arXiv:1102.2163v1 [math.PR] 10 Feb 2011

# Competitive Lotka-Volterra Population Dynamics with Jumps

Jianhai Bao[1,3], Xuerong Mao[2], Geroge Yin[3], Chenggui Yuan[4]

[1]School of Mathematics, Central South University,
Changsha, Hunan 410075, P.R.China
jianhaibao@yahoo.com.cn

[2]Department of Statistics and Modelling Science,
University of Strathclyde, Glasgow G1 1XH, UK
xuerong@stams.strath.ac.uk

[3]Department of Mathematics,
Wayne State University, Detroit, Michigan 48202.
gyin@math.wayne.edu

[4]Department of Mathematics, Swansea University,
Swansea SA2 8PP, UK
C.Yuan@swansea.ac.uk

**Abstract**

This paper considers competitive Lotka-Volterra population dynamics with jumps. The contributions of this paper are as follows. (a) We show stochastic differential equation (SDE) with jumps associated with the model has a unique global positive solution; (b) We discuss the uniform boundedness of $p$th moment with $p > 0$ and reveal the sample Lyapunov exponents; (c) Using a variation-of-constants formula for a class of SDEs with jumps, we provide explicit solution for 1-dimensional competitive Lotka-Volterra population dynamics with jumps, and investigate the sample Lyapunov exponent for each component and the extinction of our $n$-dimensional model.

**Keywords.** Lotka-Volterra Model, Jumps, Stochastic Boundedness, Lyapunov Exponent, Variation-of-Constants Formula, Stability in Distribution, Extinction.

**Mathematics Subject Classification (2010).** 93D05, 60J60, 60J05.

# 1 Introduction

The differential equation

$$\begin{cases} \dfrac{dX(t)}{dt} & = X(t)[a(t) - b(t)X(t)], \quad t \geq 0, \\ X(0) & = x, \end{cases}$$

has been used to model the population growth of a single species whose members usually live in proximity, share the same basic requirements, and compete for resources, food, habitat, or territory, and is known as the competitive Lotka-Volterra model or logistic equation. The competitive Lotka-Volterra model for $n$ interacting species is described by the $n$-dimensional differential equation

$$\frac{dX_i(t)}{dt} = X_i(t)\left[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)X_j(t)\right], i = 1, 2, \cdots, n, \tag{1.1}$$ eq08

where $X_i(t)$ represents the population size of species $i$ at time $t$, $a_i(t)$ is the rate of growth at time $t$, $b_{ij}(t)$ represents the effect of interspecific (if $i \neq j$) or intraspecific (if $i = j$) interaction at time $t$, $a_i(t)/b_{ij}(t)$ is the carrying capacity of the $i$th species in absence of other species at time $t$. Eq. (1.1) takes the matrix form

$$\frac{dX(t)}{dt} = \text{diag}(X_1(t), \cdots, X_n(t))\left[a(t) - B(t)X(t)\right], \tag{1.2}$$ eq09

where

$$X = (X_1, \cdots, X_n)^T, a = (a_1, \cdots, a_n)^T, B = (b_{ij})_{n\times n}.$$

There is an extensive literature concerned with the dynamics of Eq. (1.2) and we here only mention Gopalsamy [4], Kuang [7], Li et al. [9], Takeuchi and Adachi [22, 23], Xiao and Li [24]. In particular, the books by Gopalsamy [4], and Kuang [7] are good references in this area.

On the other hand, the deterministic models assume that parameters in the systems are all deterministic irrespective environmental fluctuations, which, from the points of biological view, has some limitations in mathematical modeling of ecological systems. While, population dynamics in the real world is affected inevitably by environmental noise, see, e.g., Gard [2, 3]. Therefore, competitive Lotka-Volterra models in random environments are becoming more and more popular. In general, there are two ways considered in the literature to model the influence of environmental fluctuations in population dynamics. One is to consider the random perturbations of interspecific or intraspecific interactions by white noise. Recently, Mao et al. [13] investigate stochastic $n$-dimensional Lotka-Volterra system

$$dX(t) = \text{diag}(X_1(t), \cdots, X_n(t))\left[(a + BX(t))dt + \sigma X(t)dW(t)\right], \tag{1.3}$$ eq0

where $W$ is a one-dimensional standard Brownian motion, and reveal that the environmental noise can suppress a potential population explosion (see, e.g., [14, 15] among others in this

