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AN EXPANSION ESTIMATE FOR DISPERSING PLANAR BILLIARDS WITH CORNER POINTS

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Abstract. It is known that the dynamics of planar billiards satisfies strong mixing properties (e.g. exponential decay of correlations) provided that some expansion condition on unstable curves is satisfied. This condition has been shown to always hold for smooth dispersing planar billiards, but it needed to be assumed separately in the case of dispersing planar billiards with corner points.

We prove that this expansion condition holds for any dispersing planar billiard with corner points, no cusps and bounded horizon.

1. Introduction

In the study of ergodic and statistical properties of hyperbolic dynamical systems with singularities, it is essential to ensure a growth condition for manifolds tangent to the unstable cone field (or *u-manifolds*); in fact, even if hyperbolicity guarantees their expansion on a local scale, singularities may cut u-manifolds in arbitrarily small pieces, and this fact could effectively prevent us from obtaining any global result on the system (see e.g. [15] for a concrete realization of this scenario). Growth conditions of this kind are usually stated in the form of a "Growth Lemma", which ensures in precise terms that any sufficiently small u-manifold will be cut by singularities in pieces that are, typically, large enough.

To fix ideas, consider a piecewise smooth map $F : \mathcal{M} \to \mathcal{M}$; we assume that $\mathcal{M}$ is a two-dimensional manifold and that F has one-dimensional stable and unstable subspaces; in systems under consideration, the singularity set is given by a union of smooth curves of $\mathcal{M}$; assume that we can find a uniform upper bound –as a function of n– on the number of smooth components of the singularity set of F^n which join at any given point $z \in \mathcal{M}$: this is called a *complexity bound* for the map. Assuming some uniform transversality condition between u-manifolds and singularity manifolds, a complexity bound immediately implies a bound on the number of connected components of the n-th image of a sufficiently small u-manifold. Then it is possible to argue, by general arguments, that a subexponential complexity bound implies the Growth Lemma (see e.g. [9]).

When it is not possible, or not feasible, to obtain a complexity bound, one can look for a more sophisticated condition, which contains more dynamical information: this strategy relies on obtaining a so-called *expansion estimate*: let $W \subset \mathcal{M}$ be a small u-manifold and let us denote by W_i the connected components of its image FW, as they are cut by singularities; let Λ_i denote the minimum expansion rate of F along W in the corresponding preimage $F^{-1}W_i$; then we say F satisfies

a one-step expansion estimate if

$$\liminf_{\delta\to 0} \sup_{W:|W|<\delta} \sum_i \frac{1}{\Lambda_i} < 1, \tag{1.1}$$

where $|W|$ is the length of W in a convenient norm. Notice that a subexponential complexity bound for the map F immediately implies an expansion estimate for some iterate F^n (or a n-step expansion estimate); therefore, condition (1.1) is indeed weaker than a complexity bound. Nevertheless, this condition is sufficient to prove the Growth Lemma by general arguments, provided that we have (mild) control on the distortion of F (see e.g. [7, Theorem 5.52]).

In this paper we obtain an expansion condition for planar billiards with corner points, no cusps and bounded horizon. Such condition is well known to hold (morally since [2, 3]) for planar billiards with no corner points, but it needed to be assumed separately –in a somewhat artificial fashion– for billiard with corner points. Our result implies that this additional assumption is unnecessary, and thus allows to conclude that, for instance, the dynamics of planar billiards with corner points enjoys exponential decay of correlation.

Let us remark that a subexponential complexity bound for finite horizon planar billiards with corner points had been announced in [1], along with an outline of the proof. In the present work, on the other hand, we directly obtain an expansion estimate, without proving a complexity bound. This approach has, in our opinion, two main advantages. First, it does not seem to be possible to obtain a complexity bound sharper than $\sim \exp(n - \log n)$, which is likely to be largely sub-optimal. In our work we obtain an expansion estimate which is more efficient than the one which would be obtained using such a complexity bound. Moreover –and in our opinion more importantly– we believe that our result can in principle be generalized to the case of unbounded horizon billiards.

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2. Definitions

In this section we provide all definitions and facts which are necessary for our exposition; the reader might refer to [2, 3, 4, 7] for (a wealth of) additional details; our notations mostly follow, whenever possible, the ones used in the given references. We will state a number of lemmata about billiard dynamics, whose proofs, unless better specified, can be found in the above references.

A *billiard table* Ω is the closure of a connected domain of $\mathbb{R}^2$ (or $\mathbb{T}^2$) so that $\partial\Omega$ is a *finite* union of smooth curves Γ_i, with pairwise disjoint interiors, that we call *boundary curves* or *walls*. Fix the standard orientation on $\partial\Omega$ so that, walking along the boundary in the positive direction, the interior of Ω lies on the left hand side. If a point $x \in \partial\Omega$ belongs to the interior of some Γ_i we say that x is a *regular point*, otherwise we call it a *corner point*. We assume that every corner point x is *simple*, that is, any sufficiently small neighborhood of x intersects at most 2

boundary curves[1]. We assume the billiard table to be *dispersing*, that is, all Γ_i to be outward convex (i.e., given any two points x, y on Γ_i, the interior of the segment joining x with y does not intersect Ω): moreover, at any regular point $x \in \partial\Omega$ the curvature $\kappa(x)$ is uniformly bounded away from zero.

We consider the dynamics of a point particle which moves with unit speed in the interior of Ω with elastic reflections on $\partial\Omega$; we refer to this system as the *billiard flow*. Notice that the flow is not well defined after a collision with a corner point: we will resolve this issue later in this section. Let $\mathcal{M} = \partial\Omega \times [-\pi/2, \pi/2]$ denote the usual cross-section of the phase space of the billiard flow; we employ standard coordinates (r, φ) on $\mathcal{M}$, where r is the arc-length parameterization of $\partial\Omega$ and $\varphi \in [-\pi/2, \pi/2]$ is the angle between the inward normal to the boundary and the outgoing billiard trajectory. We denote by $|\cdot|$ the Euclidean metric in $\mathcal{M}$. In phase portraits, we follow the convention of considering the r coordinate as *horizontal* and the φ coordinate as *vertical* (see Figure 1).

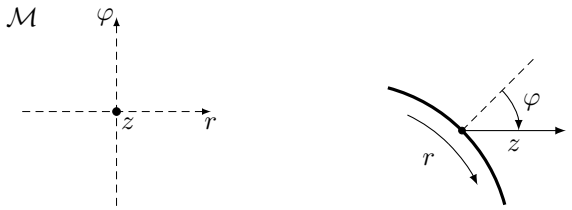

FIGURE 1. Sketch of phase space coordinates on the billiard table: the phase point z corresponds to the trajectory leaving $\partial\Omega$ at the point identified by the boundary coordinate r with angle φ from the normal pointing inside the billiard table.

