O}_U[-2] \rightarrow \mathbb{E}'_{U/\mathfrak{D}_g}$, which implies $\Delta = 0$; this is because when applying the truncation functor $\tau_{\leq 1}$ one obtains $\tau_{\leq 1}(\Delta) = \Delta$ and $\tau_{\leq 1}(\mathcal{O}_U[-2]) = 0$. \square

We now quote the virtual pull-back construction of Manolache in [Ma]. First the compatibility diagram (5.6) fits into the condition two in the construction of Manolache in [Ma]. Let $\mathbf{C}_{\mathcal{N}/\mathcal{Q}}$ be the intrinsic normal cone of \mathcal{N} relative to \mathcal{Q} and let $i : \mathbf{C}_{\mathcal{N}/\mathcal{Q}} \rightarrow h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})$ be the inclusion by the relative perfect obstruction theory (5.4). We let G' be the kernel of the morphism of bundle stacks

$$G' = \ker\{\sigma'_0 : G := h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}}) \longrightarrow \mathbf{C}_{\mathcal{N}}\},$$

where the arrow in the bracket is induced by the cosection σ'_0 , and the kernel is defined in (3.16). By the Cosection lemma in [KL2] and Lemma 5.4, we have

$$(5.9) \quad i(\mathbf{C}_{\mathcal{N}/\mathcal{Q}}) \subset i(\gamma^* \mathbf{C}_{\mathfrak{E}/\mathfrak{D}_g}) \subset G'.$$

Note that here the virtual rank of the bundle stack G is zero.

We generalize the construction in [Ma] and give a virtual pullback morphism of cosection localized classes

$$i_{G'}^! : A_* \mathcal{Q} \longrightarrow A_* \mathcal{Q}$$

defined as the composite of

$$(5.10) \quad A_* \mathcal{Q} \xrightarrow{\zeta} A_* \mathbf{C}_{\mathcal{N}/\mathcal{Q}} \xrightarrow{i_*} A_* G' \xrightarrow{0_{\sigma_0, \text{loc}}^!} A_* \mathcal{Q},$$

where $0_{\sigma_0, \text{loc}}^!$ is the localized Gysin map defined in [KL2], and ζ defined by first sending a cycle $\sum n_i [V_i]$ to $\sum n_i [\mathbf{C}_{V_i \times_{\mathcal{Q}} \mathcal{N}/V_i}]$, and then descending it to cycle class group. Note that i_* maps to $A_* G'$ is due to (5.9).

Following the same argument as in Corollary 4 in [Ma], we have

Lemma 5.5.

$$i_{G'}^!([\mathcal{Q}]^{\text{vir}}) = [\overline{M}_g(N, d)^p]_{\sigma_0}^{\text{vir}} \in A_0 \mathcal{Q}.$$

Proof. One needs to show that the KKP's deformation to normal cone lies inside the kernel of the cosection, which follows from the Lemma 6.3 in Appendix and Lemma 5.4. \square

We now prove our main Theorem.

Theorem 5.6. *We have*

$$N_g(d)_{\mathbb{P}^4}^p = N_g(d)_{K_{N\mathcal{Q}}} = (-1)^{5d+1-g} \cdot N_g(d)_Q.$$

Proof. We first compute the degree of the zero cycle $i_{G'}^1([\xi]) \in A_0\mathcal{Q}$, where ξ is any closed point in \mathcal{Q} . Let ξ be $[u, C] \in \mathcal{Q}$. We denote

$$V_1 = H^0(C, u^* \mathcal{O}(5)), \quad V_2 = H^0(C, u^* \mathcal{O}(-5) \otimes \omega_C) \cong V_1^\vee, \quad V = V_1 \oplus V_2.$$

It is direct to check (v is defined in diagram (5.3))

$$v^{-1}\xi := \mathcal{N} \times_{\mathcal{Q}} \xi \cong V, \quad G|_{v^{-1}\xi} \cong [V \times V^\vee / V],$$

where the action of V on $V \times V^\vee$ is via the zero homomorphism $0 : V \rightarrow V \times V^\vee$. One also checks that the cosection σ_0 restricted to $v^{-1}\xi$ is induced by

$$\sigma_\xi : V \times V^\vee = (V_1 \oplus V_2) \times (V_1^\vee \oplus V_2^\vee) \longrightarrow \mathbb{C},$$

given by dual pairings $V_i \times V_i^\vee \rightarrow \mathbb{C}$.