connection). Another is to consider the stochastic perturbation of growth rate $a(t)$ by the white noise with

$$a(t) \to a(t) + \sigma(t)\dot{W}(t),$$

where $\dot{W}(t)$ is a white noise, namely, $W(t)$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). As a result, Eq. (1.2) becomes a competitive Lotka-Volterra model in random environments

$$dX(t) = \mathrm{diag}(X_1(t), \cdots, X_n(t)) \left[(a(t) - B(t)X(t))dt + \sigma(t)dW(t)\right]. \tag{1.4}$$

eq41

There is also extensive literature concerning all kinds of properties of model (1.4), see, e.g., Hu and Wang [5], Jiang and Shi [6], Liu and Wang [11], Zhu and Yin [25, 26], and the references therein.

Furthermore, the population may suffer sudden environmental shocks, e.g., earthquakes, hurricanes, epidemics, etc. However, stochastic Lotka-Volterra model (1.4) cannot explain such phenomena. To explain these phenomena, introducing a jump process into underlying population dynamics provides a feasible and more realistic model. In this paper, we develop Lotka-Volterra model with jumps

$$\begin{aligned} dX(t) = \mathrm{diag}(X_1(t^-), \cdots, X_n(t^-)) \Big[&(a(t) - B(t)X(t))dt \\ &+ \sigma(t)dW(t) + \int_{\mathbb{Y}} \gamma(t,u)\tilde{N}(dt,du)\Big]. \end{aligned} \tag{1.5}$$

eq42

Here $X, a, B$ are defined as in Eq. (1.2),

$$\sigma = (\sigma_1, \cdots, \sigma_n)^T, \gamma = (\gamma_1, \cdots, \gamma_n)^T,$$

$W$ is a real-valued standard Brownian motion, $N$ is a Poisson counting measure with characteristic measure $\lambda$ on a measurable subset $\mathbb{Y}$ of $[0, \infty)$ with $\lambda(\mathbb{Y}) < \infty$, $\tilde{N}(dt, du) := N(dt, du) - \lambda(du)dt$. Throughout the paper, we assume that $W$ and $N$ are independent.

As we know, for example, bees colonies in a field [20]. In particular, they compete for food strongly with the colonies located near to them. Similar phenomena abound in the nature, see, e.g., [21]. Hence it is reasonable to assume that the self-regulating competitions within the same species are strictly positive, e.g., [25, 26]. Therefore we also assume

(**A**) For any $t \geq 0$ and $i, j = 1, 2, \cdots, n$ with $i \neq j$, $a_i(t) > 0, b_{ii}(t) > 0, b_{ij}(t) \geq 0, \sigma_i(t)$ and $\gamma_i(t, u)$ are bounded functions, $\hat{b}_{ii} := \inf_{t\in\mathbb{R}_+} b_{ii}(t) > 0$ and $\gamma_i(t, u) > -1, u \in \mathbb{Y}$.

In reference to the existing results in the literature, our contributions are as follows:

- We use jump diffusion to model the evolutions of population dynamics;
- We demonstrate that if the population dynamics with jumps is self-regulating or competitive, then the population will not explode in a finite time almost surely;

- We discuss the uniform boundedness of $p$-th moment for any $p > 0$ and reveal the sample Lyapunov exponents;
- We obtain the explicit expression of 1-dimensional competitive Lotka-Volterra model with jumps, the uniqueness of invariant measure, and further reveal precisely the sample Lyapunov exponents for each component and investigate its extinction.

# 2 Global Positive Solutions

As the $i$th state $X_i(t)$ of Eq. (1.5) denotes the size of the $i$th species in the system, it should be nonnegative. Moreover, in order to guarantee SDEs to have a unique global (i.e., no explosion in a finite time) solution for any given initial data, the coefficients of the equation are generally required to satisfy the linear growth and local Lipschitz conditions, e.g., [15]. However, the drift coefficient of Eq. (1.5) does not satisfy the linear growth condition, though it is locally Lipschitz continuous, so the solution of Eq. (1.5) may explode in a finite time. It is therefore requisite to provide some conditions under which the solution of Eq. (1.5) is not only positive but will also not explode to infinite in any finite time.