Let $F : \mathcal{M} \to \mathcal{M}$ be the Poincaré map of the billiard flow, which is commonly called *billiard map*. For $z \in \mathcal{M}$, denote by $\tau(z)$ the return time of the flow to $\mathcal{M}$. The map F preserves the smooth probability measure $\mathrm{d}\nu = C_\nu \cos\varphi \mathrm{d}r\mathrm{d}\varphi$, where C_ν is a normalization constant.

We can write $\mathcal{M} = \bigcup_i \mathcal{M}_i = \bigcup_i \Gamma_i \times [-\pi/2, \pi/2]$; each $\mathcal{M}_i$ is thus diffeomorphic to either a cylinder (if Γ_i is a closed curve surrounding a scatterer with no corner points, i.e. $\partial\Gamma_i = \emptyset$) or a square (otherwise). Let us define

$$S_0 = \bigcup_i S_{0,i} = \bigcup_i \Gamma_i \times \{\pm\pi/2\} \qquad V_0 = \bigcup_i V_{0,i} = \bigcup_i \partial\Gamma_i \times [-\pi/2, \pi/2].$$

The set S_0 contains all grazing collisions, whereas V_0 contains all collisions with a corner point; let $R_0 = S_0 \cup V_0$. The maps F and F^{-1} are piecewise smooth: their singularities lie on the set $F^{-1}R_0$ and FR_0, respectively. We denote by $R_l = F^l R_0$ and $R_{m,n} = \bigcup_{l=m}^n R_l$ so that the singularities of F^n and F^{-n} lie, respectively, on the sets $R_{-n,0}$ and $R_{0,n}$. Similarly, we define S_l, $S_{m,n}$ and V_l, $V_{m,n}$.

Remark 2.1. Note that our definition of the sets R_l is not really precise at the moment. Indeed, we have not yet fixed the definition of F at corner points, and we definitely need to do that for R_l to make sense. We will fill this gap later, making sure that $R_{-n,0}$ indeed becomes the singularity set for F^n.

[1] This assumption is convenient, although inessential: our argument can be adapted to the non-simple case.

If $\partial\Omega$ has no corner points[2], then $\mathcal{M}$ is given by the disjoint union of smooth simple closed curves, and $V_0 = \emptyset$. On the other hand, if $\partial\Omega$ has corner points, then some of the $\mathcal{M}_i$ will be rectangles which share a portion of both vertical edges with some other rectangles in $\{\mathcal{M}_j\}$. It is instead convenient to cut the phase space along corner points and redefine[3] $\mathcal{M}$ as the *disjoint* union $\mathcal{M} = \bigsqcup_i \mathcal{M}_i$, in such a way that each $\mathcal{M}_i$ is a connected component of $\mathcal{M}$: this allows us to write $\partial\mathcal{M}_i = S_{0,i} \cup V_{0,i}$ and $\partial\mathcal{M} = S_0 \cup V_0$. More importantly, for each i, $F|_{F^{-1}\mathcal{M}_i}$ uniquely extends to the closure $\overline{F^{-1}\mathcal{M}_i}$ by continuation: the map F can thus be multi-valued at points belonging to the closure of several $F^{-1}\mathcal{M}_i$'s.

We now introduce the notion of *proper* and *improper* collisions of a billiard trajectory with a boundary point $x \in \partial\Omega$: intuitively, improper collisions are such that they can be avoided by perturbing the trajectory; for smooth billiard tables, only tangential collisions can be improper, and so there is no need to introduce a separate notion. In our case, we first need to introduce a few auxiliary definitions. Let $x \in \partial\Omega$; denote with w_- (resp. w_+) the limit of tangent vectors to the boundary from the left (resp. right) according to the orientation chosen in at the beginning of this section (they can coincide if, for instance, x is regular). Then w_+ and $-w_-$ cut the tangent space (that is $\mathbb{R}^2$) in two open sectors: we call *internal* the sector bounded (going clockwise) by $-w_-$ and w_+ and *external* the opposite sector. We denote with γ the angle of the internal sector; if x is regular, then necessarily $\gamma = \pi$; otherwise a corner point is said to be *acute* if $0 < \gamma < \pi$, *flat* if $\gamma = \pi$ and *obtuse* if $\gamma > \pi$. If $\gamma = 0$ we have a *cusp*; we assume our billiard table to have no cusps.

Definition 2.2. We say that a collision is *proper* if the incoming velocity vector lies in the external sector and *improper* otherwise.

Notice that if a corner point x is acute, then all collisions hitting x are proper. We assume the *bounded horizon* condition, that is, that the maximal length of a straight billiard trajectory (i.e. between two proper collisions) is uniformly bounded above: $\tau(z) \leq \tau_{\max} < \infty$. Let us summarize our

Standing Assumptions. We assume that our billiard table Ω is such that:

(A0) all corner points are simple;
(A1) all corner points have strictly positive internal angle, i.e. there are no cusps;
(A2) the bounded horizon condition is satisfied.

It is well known that the billiard flow admits a unique (non smooth) continuation after a grazing collision; on the other hand, as we mentioned before, the trajectory is not –in general– well defined after a collision with a corner point. Since we are interested in statistical properties of the smooth invariant measure ν –or, more generally, in describing orbits of Lebesgue-typical phase points– we could in principle define the dynamics arbitrarily for the zero measure set of phase points whose trajectories hit a corner point, or even leave it undefined: the choice of definition does not influence the statistical properties of the system. However, understanding the possible trajectories occurring *near* these singular points is of utmost importance in our study, so it is convenient to define the dynamics at (corner) singularities so that it reflects –at least to some extent– these possibilities. In particular, it is convenient to make sure that singularities of higher iterates of F (which can naturally

[2] Note that dispersing billiards without corner points are only realizable as subsets of $\mathbb{T}^2$
[3] The reader will excuse our abuse of notation

be defined as the boundaries of the domains of smoothness consisting of regularly colliding phase points) can be obtained as (inverse) images by F of the singularities (recall Remark 2.1). This is why we define the dynamics F at corner points as a *possibly multi-valued* function, having *branches* corresponding to all possible limits of nearby trajectories.