Applying the composition (5.10) step by step, from

$$\zeta([\xi]) = [\mathbf{C}_{V/\xi}] \in A_*(G'|_\xi),$$

we have

$$i_{G'}^1([\xi]) = 0_{\sigma_\xi, \text{loc}}^1(\mathbf{C}_{V/\xi}) = (-1)^{\text{rank } V} [\xi] = (-1)^{5d+1-g} [\xi] \in A_0\mathcal{Q}.$$

Here the second equality follows from

$$\mathbf{C}_{V/\xi} = [V \times 0 / V] \subset [V \times V^\vee / V] = G|_{v^{-1}\xi},$$

and [KL2, Example 2.9]. Finally $\text{rank } V = 5d + 1 - g$ by Riemann-Roch theorem.

Taking degree,

$$\deg i_{G'}^1([\xi]) = (-1)^{5d+1-g}.$$

Since both $[\mathcal{Q}]^{\text{vir}}$ and $[\overline{M}_g(N, d)^P]_{\sigma_0}^{\text{vir}}$ in Lemma 5.5 are of zero dimensions, taking degrees we obtain

$$\deg [\overline{M}_g(N, d)^P]_{\sigma_0}^{\text{vir}} = \deg i_{G'}^1([\xi]) \cdot \deg [\mathcal{Q}]^{\text{vir}} = (-1)^{5d+1-g} N_g(d)_Q.$$

This proves the second identity in the statement of the theorem. The first identity is Theorem 5.3. \square

6. APPENDIX

We recall some useful facts known to the experts.

6.1. Kresch-Kim-Pantev's construction. Let S be a stack.

Convention. For a complex (derived object) \mathbb{G} on S , we denote $\mathbb{G}(k)$ without further commenting to be

$$\mathbb{G}(k) := p_S^* \mathbb{G} \otimes p_{\mathbb{P}^1}^* \mathcal{O}(k);$$

further, whenever we see a complex over S appearing in a sequence involving complexes over $S \times \mathbb{P}^1$, we understand the complex as its pull-back from S .

Definition 6.1. Let $\mathbb{E}_1 \xrightarrow{b} \mathbb{E}_2 \longrightarrow \mathbb{E}_3 \xrightarrow{+1}$ be a distinguished triangle of objects in $\mathbf{D}(S)$ whose cohomologies concentrated at non-positive degrees. Assume \mathbb{E}_1 is of amplitude in $[-1, \infty]$. Let $[x, y]$ be the homogeneous coordinates of \mathbb{P}^1 , and let

$$\tilde{b} : \mathbb{E}_1(-1) \rightarrow \mathbb{E}_1 \oplus \mathbb{E}_2$$

be defined by $(x \cdot 1, y \cdot b)$. We form the mapping cone $c(\tilde{b})$ of \tilde{b} , which fits into the distinguished triangle

$$\mathbb{E}_1(-1) \xrightarrow{\tilde{b}} \mathbb{E}_1 \oplus \mathbb{E}_2 \longrightarrow c(\tilde{b}) \xrightarrow{+1}.$$

Applying the h^1/h^0 construction to $c(\bar{b})^\vee$, we obtain $h^1/h^0(c(\bar{b})^\vee)$, which is a cone-stack over $S \times \mathbb{P}^1$ [BF]. Following [KKP] we call it the deformation of $h^1/h^0(\mathbb{E}_2^\vee)$ to $h^1/h^0(\mathbb{E}_1^\vee) \times_S h^1/h^0(\mathbb{E}_3^\vee)$.

Let $i : X \rightarrow Y$ and $j : Y \rightarrow Z$ be morphisms of relative Deligne-Mumford type, between stacks. Let

$$(6.1) \quad i^* \mathbb{L}_{Y/Z} \xrightarrow{\beta} \mathbb{L}_{X/Z} \longrightarrow \mathbb{L}_{X/Y} \xrightarrow{+1}$$

be the induced distinguished triangle of cotangent complexes. We quote the main theorem of [KKP].

Proposition 6.2. [KKP] *We have a natural isomorphism*

$$N_{X \times \mathbb{P}^1 / M_{Y/Z}^c} \cong h^1/h^0(c(\tilde{\beta})^\vee).$$

Now we stated a truncated version which is dual to Definition 6.1.