Throughout this paper, $K$ denotes a generic constant whose values may vary for its different appearances. For a bounded function $\nu$ defined on $\mathbb{R}_+$, set

$$\hat{\nu} := \inf_{t\in\mathbb{R}_+} \nu(t) \text{ and } \check{\nu} := \sup_{t\in\mathbb{R}_+} \nu(t).$$

For convenience of reference, we recall some fundamental inequalities stated as a lemma.

**Lemma 2.1.**

$$x^r \le 1 + r(x-1), \quad x \ge 0, \quad 1 \ge r \ge 0, \tag{2.1}$$ eq100

$$n^{(1-\frac{p}{2})\wedge 0}|x|^p \le \sum_{i=1}^n x_i^p \le n^{(1-\frac{p}{2})\vee 0}|x|^p, \forall p > 0, x \in \mathbb{R}^n_+, \tag{2.2}$$ eq101

where $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i > 0, 1 \le i \le n\}$, and

$$\ln x \le x - 1, \qquad x > 0. \tag{2.3}$$ eq102

solution **Theorem 2.1.** Under assumption (**A**), for any initial condition $X(0) = x_0 \in \mathbb{R}^n_+$, Eq. (1.5) has a unique global solution $X(t) \in \mathbb{R}^n_+$ for any $t \ge 0$ almost surely.

*Proof.* Since the drift coefficient does not fulfil the linear growth condition, the general theorems of existence and uniqueness cannot be implemented to this equation. However, it is locally Lipschitz continuous, therefore for any given initial condition $X(0) \in \mathbb{R}^n_+$ there is a unique local solution $X(t)$ for $t \in [0, \tau_e)$, where $\tau_e$ is the explosion time. By Eq. (1.5) the $i$th component $X_i(t)$ of $X(t)$ admits the form for $i = 1, \cdots, n$

$$dX_i(t) = X_i(t^-)\Big[\Big(a_i(t) - \sum_{j=1}^n b_{ij}(t)X_j(t)\Big)dt + \sigma_i(t)dW(t) + \int_{\mathbb{Y}} \gamma_i(t,u)\tilde{N}(dt,du)\Big].$$

Noting that for any $t \in [0, \tau_e)$

$$X_i(t) = X_i(0) \exp\Big\{ \int_0^t \Big(a_i(s) - \sum_{j=1}^n b_{ij}(s) X_j(s) - \frac{1}{2}\sigma_i^2(s) + \int_{\mathbb{Y}} (\ln(1+\gamma_i(s,u)) - \gamma_i(s,u))\lambda(du)\Big) ds + \int_0^t \sigma_i(s) dW(s) + \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(s,u))\tilde{N}(ds,du)\Big\},$$

together with $X_i(0) > 0$, we can conclude $X_i(t) \geq 0$ for any $t \in [0, \tau_e)$. Now consider the following two auxiliary SDEs with jumps

$$\begin{aligned} dY_i(t) &= Y_i(t^-)\Big[\Big(a_i(t) - b_{ii}(t)Y_i(t)\Big)dt + \sigma_i(t)dW(t) + \int_{\mathbb{Y}} \gamma_i(t,u)\tilde{N}(dt,du)\Big], \\ Y_i(0) &= X_i(0), \end{aligned} \tag{2.4}$$

eq103

and

$$\begin{aligned} dZ_i(t) &= Z_i(t^-)\Big[\Big(a_i(t) - \sum_{i\neq j} b_{ij}(t)Y_j(t) - b_{ii}(t)Z_i(t)\Big)dt + \sigma_i(t)dW(t) + \int_{\mathbb{Y}} \gamma_i(t,u)\tilde{N}(dt,du)\Big], \\ Z_i(0) &= X_i(0). \end{aligned} \tag{2.5}$$

eq112

Due to $1 + \gamma_i(t,u) > 0$ by (**A**), it follows that for any $x_2 \geq x_1$

$$(1+\gamma_i(t,u))x_2 \geq (1+\gamma_i(t,u))x_1.$$

Then by the comparison theorem [17, Theorem 3.1] we can conclude that

$$Z_i(t) \leq X_i(t) \leq Y_i(t), t \in [0, \tau_e). \tag{2.6}$$

eq111

By Lemma 4.2 below, for $Y_i(0)(= X_i(0)) > 0$, we know that $Y_i(t)$ will not be expolded in any finite time. Moreover, similar to that of Lemma 4.2 below for $Z_i(0)(= X_i(0)) > 0$, we can show

$$\mathbb{P}(Z_i(t) > 0 \text{ on } t \in [0, \tau_e)) = 1.$$

Hence $\tau_e = \infty$ and $X_i(t) > 0$ almost surely for any $t \in [0, \infty)$. The proof is therefore complete. □

# 3 Boundedness, Tightness, and Lyapunov-type Exponent

In the previous section, we see that Eq. (1.5) has a unique global solution $X(t) \in \mathbb{R}^n_+$ for any $t \geq 0$ almost surely. In this part we shall show for any $p > 0$ the solution $X(t)$ of Eq. (1.5) admits uniformly finite $p$-th moment, and discuss the long-term behaviors.

te moment **Theorem 3.1.** Let assumption (**A**) hold.