Let us consider a trajectory having a collision with a corner point $x \in \partial\Omega$; we call this the *reference trajectory*. If the collision is improper, we have three possibilities: trajectories close to the reference one may hit either one of the two walls which join at x, or miss both of them. However, it is easy to see (see e.g. Figure 2) that if there exist nearby trajectories colliding with the back wall, then the latter is necessarily tangent to the reference trajectory, and thus the corresponding continuation coincides with the one relative to trajectories missing the walls.

In case of a proper collision, there is no possibility of a nearby trajectory missing both walls; by arguments similar to the ones above we can prove the following

Lemma 2.3 (see [7, Section 2.8])**.** *Let $x \in \partial\Omega$ be a corner point of interior angle γ; a trajectory of the billiard flow hitting x has at most 2 possible continuations. Every trajectory colliding with x admits a unique continuation if and only if $\gamma = \pi/n$, for some natural number n.*

We emphasize that the above lemma refers to possible continuations of the *flow*; indeed, the continuation of the map is also two-valued, but the "unique if $\gamma = \pi/n$" part of the statement is not true for the map.

After a single collision with a wall occurring at a corner point, the trajectory might immediately collide with the adjacent wall; in this case we say that we have an *immediate collision*; if, in addition, the collision is grazing, we say we have an *immediate grazing collision*. In principle, a trajectory may undergo several subsequent immediate collisions before leaving the corner; this is usually referred to as a *corner sequence*[4]. Assumption (A1) implies that the number of immediate collisions in a corner sequence is uniformly bounded above for any given billiard table (see e.g. [7]). Notice that an immediate grazing collision is necessarily the last one in a collision sequence (see also [4, Section 9]).

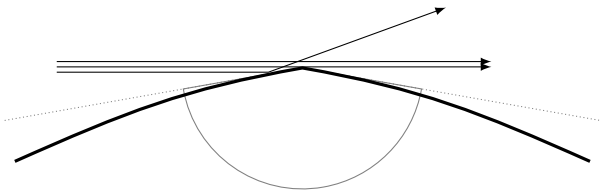

FIGURE 2. Trajectories in a neighborhood of a reference trajectory undergoing improper collision: some trajectories miss the scatterer and fly by, others will hit the scatterer frontally with some angle. No collision with the back of the scatterer is possible unless the back wall is tangent to the reference trajectory.

One of the peculiar technical difficulties in the theory of dispersing billiards is the fact that derivatives blow up at grazing singularities: this fact prevents simple

[4] Indeed, in the literature, one generally refers to a *corner sequence* as a set of consecutive collisions occurring in a neighborhood of a corner point rather than at the corner point.

distortion control, which is crucial to obtain ergodic and statistical results. In order to provide an elegant solution to this problem, Sinai (see [3]) introduced so-called *homogeneity strips* $\mathbb{H}_{\pm k} \subset \mathcal{M}$ defined as follows:

$$\mathbb{H}_k = \{(r,\varphi) \in \mathcal{M} \text{ s.t. } \varphi \in (\pi/2 - k^{-2}, \pi/2 - (k+1)^{-2}]\}$$
$$\mathbb{H}_{-k} = \{(r,\varphi) \in \mathcal{M} \text{ s.t. } \varphi \in [-\pi/2 + (k+1)^{-2}, -\pi/2 + k^{-2})\},$$

for $k \geq k_0$ where $k_0 > 0$ is fixed large enough (it will be specified below, after the statement of Lemma 2.13); for notational convenience, let us also define

$$\mathbb{H}_0 = \{(r,\varphi) \in \mathcal{M} \text{ s.t. } \varphi \in [-\pi/2 + k_0^{-2}, \pi/2 - k_0^{-2}]\}.$$

Then on each $\mathbb{H}_k$, the derivatives of F will be roughly comparable and we will be able to control its distortion. We let $\widehat{S}_0 = S_0 \cup \bigcup_k \partial\mathbb{H}_k$ and $\widehat{R} = \widehat{S}_0 \cup V_0$. The boundaries $\partial\mathbb{H}_k$ of homogeneity strips are called *secondary singularities*. Our goal is now to prove the one-step expansion estimate (1.1) for the billiard system in which these (infinitely many) secondary singularities are also taken into account. As usual in the case of planar billiards, this requires some extra technical work, but means no real extra difficulty, because expansion in the homogeneity strips with $k \geq k_0$ is huge and their total contribution to the sum in (1.1) can be made arbitrarily small by choosing k_0 large enough (this will in fact be the content of Lemma 2.13).

Lemma 2.4 (Invariant cones)**.** *At each point (r,φ) let us consider nonzero vectors $(\delta r, \delta\varphi)$ belonging to the tangent space $T\mathcal{M} = \mathbb{R}^2$. Let us call* increasing cone *the cone given by $\{\delta r \delta\varphi \geq 0\}$ and* decreasing cone *the cone $\{\delta r \delta\varphi \leq 0\}$. Then the differential $\mathrm{d}F$ maps the increasing cone strictly into itself and likewise $\mathrm{d}F^{-1}$ maps the decreasing cone strictly into itself.*

The push-forward (resp. pull-back) of the increasing cone (resp. decreasing cone) by F defines a cone field, that is called *unstable cone* (resp. *stable cone*) and denoted with $\mathcal{C}^{\mathrm{u}}$ (resp. $\mathcal{C}^{\mathrm{s}}$). A vector is said to be unstable (resp. stable) if it belongs to the unstable (resp. stable) cone; likewise a smooth curve $W \subset \mathcal{M}$ is said to be *unstable* or a *u-curve* (resp. *stable* or a *s-curve*) if the tangent vector at any point of W is unstable (resp. stable). We now collect a few known important results about dispersing billiards with corner points. We closely follow the exposition of [4, Section 9].

Lemma 2.5 (Transversality)**.** *For any $z \in \mathcal{M}$, the angle between stable and unstable cones at z is uniformly bounded away from zero.*

Remark 2.6. If the billiard table Ω has no corner points, a stronger version of the above lemma holds, that is, for any $z, z' \in \mathcal{M}$, the angle between the stable cone at z and the unstable cone at z' is uniformly bounded away from zero. In any case the transversality condition stated in Lemma 2.5 suffices for our purposes.