Lemma 6.3. *Let*

$$\mathbb{T}_{X/Y}^{\leq 1} \longrightarrow \mathbb{T}_{X/Z}^{\leq 1} \xrightarrow{k} i^* \mathbb{T}_{Y/Z}^{\leq 1}$$

be the truncation by $\tau_{\leq 1}$ of the dual of the distinguished triangle (6.1). (It is not a distinguished triangle.) Let $c_0(\tilde{k})$ be defined by making

$$c_0(\tilde{k}) \longrightarrow i^* \mathbb{T}_{Y/Z}^{\leq 1} \oplus \mathbb{T}_{X/Z}^{\leq 1} \xrightarrow{\tilde{k}} i^* \mathbb{T}_{Y/Z}^{\leq 1} \otimes \mathcal{O}_{\mathbb{P}^1}(1)$$

a distinguished triangle, where $\tilde{k} = (x, y \cdot k)$ as in Definition 6.1. Then there is a natural isomorphism

$$h^1/h^0(c(\tilde{\beta})^\vee) \cong h^1/h^0(c_0(\tilde{k})).$$

Proof. Using simplicial resolution of Illusid, we can represent $i^* \mathbb{L}_{Y/Z}$ and $\mathbb{L}_{X/Z}$ by perfect complex (over X globally) of amplitude $[-\infty, 0]$ and represent $\beta : i^* \mathbb{L}_{Y/Z} \rightarrow \mathbb{L}_{X/Z}$ by a homomorphism of between these two complexes. From this it is direct to show that the canonical morphism

$$(6.2) \quad c_0(\tilde{k}) \longrightarrow c(\tilde{\beta})^\vee$$

induces isomorphisms on H^1 and H^0 of the two complexes in (6.2). Hence their truncations by $\tau_{\leq 1}$ are isomorphic under this arrow, which shows that the cone-stacks of the h^1/h^0 constructions of the two complexes in (6.2) are isomorphic under the arrow induced by (6.2). \square

6.2. Application. We recall the rational equivalence inside the deformations of ambient conestacks constructed by Kim-Kresch-Pantev [KKP].

Let Z be an Artin stack, locally of finite type and of pure-dimension. Let Y be a stack and $Y \rightarrow Z$ be a morphism of relative Deligne-Mumford type. in the derived category of coherent sheaves on X . Let \mathbb{E}^\vee (resp. $\mathbb{F}^\vee, \mathbb{V}^\vee$) be a perfect relative obstruction theory of X/Z (resp. $Y/Z, X/Y$).

Definition 6.4. *We say \mathbb{F} and \mathbb{E} are truncated-compatible (verses (V, s)) if there exists a commutative diagram*

$$(6.3) \quad \begin{array}{ccccccc} \mathbb{V} & \longrightarrow & \mathbb{E} & \xrightarrow{g} & \mathbb{F}|_X & \longrightarrow & \\ & & & & & +1 & \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathbb{T}_{X/Y}^{\leq 1} & \longrightarrow & \mathbb{T}_{X/Z}^{\leq 1} & \xrightarrow{k} & \mathbb{T}_{Y/Z}|_X^{\leq -1} & & \end{array}$$

such that its top row is a distinguished triangle, and its bottom row is the first line in Lemma 6.3.

Accordingly, the morphisms g and k in (6.3) induces homomorphism \tilde{g} and \tilde{k} that fit into a homomorphism of distinguished triangle's

$$\begin{array}{ccccccc} c_0(\tilde{g}) & \longrightarrow & \mathbb{F}|_X \oplus \mathbb{E} & \xrightarrow{\tilde{g}} & \mathbb{F}|_X(1) & \xrightarrow{+1} & \\ \uparrow & & \uparrow & & \uparrow & & \\ c_0(\tilde{k}) & \longrightarrow & \mathbb{T}_{Y/Z}^{\leq 1}|_X \oplus \mathbb{T}_{X/Z}^{\leq 1} & \xrightarrow{\tilde{k}} & \mathbb{T}_{Y/Z}^{\leq 1}|_X \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \xrightarrow{+1} & , \end{array}$$

where $c_0(\tilde{g})$ is to make the first row a distinguished triangle as $c_0(\tilde{k})$ did in Lemma 6.3. We let $M_{Y/Z}^0$ be the deformation of Z to the normal cone $\mathbf{C}_{Y/Z}$; let $\mathbf{C}_{X \times \mathbb{P}^1 / M_{Y/Z}^0}$ be the normal cone to $X \times \mathbb{P}^1$ in $M_{Y/Z}^0$, and let $N_{X \times \mathbb{P}^1 / M_{Y/Z}^0}$ be the normal sheaf of $X \times \mathbb{P}^1$ in $M_{Y/Z}^0$. By the functoriality of the h^1/h^0 construction, we have