**(1)** For any $p \in [0, 1, ]$ there is a constant $K$ such that

$$\sup_{t\in\mathbb{R}_+} \mathbb{E}|X(t)|^p \le K. \tag{3.1}$$ eq9501

**(2)** Assume further that there exists a constant $\bar{K}(p) > 0$ such that for some $p > 1, t \ge 0, i = 1, \cdots, n$

$$\int_{\mathbb{Y}} |\gamma_i(t, u)|^p \lambda(du) \le \bar{K}(p). \tag{3.2}$$ eq90

Then there exists a constant $K(p) > 0$ such that

$$\sup_{t\in\mathbb{R}_+} \mathbb{E}|X(t)|^p \le K(p). \tag{3.3}$$ eq95

*Proof.* We shall prove (3.3) firstly. Define a Lyapunov function for $p > 1$

$$V(x) := \sum_{i=1}^{n} x_i^p, x \in \mathbb{R}^n_+. \tag{3.4}$$ eq43

Applying the Itô formula, we obtain

$$\mathbb{E}(e^t V(X(t))) = V(x_0) + \mathbb{E}\int_0^t e^s[V(X(s)) + \mathcal{L}V(X(s), s)]ds,$$

where, for $x \in \mathbb{R}^n_+$ and $t \ge 0$,

$$\begin{aligned} \mathcal{L}V(x,t) := p\sum_{i=1}^{n}\left[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)x_j - \frac{(1-p)\sigma_i^2(t)}{2}\right]x_i^p \\ + \sum_{i=1}^{n}\int_{\mathbb{Y}}\left[(1+\gamma_i(t,u))^p - 1 - p\gamma_i(t,u)\right]\lambda(du)x_i^p. \end{aligned} \tag{3.5}$$ eq50

By assumption (**A**) and (3.2), we can deduce that there exists constant $K > 0$ such that

$$\begin{aligned} V(x) + \mathcal{L}V(x,t) &\le \sum_{i=1}^{n}\left[-pb_{ii}(t)x_i^{p+1} + \left(1 + pa_i(t) + \frac{p(p-1)\sigma_i^2(t)}{2}\right)x_i^p\right] \\ &\quad + \sum_{i=1}^{n}\int_{\mathbb{Y}}\left[(1+\gamma_i(t,u))^p - 1 - p\gamma_i(t,u)\right]\lambda(du)x_i^p \\ &\le K. \end{aligned}$$

Hence

$$\mathbb{E}(e^t V(X(t))) \le V(x_0) + \int_0^t Ke^s ds = V(x_0) + K(e^t - 1),$$

which yields the desired assertion (3.3) by the inequality (2.2).

For any $p \in [0,1]$, according to the inequality (2.1),

$$\int_{\mathbb{Y}} \left[(1+\gamma_i(t,u))^p - 1 - p\gamma_i(t,u)\right] \lambda(du) \le 0.$$

Consequently

$$V(x) + \mathcal{L}V(x,t) \le \sum_{i=1}^{n} \left[-pb_{ii}(t)x_i^{p+1} + (1+pa_i(t))\, x_i^p\right],$$

which has upper bound by (**A**). Then (3.1) holds with $p \in [0,1]$ under (**A**). □

exin **Corollary 3.1.** Under assumption (**A**), there exists an invariant probability measure for the solution $X(t)$ of Eq. (1.5).