Lemma 2.7 (Structure of singularities)**.** *For any $l \in \mathbb{Z}$, the set R_l is a finite union of C^3 smooth curves; if $l > 0$, then such curves are unstable, otherwise if $l < 0$ they are stable. A point $z \in R_{n,m}$ is said to be* simple *if a sufficiently small neighborhood of z intersects non-trivially only one smooth curve of $R_{n,m}$ and* multiple *otherwise. For fixed $n < m$ there exist only finitely many multiple points.*

Lemma 2.8 (Expansion of unstable vectors (see [4, Lemma 9.1]))**.** *Let $z = (r, \varphi) \in \mathcal{M}$ and $Fz = (r', \varphi') \in \mathcal{M}$. There exists a constant $C_\# > 0$, which depends on the billiard table only, so that, for any $v \in \mathcal{C}^u(z)$:*

$$|\mathrm{d}Fv| \geq \frac{C_\#}{\cos\varphi'}|v|$$

In particular the above lemma ensures that, if $Fz \in \mathbb{H}_k$, then the expansion rate along unstable vectors at z is bounded below by $C_\# k^2$. This fact will be crucial for the proof of Lemma 2.13. Notice that, even if the expansion rate of unstable vectors diverges as $\varphi \to \pm\pi/2$, this divergence is in fact integrable; more precisely we can show that

Lemma 2.9 (Maximal expansion of u-curves (see [7, Exercise 4.50]))**.** *There exists a constant C_*, which depends on Ω only, so that for any u-curve W and any connected component $W' \subset FW$, we have:*

$$|W'| \leq C_*|W|^{1/2}.$$

Lemma 2.10 (Hyperbolicity, see [4, Lemma 9.2])**.** *The map F is uniformly hyperbolic in the Euclidean metric, that is, there exist $c > 0$ and $\Lambda_* > 1$ so that for any $v \in \mathcal{C}^u$ we have*

$$|\mathrm{d}F^n(v)| \geq c^{-1}\Lambda_*^n.$$

A similar statement holds for stable vectors: for any $v \in \mathcal{C}^s$ we have $|\mathrm{d}F^{-n}(v)| \geq c^{-1}\Lambda_*^n$.

Consider a u-curve W: the image of W by F is given by the union of a finite number of connected components, since W is cut by singularities R; each component might be further subdivided by singularities $\widehat{R}$ into countably many pieces, which we call *H-components*. We denote by $\{W_i\}$ the H-components of FW and by $\{W_{i,n}\}$ the H-components of F^nW.

Definition 2.11. If $W_i \subset \mathbb{H}_0$, we say that W_i is *regular*; otherwise we call W_i *nearly grazing*. Likewise, we say that $W_{i,n}$ is *regular* if for any $0 \leq l < n$ we have that $F^{-l}W_{i,n} \subset \mathbb{H}_0$ and *nearly grazing* otherwise; if $W_{i,n}$ is nearly grazing, define

$$\mathrm{rank}(W_{i,n}) = \min\{p \in \{1, \cdots, n\} \text{ s.t. } F^{-(n-p)}W_{i,n} \cap \mathbb{H}_0 = \emptyset\}.$$

For $n > 0$, define the *regular n-complexity of W*, denoted with $K_n^{\mathrm{reg}}(W)$, as the number of regular H-components of F^nW; if $n = 0$ we set conventionally $K_0^{\mathrm{reg}}(W) = 1$. Finally, define:

$$K_n^{\mathrm{reg}} = \liminf_{\delta\to 0} \sup_{W:\,|W|\leq\delta} K_N^{\mathrm{reg}}(W).$$

Lemma 2.12. *There exists $N \in \mathbb{N}$, which depends only on Ω, so that*

$$K_N^{reg} < \frac{1}{3}c^{-1}\Lambda_*^N, \tag{2.1}$$

where c and Λ_* are the ones obtained by Lemma 2.10.

Lemma 2.13. *For any $\varepsilon > 0$ we can choose k_0 large enough in the definition of homogeneity strips so that*

$$\liminf_{\delta\to 0} \sup_{W:\,|W|\leq\delta} \sum_i{}^* \frac{1}{\Lambda_i} < \varepsilon, \tag{2.2}$$

where $\sum^*$ denotes that the sum is restricted to nearly grazing components.

We emphasize that Lemma 2.13 above is stated for a single iteration of F; this will be sufficient for our purposes. We fix k_0 so that Lemma 2.13 holds for $\varepsilon = \frac{1}{2}cN^{-1}\Lambda_*^{-2N}$, where N is the one provided in Lemma 2.12 and c and Λ_* are the ones obtained by Lemma 2.10; the reader will find the reason for this choice in the proof of our Main Theorem.

We will prove Lemmata 2.12 and 2.13 in the next section. Given for granted the above two statements, we can now state and prove our

Main Theorem. Let Ω satisfy our Standing Assumptions (A0-A2), then

$$\liminf_{\delta\to 0} \sup_{W:\,|W|\le\delta} \sum_i \frac{1}{\Lambda_{i,N}} < 1, \tag{2.3}$$

where N is the one obtained by Lemma 2.12 and we denote by $\Lambda_{i,N}$ the minimum expansion of F^N on $F^{-N}W_{i,N}$.

Proof. For any $n > 0$ and u-curve W define

$$\mathcal{L}_n(W) = \sum_i \frac{1}{\Lambda_{i,n}} \qquad\qquad \mathcal{L}_n(\delta) = \sup_{W:|W|\le\delta} \mathcal{L}_n(W);$$

and set $\mathcal{L}_0(\cdot) = 1$ by convention. By Lemmata 2.12 and 2.13 and our choice of k_0 made above, we know that there exists a δ_0 so that

$$\sup_{W:\,|W|\le\delta_0} K_N^{\mathrm{reg}}(W) < \frac{1}{2}c^{-1}\Lambda_*^N \qquad\qquad \sup_{W:\,|W|\le\delta_0} \sum_i{}^* \frac{1}{\Lambda_i} < cN^{-1}\Lambda_*^{-2N}.$$

Recall that Lemma 2.9 gives an a priori bound on the length of H-components: $|W_i^n| \le |F^nW| \le C_*^n|W|^{2^{-n}}$; let us define $\delta_n = (\delta_0 C_*^{-n})^{2^n}$. We claim that for any u-curve W with $|W| < \delta_n$, the following estimate holds:

$$\mathcal{L}_n(W) \le K_n^{\mathrm{reg}}(W)\cdot c\Lambda_*^{-n} + cN^{-1}\Lambda_*^{-2N}\sum_{r=1}^{n} K_{r-1}^{\mathrm{reg}}(W)\mathcal{L}_{n-r}(\delta_{n-r}). \tag{2.4}$$