$$(6.4) \quad \mathcal{D} := C_{X \times \mathbb{P}^1 / M_{Y/Z}^0} \subset N_{X \times \mathbb{P}^1 / M_{Y/Z}^0} \cong h^1/h^0(c_0(\tilde{k})),$$

where the isomorphism is proved in [KKP] and Lemma 6.3. We also have the inclusion

$$(6.5) \quad h^1/h^0(c_0(\tilde{k})) \subset h^1/h^0(c_0(\tilde{g})) \cong h^1/h^0(c(\overline{g^\vee})^\vee),$$

where $h^1/h^0(c(\overline{g^\vee})^\vee)$ is the deformation of $h^1/h^0(\mathbb{E})$ to $h^1/h^0(\mathbb{F}|_X) \times_X h^1/h^0(\mathbb{V})$ as in Definition 6.1. This shows that the truncated compatibility (6.3) is sufficient to apply Kresch-Kim-Pantev construction of rational equivalence.

6.3. Obstruction class assignments. Assume there is a smooth morphism of Artin stacks $H \rightarrow W$. Suppose $T \subset T'$ is a pair of affine schemes such that $J := I_{T/T'}$ and $J^2 = 0$. Fix a morphism $T' \rightarrow \mathfrak{D}_g$, which pulls back $\pi_{\mathfrak{D}_g} : \mathcal{C}_{\mathfrak{D}_g} \rightarrow \mathfrak{D}_g$ to $\pi_T : \mathcal{C}_T \rightarrow T$ and $\pi_{T'} : \mathcal{C}_{T'} \rightarrow T'$. Assume there is a commutative diagram

$$(6.6) \quad \begin{array}{ccc} \mathcal{C}_T & \xrightarrow{\mathfrak{e}} & H \\ \downarrow & & \downarrow \\ \mathcal{C}_{T'} & \longrightarrow & W. \end{array}$$

Since the ideal sheaf of $\mathcal{C}_T \subset \mathcal{C}_{T'}$ is $\pi_{T'}^* J$, it is a square zero extension. We denote $V_T := \mathfrak{e}^* \Omega_{H/W}^\vee$ then V_T is a locally free sheaf over \mathcal{C}_T . The diagram (6.6) provides a morphism

$$(6.7) \quad V_T^\vee \cong \mathfrak{e}^* \mathbb{L}_{H/W} \longrightarrow \mathbb{L}_{\mathcal{C}_T/\mathcal{C}_{T'}} = \pi_{T'}^* \mathbb{L}_{T/T'} \longrightarrow \mathbb{L}_{\mathcal{C}_T/\mathcal{C}_{T'}}^{\geq -1} = \pi_{T'}^* J[1],$$

(here \mathfrak{e}^* denotes derived pull back) which defines an element

$$\omega(\mathfrak{e}, H, W) \in \text{Ext}_{\mathcal{C}_T}^1(V_T^\vee, \pi_{T'}^* J) \cong H^1(\mathcal{C}_T, V_T \otimes \pi_{T'}^* J).$$

Lemma 6.5. $\omega(\mathfrak{e}, H, W) = 0$ if and only if the diagram (6.6) admits a lifting $\mathcal{C}_{T'} \rightarrow H$ that commutes with the diagram.

Proof. We form the diagram

$$(6.8) \quad \begin{array}{ccc} X_0 := \mathcal{C}_T & \xrightarrow{i} & X := \mathcal{C}_{T'} \\ \downarrow \bar{\epsilon} & & \\ Y_0 := H \times_W \mathcal{C}_T & \xrightarrow{j} & Y := H \times_W \mathcal{C}_{T'} \\ \downarrow \Delta & & \downarrow \\ \mathcal{C}_T & \xrightarrow{\subset} & S := \mathcal{C}_{T'} \end{array}$$

where i and j are extensions over S . By construction, the associated homomorphism of sheaves

$$v : \bar{\epsilon}^* I_{Y_0/Y} \longrightarrow I_{X_0/X} = \pi_T^* J$$

is an isomorphism. If $\bar{\epsilon}$ lifts to a $\mathcal{C}_{T'}$ -morphism $\bar{\epsilon}' : X \rightarrow Y$, then the $X \rightarrow \mathcal{C}_{T'}$ is an isomorphism.