In fact, a H-component $W_{i,n}$ can either be regular, or not. By definition, the number of regular H-components is $K_n^{\mathrm{reg}}(W)$: hence their contribution to $\mathcal{L}_n$ is bounded by the first term in (2.4). On the other hand, the contribution of all nearly grazing H-components of rank r is bounded by the r-th term in the sum in (2.4) using Lemma 2.13. In particular, since $\Lambda_* > 1$ and $K_{r-1}^{\mathrm{reg}}(W) \le K_n^{\mathrm{reg}}(W)$ we have:

$$\mathcal{L}_n(W) \le K_n^{\mathrm{reg}}(W)c\left[1 + N^{-1}\Lambda_*^{-2N}\sum_{r=1}^{n}\mathcal{L}_{n-r}(\delta_{n-r})\right].$$

We can thus ensure, by induction, that $\mathcal{L}_n(\delta_n) < \Lambda_*^n$ for all $0 \le n < N$; hence we use (2.4) one final time to obtain

$$\mathcal{L}_N(W) \le 2K_N^{\mathrm{reg}}(W)c\Lambda_*^{-N} < 1,$$

which concludes the proof. □

Corollary 2.14. *Let Ω satisfy our Standing Assumptions (A0-A2), then the billiard map features exponential decay of correlations and the central limit theorem for Hölder-continuous observables.*

Proof. The statement is proven in [4] under the assumption that the complexity is sub-exponential, but if fact only the statement of our Main Theorem is used. □

As further corollaries, many other strong statistical properties are satisfied by the billiard map under our assumptions. In fact, in [4] a *Young tower* (introduced in the seminal work of Young [16]) with an exponential tail of the return times is constructed. For such "Young systems", many further statistical properties have been proved, including large deviations ([13], [11]), local limit laws ([14]), almost sure invariance principles ([10]), and Berry-Esséen type theorems ([12]). It is worthwhile to mention that the same strong statistical properties can also be obtained by means of the more geometrical *coupling* approach (introduced in [17] and further developed in [6, 5]); the reader can find a detailed exposition of the application of this technique to billiard systems in [7, Section 7].

3. Proof of main technical lemmata

In smooth dispersing billiards, the free path $\tau(r,\varphi)$ is always bounded away from zero; however, in our situation, $\tau(r,\varphi)$ can become arbitrarily small if r approaches an acute corner point, where corner series may occur. On the other hand, for any fixed small $\varrho > 0$, there exists $\tau_*(\varrho) > 0$ so that $\tau(r,\varphi) \geq \tau_*(\varrho)$ if r does not belong to the ϱ-neighborhood of any acute corner point; in particular

Remark 3.1. The free path between two improper collisions is uniformly bounded away from zero by some $\tau_* > 0$.

In order to prove Lemma 2.12 we study how singularity curves can join at a multiple point. For $\rho > 0$, and $z \in \mathcal{M}$, let us denote by $\mathcal{U}_{z,\rho}$ the open ρ-ball around z. By the facts stated in the previous section, we obtain

Lemma 3.2 (Local singularity portrait (see also [1, Theorem 6.1] or [2, Lemma 8.6]))**.**

There exists a strictly positive (non-increasing) sequence $\{\rho_n\}$ so that for any $n > 0$:

1. *for any z, z' distinct multiple points of $R_{-n,0}$ we have $\mathrm{cl}\mathcal{U}_{z,\rho_n} \cap \mathrm{cl}\mathcal{U}_{z',\rho_n} = \emptyset$;*
2. *fix a multiple point z: the neighborhood $\mathcal{U} = \mathcal{U}_{z,\rho_n}$ is cut by $R_{-n,0}$ in a finite[5] number of sectors, which we call sectors* of order n *and denote with $\{\mathcal{V}_i\}_{i\in\{1,\cdots,k(n,z)\}}$; then for each i the map $F^n|\mathcal{V}_i$ is smooth. By Lemma 2.7 we obtain that each sector is bounded by stable curves (see Figure 3).*

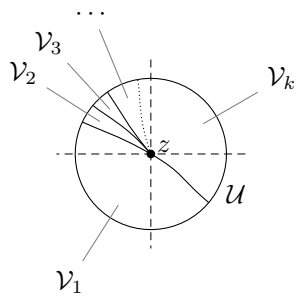

FIGURE 3. Singularity portrait around a point $z \in \operatorname{int}\mathcal{M}$.

The description given by the above lemma is valid also for $z \in \partial\mathcal{M}$, with the difference that, in this case, the neighborhood $\mathcal{U}$ can be either a half-ball or a

[5] Recall that the set R does not contain the boundaries of homogeneity strips

quarter-ball, depending on z. In particular, if $z \in V_0$, we could have an immediate collision; however, no singularity curves other than R_0 can join at z in this case. On the other hand, any neighborhood of an immediate grazing collision will necessarily contain a curve belonging to $R_{-1,0}$ which joins V_0 tangentially.

Observe that the increasing quadrants (i.e. the North-East (NE) and the South-West (SW) quadrant) cannot be cut by any future singularity; we call them *inactive quadrants*. On the other hand, both decreasing quadrants (i.e. NW and SE) might be cut by a future singularity; we call them *active quadrants*. Let us fix $\widehat{\mathbb{H}}_0$ to be the closure of a neighborhood of $\mathbb{H}_0$; for definiteness we let $\widehat{\mathbb{H}}_0 = \mathbb{H}_{-k_0} \cup \mathbb{H}_0 \cup \mathbb{H}_{k_0}$. We say that a sector $\mathcal{V}_i$ is *regular* if

$$\lim_{\mathcal{V}_i \ni z' \to z} F^l z' \in \widehat{\mathbb{H}}_0 \text{ for all } 0 < l \leq n;$$

otherwise we say that $\mathcal{V}_i$ is *nearly grazing*. Let us denote with $\mathcal{K}_n(z)$ the number of regular sectors of $\mathcal{U}_{z,\rho_n}$ (meeting at z); notice that $\mathcal{K}_n(z)$ makes sense also if z is a simple point of $R_{-n,0}$, and it can be at most 2. We introduce the notation $\mathcal{K}_n = \sup_{z \in \mathcal{M}} \mathcal{K}_n(z)$.

In Section 1 we introduced the *complexity* of the singularity set of F^n as the "number of smooth components of the singularity set of F^n which join at a given point z". However, from what we did so far, it is clear that it is not really the number of singularities that matters, but rather the "number of domains of smoothness of F^n" which join at z (in fact, even defining the dynamics at singular points is just an auxiliary tool to count these). So, in the sequel, when considering complexity growth, we will always think of these "possible trajectories of non-singular phase points near z", and not the singularity set itself.

One of the key ideas in our present approach to studying growth of u-curves is that we make advantage of the strong expansion occuring at nearly grazing collisions. As a result, we do not need to count every component (near some $z \in \mathcal{M}$), into which the phase space is cut by the singularities of F^n, but we can consider only those which never experience such a strong expansion. This is the content of the following lemma.

Lemma 3.3. *There exists Ξ depending only on Ω so that for any $n > 0$*

$$\mathcal{K}_n \leq \Xi n.$$

As we will mention later, in presence of corner points, such a linear bound is not true for the total complexity of the singularity set. Indeed, our "regular complexity" turns out to be *much smaller*, and especially much easier to control, than the total complexity. This is true despite the fact that corner points, which are responsible for branching of the trajectories, do not themselves cause trouble. In our detailed study of the mechanism of complexity growth, we will see that it is only corner points and grazing collisions *together* that make a superlinear growth possible.

Assuming the above lemma, the proof of Lemma 2.12 follows from a variation on rather standard arguments.

Proof of Lemma 2.12. Let us fix N so that $\Xi N < \frac{1}{3} c^{-1} \Lambda_*^N$; notice that since Ξ, c and Λ_* depend only on Ω, then so does N. Let $h = k_0^{-2} - k_0 + 1^{-2}$ denote the height of the homogeneity strip $\mathbb{H}_{k_0}$. We will show that there exists a δ so that

$$\sup_{W \colon |W| \leq \delta} K_N^{\text{reg}}(W) < \frac{1}{3} c^{-1} \Lambda_*^N.$$

Let $X_N \subset R_{-N,0}$ denote the (finite) set of multiple points of $R_{-N,0}$. We fix a small ρ so that $\rho < \rho_N$, where $\{\rho_n\}$ is the sequence given by Lemma 3.2 and for any $0 < l \leq N$, the diameter of every connected component of $F^l\mathcal{U}_{z,\rho}$ is bounded above by $h/3$ for any $z \in X_N$.

We choose δ so that for any u-curve W with $|W| < \delta$:

- for any $0 < l \leq N$, each component of F^lW is shorter than $h/3$; we can ensure this by Lemma 2.9;
- for any $z, z' \in X_N$, $z' \neq z$, if $W \cap \mathcal{U}_{z,\rho} \neq \emptyset$, then $W \cap \mathcal{U}_{z',\rho} = \emptyset$ (by Lemma 3.2);
- if $W \cap \mathcal{U}_{z,\rho} = \emptyset$ for each $z \in X_N$, then FW has at most 2 components (by Lemma 2.5);
- if $W \cap \mathcal{U}_{z,\rho} \neq \emptyset$ for some $z \in X_N$, then W is cut by $R_{-N,0}$ in at most ΞN components which are contained in a regular sector (by Lemma 3.3 and Lemma 2.5).

All components which belong to a nearly grazing sector are cut by $\widehat{R}_{-N,0}$ in H-components which are necessarily nearly grazing (by our assumptions on ρ and δ): as such they do not contribute to $K_N^{\text{reg}}(W)$. On the other hand, a component belonging to a regular sector can be further split in H-components, but only at most one of them will be regular (the case $N = 1$ is trivial; the case $N > 1$ can be obtained by induction). This concludes the proof of our lemma. □

Fix $z \in \mathcal{M}$; let us call the straight billiard trajectory emanating from z the *reference trajectory*; we denote by $x \in \partial\Omega$ the starting configuration point of the reference trajectory. After –perhaps– one or more improper collisions, the reference trajectory will eventually properly collide with $\partial\Omega$ at some point $x' \in \partial\Omega$. By Remark 3.1 we conclude that the number of such improper collisions is uniformly bounded by $\tau_{\max}/\tau_*$. A trajectory close to the reference trajectory could experience its first collision with different boundary walls; any such possibility corresponds to a sector $\mathcal{V}_i$ of order one (see Figure 4).

Lemma 3.4. *The number of sectors of order* 1 *is uniformly bounded by* $2(\tau_{max}/\tau_* + 1)$.

Proof. Note that we are only following the perturbed trajectories *until the first collision*, so the question reduces to counting the possible ways in which this first collision can occur. An improper collision can create at most two sectors, one corresponding to collisions occurring on the left (with respect to the selected orientation) of the improper collision, and one corresponding to collisions occurring on the right (as already noted earlier, only one is possible unless we have a tangential collision). The remaining proper collision can create, for the same reason, at most two sectors. □

We can now give the

Proof of Lemma 2.13. As in the proof of Lemma 2.12, let h denote the height of the homogeneity strip $\mathbb{H}_{k_0}$; choose δ so that, for any u-curve W shorter than δ:

- each component of FW is shorter than $h/3$ (by Lemma 2.9);
- W is cut in at most $2(\tau_{\max}/\tau_* + 1)$ components (by Lemma 3.4).

A nearly grazing component might be further split into H-components and will contribute with at most $C_\# k_0^{-1} = \sum_{k \geq k_0} C_\# k^{-2}$; the contribution of all nearly

grazing components is thus bounded by $2C_{\#}(\tau_{\max}/\tau_* + 1)k_0^{-1}$, which can be made arbitrarily small by taking k_0 to be large enough. □

We now proceed with the proof of Lemma 3.3. Define the image sector $\mathcal{V}_i' = F\mathcal{V}_i$ and let z_i', which we call the *center* of $\mathcal{V}_i'$, be the image of z by the corresponding branch of the dynamics. The next lemma refers to the key phenomenon which

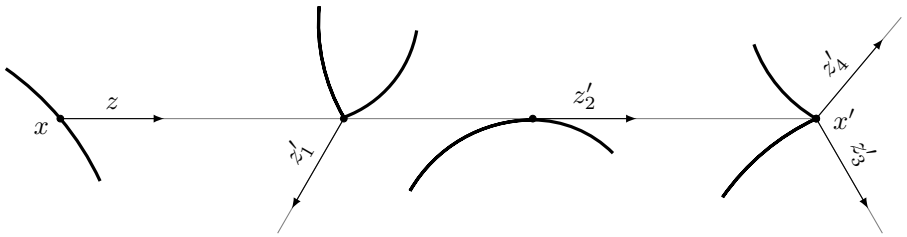

FIGURE 4. The straight billiard trajectory emanating from z might encounter several improper collisions before properly hitting the boundary. For each improper collision we have one or two possible new images z'. We stress that z' are images of z by a single iteration of F (in particular notice that z_3' and z_4' in the picture will undergo several other immediate collisions before leaving the corner point x'). Notice that z_4' is only a theoretical possibility that we count, but it does not occur in reality.

prevents complexity to grow fast in the class of billiards we are considering. To understand it, it is worth to spend some time to explain the mechanism of complexity growth. We are counting the number of *sectors* into which a small neighborhood of z is cut by singularities. *Future time* singularities are *stable* curves, so a sector $\mathcal{V}$ bordered by two such singularities is either contained entirely in an active quadrant, or it contains an entire inactive quadrant. On the other hand, the *image* of such a sector is bordered by the *images* of singularities, which are either the boundary of the phase space or *past time* singularities, both being in (or on the boundary of) the *unstable* quadrants. As a result, the *image* $F\mathcal{V}$ is either contained entirely in an *inactive* quadrant –then we call it an *inactive sector*– or it contains an entire *active* quadrant and then we call it an *active sector*.