Following the steps in the proof of [II, Thm 2.1.7], the obstruction to the existence of such $\bar{\epsilon}'$ (in the notation of [II]) are constructed as follows. First one has a sequence of cotangent complexes

$$(6.9) \quad \mathbb{L}_{X_0/Y_0}[-1] \longrightarrow \bar{\epsilon}^* \mathbb{L}_{Y_0/S} \longrightarrow \bar{\epsilon}^* \mathbb{L}_{Y_0/Y} \longrightarrow \bar{\epsilon}^* \mathbb{L}_{Y_0/Y}^{\geq -1} \longrightarrow \pi_T^* J[1],$$

where the first (left) morphism comes from the triple $X_0 \rightarrow Y_0 \rightarrow S$; the middle morphism is induced by $\mathbb{L}_{Y_0/S} \rightarrow \mathbb{L}_{Y_0/Y}$.

Using $\mathbb{L}_{X_0/Y_0} = V_T^\vee[1]$, this sequence associates an element

$$\omega(\bar{\epsilon}, j) \in \text{Ext}_{X_0}^2(\mathbb{L}_{X_0/Y_0}, \pi_T^* J) = \text{Ext}_{X_0}^1(V_T^\vee, \pi_T^* J) = H^1(\mathcal{C}_T, V_T \otimes \pi_T^* J).$$

The argument in [II, Thm 2.1.7] shows that $\omega(\bar{\epsilon}, j) = 0$ if and only if a lift $\bar{\epsilon}' : X \rightarrow Y$ exists in the diagram (6.8). Such lift exists if and only if a lift $\epsilon' : \mathcal{C}_{T'} \rightarrow H$ exists in the diagram (6.6). Hence we only need to verify that $\omega(\bar{\epsilon}, j) = \omega(\epsilon, H, W)$.

To this end, we verify the commutativity of the following diagram

$$(6.10) \quad \begin{array}{ccccc} \mathbb{L}_{X_0/Y_0}[-1] & \longrightarrow & \bar{\epsilon}^* \mathbb{L}_{Y_0/S} & \longrightarrow & \bar{\epsilon}^* \mathbb{L}_{Y_0/Y} \\ \downarrow \cong & & \uparrow & & \uparrow \cong \\ \bar{\epsilon}^* \mathbb{L}_{Y_0/X_0} & \xleftarrow{\cong} & \bar{\epsilon}^* j^* \mathbb{L}_{Y/S} & \longrightarrow & \mathbb{L}_{X_0/S}, \end{array}$$

where the first vertical arrow is an isomorphism because $\mathbb{L}_{X_0} = 0$; the left square is commutative because the canonical $\mathbb{L}_{Y_0/S} \rightarrow \mathbb{L}_{Y_0/X_0}$ induces a $\bar{\epsilon}^* \mathbb{L}_{Y_0/S} \rightarrow \bar{\epsilon}^* \mathbb{L}_{Y_0/X_0}$ that splits the left square into two commutative triangles of cotangent complexes; the third vertical arrow is composing $\mathbb{L}_{X_0/S} \cong \bar{\epsilon}^* \Delta^* \mathbb{L}_{X_0/S}$ with the isomorphism $\Delta^* \mathbb{L}_{X_0/S} \xrightarrow{\cong} \mathbb{L}_{Y_0/Y}$. The right square is commutative because one has a canonical pullback $\bar{\epsilon}^* \mathbb{L}_{Y_0/S} \rightarrow \mathbb{L}_{X_0/S}$ and a commutative diagram

$$\begin{array}{ccc} \bar{\epsilon}^* \mathbb{L}_{Y_0/S} & \longrightarrow & \bar{\epsilon}^* \mathbb{L}_{Y_0/Y} \\ \downarrow & & \uparrow \cong \\ \mathbb{L}_{X_0/S} & \xleftarrow{\cong} & \bar{\epsilon}^* \Delta^* \mathbb{L}_{X_0/S}. \end{array}$$

The upper and lower rows of the diagram (6.10) are respectively sequence (6.7) and (6.9). Thus the commutative diagram (6.10) implies $\omega(\bar{\epsilon}, j) = \omega(\epsilon, H, W)$. \square

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