When considering higher iterates, we look at how $F(\mathcal{V})$ is further cut by future singularities. Clearly, it can only be further cut if it is active. To bound complexity, we first need to understand the number of active sectors as time evolves.

If there are no corner points, one can make use of the continuity of the flow to see that the number of active sectors is always 2 (resulting in linear complexity): there is simply no space for more. However, in the presence of corner points, singular trajectories can branch, and after a branching there is in principle room for 4 active sectors. When we look at the possible collisions in detail, we will see that a corner collision –maybe somewhat surprisingly– does not increase the number of active sectors, but a *combination of corner and grazing collisions* is able to do that. In fact, a grazing collision is able to turn an inactive sector into an active one, which means that the number of active sectors *can grow*, and complexity *can be superlinear*. We can (and did) avoid going into this in detail by making use of

strong expansion near grazing collisions and counting *regular* sectors only. The key lemma that follows is about conservation of the number of *regular active sectors*.

Lemma 3.5. *Let $\mathcal{Q}$ be an active quadrant of $\mathcal{U} \ni z$ and $\{\mathcal{V}_{reg,j}\}$ denote regular sectors of order 1 in which $\mathcal{Q}$ is subdivided by singularities. Then, at most one of the regular image sectors $\{\mathcal{V}'_{reg,j}\}$ contains an active quadrant.*

Assuming the above lemma, we can now give the

Proof of Lemma 3.3. The proof follows from an argument that is similar to the one presented at the end of [8, Section 4]. Let us fix a multiple point $z \in R_{-n,0}$ and let $\mathcal{U} = \mathcal{U}_{z,\rho_n}$; for an arbitrary sector $\mathcal{V} \subset \mathcal{U}$ we can define the quantity $\mathcal{K}_n(z)|\mathcal{V}$, which is the number of regular sectors of order n meeting at z which intersect non-trivially the sector $\mathcal{V}$. Let $\{\mathcal{Q}_\alpha\}$ denote the quadrants of $\mathcal{U}$ (recall that there can be either 1, 2 or 4); then necessarily

$$\mathcal{K}_n(z) \le \sum_\alpha \mathcal{K}_n(z)|\mathcal{Q}_\alpha.$$

If $\mathcal{Q}$ is inactive then we have trivially $\mathcal{K}_n(z)|\mathcal{Q} = 1$; therefore, it suffices to obtain a linear bound on $\mathcal{K}_n(z)|\mathcal{Q}$ if $\mathcal{Q}$ is active. Let us fix an arbitrary active quadrant $\mathcal{Q} \subset \mathcal{U}$ and let $\{\mathcal{V}_j\}_{j\in\{1,\cdots,k\}}$ denote the regular sectors of order 1 which intersect non-trivially $\mathcal{Q}$; by further cutting some of the $\mathcal{V}_j$ along the vertical and horizontal axes, we can assume that all $\mathcal{V}_j$ are indeed contained in $\mathcal{Q}$. Notice that k is uniformly bounded by some K by Lemma 3.4. Recall that $\mathcal{V}'_j = F\mathcal{V}_j$ and z'_j denotes the center of sector $\mathcal{V}'_j$. Then, by definition

$$\mathcal{K}_n(z)|\mathcal{Q} = \sum_j \mathcal{K}_{n-1}(z'_j)|\mathcal{V}'_j$$

By Lemma 3.5, only one of the $\mathcal{V}'_j$s will be active, thus we have:

$$\mathcal{K}_n(z)|\mathcal{Q} \le K + \mathcal{K}_{n-1}(z'_{\text{active}})|\mathcal{V}'_{\text{active}}$$

from which we can conclude by induction that $\mathcal{K}_n(z)|\mathcal{Q} < (K+2)n$, which proves our statement with $\Xi = 2(K+2)+2$. □

We now come to the essence of this work and give the

Proof of Lemma 3.5. Recall that $x \in \partial\Omega$ and $x' \in \partial\Omega$ denote respectively the starting and ending point of our reference trajectory; for ease of exposition, assume that the reference trajectory is horizontal and that x' lies to the right of x (as in Figure 4). Additionally, assume the point x' to be a corner point[6]. The reference trajectory will intersect a number of walls; we say that a wall is of *type A* (resp. *type B*) if it lies above (resp. below) the reference trajectory. We say that a collision is of *type A* (resp. of *type B*) if it occurs with a wall of type A (resp. of type B). By definition, all trajectories belonging to the same sector $\mathcal{V} \subset \mathcal{Q}$ will have their first collision with the same wall, therefore their type depends on $\mathcal{V}$ only: we thus naturally obtain the definition of sectors of *type A* and sectors of *type B*.

To fix ideas we assume that $\mathcal{Q}$ is the NW quadrant, i.e. if $z = (x, \varphi)$, we only consider trajectories leaving from a left half-neighborhood of x (that is, above the reference trajectory according to our choice for the orientation) with an angle

[6] If x' is a regular point, we can artificially break the corresponding wall at x'; this will only make the singularity set larger.

slightly larger than φ (see Figure 5). The proof for the SE quadrant follows by the same argument, performing a few simple modifications, and it is left to the reader.

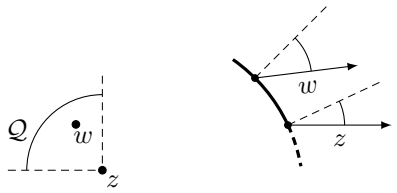

FIGURE 5. Phase points corresponding to the NW quadrant.

We now proceed to prove the following items, which immediately imply our lemma:

i) there exists exactly one sector of type A;
ii) the image of any regular sector of type B does not contain an active quadrant.

First, we claim that any trajectory emanating from $\mathcal{Q}$ and undergoing a collision of type A will necessarily hit the *leftmost* A-wall; we denote this wall by Γ_A (see Figure 6) and the corresponding sector by $\mathcal{V}_A$.

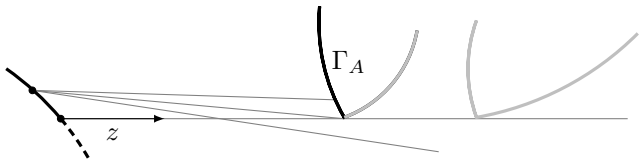

FIGURE 6. Type A collisions may only occur on the wall Γ_A; all other type A walls cannot be reached by any trajectory leaving from $\mathcal{Q}$.

In fact, by elementary geometry considerations (see once again Figure 6), any trajectory starting above the reference trajectory, and missing Γ_A will cross the reference trajectory and will consequently undergo a type B collision, i.e. it will miss all other walls of type A. This immediately implies that $\mathcal{V}_A$ is the only sector of type A, which proves item i). Notice moreover that the same considerations imply that trajectories leaving $\mathcal{Q}$ can only hit walls of type B which are either to the left or immediately to the right of Γ_A.

Let us now consider collisions of type B: the situation is as in Figure 7. As we

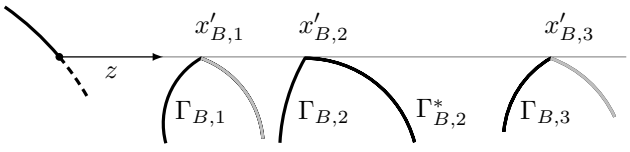

FIGURE 7. Several possibilities corresponding to type B collisions.

mentioned earlier, each scatterer generates at most two new sectors; one corresponds

to frontal collisions with the scatterer, the second (if present) to collisions with the back of the scatterer[7]. Denote with $\Gamma_{B,j}$ walls facing x (which give rise to frontal collisions) and with $\Gamma^*_{B,j}$ walls facing away from x (which give rise to back collisions); we choose the index j so that $\Gamma_{B,1}$ is the leftmost wall and respecting the left-to-right ordering of the scatterers. Let us denote with $\mathcal{V}_{B,j}$ and $\mathcal{V}^*_{B,j}$ the corresponding sectors; a sample singularity portrait is depicted in Figure 8. Finally, denote by $x'_{B,j}$ the corner point corresponding to the j-th pair of walls, that is, the intersection of $\Gamma_{B,j}$ (or $\Gamma^*_{B,j}$) with the reference trajectory.

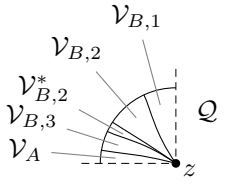

FIGURE 8. Typical singularity portrait of $\mathcal{Q}$: there is only one type A sector, and several type B sectors; sectors corresponding to collision with the back of a scatterers are nearly grazing.

Our next remark is simple, but extremely important

Remark. Our choice of ρ_n implies that we can have a back collision with $\Gamma^*_{B,j}$ only if $\Gamma^*_{B,j}$ is tangent to the reference trajectory (see e.g. $\Gamma^*_{B,2}$ in Figure 7). In particular, all non-empty $\mathcal{V}^*_{B,j}$ are *necessarily* nearly grazing and thus they can be neglected.

We now prove that every $\mathcal{V}'_{B,j}$ is inactive, which, by the above remark, implies item ii). Fix j and let us denote $\mathcal{V}_B = \mathcal{V}_{B,j}$ for ease of exposition; let Γ_B be the corresponding wall, x'_B denote the corresponding corner point and z'_B the corresponding phase point; similarly, let x'_A be the corner point corresponding to Γ_A. As a preliminary remark, notice that, by construction, $\mathcal{V}'_B$ is contained in a left half-ball centered at z'_B; $\mathcal{V}'_B$ is bounded by 3 curves; we denote the two curves which join at z'_B by ξ_1 and ξ_2 according to counterclockwise orientation (see Figure 9).

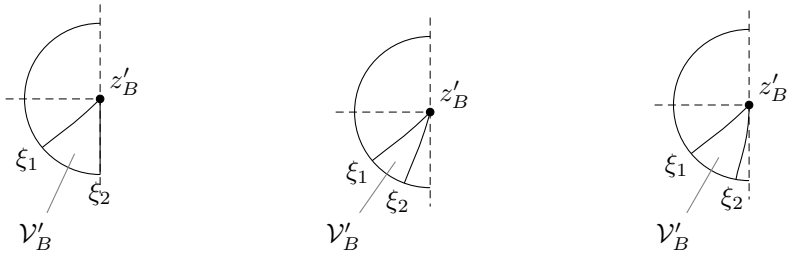

FIGURE 9. Possible portraits of $\mathcal{V}'_B$

We claim that ξ_1 is a u-curve: there are two possibilities (see once again Figure 7):

[7]If the collision is with a regular point, we can always artificially add a singularity which distinguishes front and back scatterings

- $j = 1$, then ξ_1 is the image of the fan emanating from x, i.e. a u-curve;
- if $j > 1$, then ξ_1 is the image of the fan emanating from $x'_{B,j-1}$, or the image of trajectories leaving tangentially $\Gamma^*_{B,j-1}$; in any of the two cases, we have a u-curve.

We claim that ξ_2 is either a u-curve or a vertical half-segment pointing downwards from z'_B. Then our proof is complete: since u-curves are increasing, we conclude that $\mathcal{V}'_B$ cannot contain an active quadrant. There are three possibilities (see Figure 9):

(a) x'_A lies on the right of x'_B;
(b) x'_A lies on the left of x'_B;
(c) x'_A coincides with x'_B (and thus with x');

Consider case (a): then there exist trajectories which leave $\mathcal{Q}$ and hit x'_B directly; then ξ_2 is the image of such trajectories, which corresponds to a vertical half-segment pointing downwards (this is depicted in the leftmost picture in Figure 9). In case (b), by our previous remark, Γ_B has to be the wall immediately on the right of Γ_A: in this case there are no trajectories (excluding the reference one) leaving $\mathcal{Q}$ and hitting directly x'_B and ξ_2 is either the image of the fan emanating from x'_A, or the image of trajectories leaving tangentially Γ_A (if Γ_A intersect the reference wall with a tangency). In any case ξ_2 turns out to be a u-curve (see the central picture in Figure 9). In case (c) there are two possibilities, either Γ_A is tangent to the reference trajectory, or not; in the former case we conclude by the argument described in case (b) (and this is depicted by the rightmost picture in Figure 9). The latter case follows from the argument presented in case (a) (and corresponds once again to the leftmost picture in Figure 9). This proves our claim, and concludes the proof of our main lemma. □

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