*Proof.* Let $\mathbb{P}(t,x,A)$ be the transition probability measure of $X(t,x)$, starting from $x$ at time 0. Denote

$$\mu_T(A) := \frac{1}{T}\int_0^T \mathbb{P}(t,x,A)dt$$

and $B_r := \{x \in \mathbb{R}_+^n : |x| \le r\}$ for $r \ge 0$. In the light of Chebyshev's inequality and Theorem 3.1 with $p \in (0,1)$,

$$\mu_T(B_r^c) = \frac{1}{T}\int_0^T \mathbb{P}(t,x,B_r^c)dt \le \frac{1}{r^p T}\int_0^T \mathbb{E}|X(t,x)|^p dt \le \frac{K}{r^p},$$

and we have, for any $\epsilon > 0$, $\mu_T(B_r) > 1-\epsilon$ whenever $r$ is large enough. Hence $\{\mu_T, T > 0\}$ is tight. By Krylov-Bogoliubov's theorem, e.g., [19, Corollary3.1.2, p22], the conclusion follows immediately. □

**Definition 3.1.** The solution $X(t)$ of Eq. (1.5) is called stochastically bounded, if for any $\epsilon \in (0,1)$, there is a constant $H := H(\epsilon)$ such that for any $x_0 \in \mathbb{R}_+^n$

$$\limsup_{t\to\infty} \mathbb{P}\{|X(t)| \le H\} \ge 1-\epsilon.$$

As an application of Theorem 3.1, together with the Chebyshev inequality, we can also establish the following corollary.

undedness **Corollary 3.2.** Under assumption (**A**), the solution $X(t)$ of Eq. (1.5) is stochastically bounded.

For later applications, let us cite a strong law of large numbers for local martingales, e.g., Lipster [10], as the following lemma.

e numbers

**Lemma 3.1.** Let $M(t), t \geq 0$, be a local martingale vanishing at time 0 and define

$$\rho_M(t) := \int_0^t \frac{d\langle M\rangle(s)}{(1+s)^2}, t \geq 0,$$

where $\langle M\rangle(t) := \langle M, M\rangle(t)$ is Meyer's angle bracket process. Then

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \text{ a.s. provided that } \lim_{t\to\infty} \rho_M(t) < \infty \text{ a.s.}$$

**Remark 3.1.** Let

$$\Psi^2_{\text{loc}} := \left\{ \Psi(t,z) \text{ predictable } \Big| \int_0^t \int_{\mathbb{Y}} |\Psi(s,z)|^2 \lambda(du)ds < \infty \right\}$$

and for $\Psi \in \Psi^2_{\text{loc}}$

$$M(t) := \int_0^t \int_{\mathbb{Y}} \Psi(s,z)\tilde{N}(ds,du).$$

Then, by, e.g., Kunita [8, Proposition 2.4]

$$\langle M\rangle(t) = \int_0^t \int_{\mathbb{Y}} |\Psi(s,z)|^2 \lambda(du)ds \text{ and } [M](t) = \int_0^t \int_{\mathbb{Y}} |\Psi(s,z)|^2 N(ds,du),$$

where $[M](t) := [M,M](t)$, square bracket process (or quadratic variation process) of $M(t)$.

**Theorem 3.2.** Let assumption (**A**) hold. Assume further that for some constant $\delta > -1$ and any $t \geq 0$

$$\gamma_i(t,u) \geq \delta, u \in \mathbb{Y}, i = 1, \cdots, n, \tag{3.6}$$ eq78

and there exists constant $K > 0$ such that

$$\int_0^t \int_{\mathbb{Y}} |\gamma(s,u)|^2 \lambda(du)ds \leq Kt. \tag{3.7}$$ eq99

Then the solution $X(t), t \geq 0$, of Eq. (1.5) has the property

$$\limsup_{t\to\infty} \frac{1}{t} \left[ \ln(|X(t)|) + \frac{\min\limits_{1\leq i\leq n} \hat{b}_{ii}}{\sqrt{n}} \int_0^t |X(s)|ds \right] \leq \max_{1\leq i\leq n} \check{a}_i, \quad \text{a.s.} \tag{3.8}$$ eq16

*Proof.* For any $x \in \mathbb{R}^n_+$, let $V(x) = \sum\limits_{i=1}^n x_i$, by Itô's formula

$$\begin{aligned} \ln(V(X(t))) \leq \ln(V(x_0)) &+ \int_0^t \Big( X^T(s)(a(s) - B(s)X(s)/V(X(s)) \\ &\qquad - (X^T(s)\sigma(s))^2/(2V^2(X(s))) \Big) ds \\ &+ \int_0^t X^T(s)\sigma(s)/V(X(s))dW(s) + \int_0^t \int_{\mathbb{Y}} \ln(1 + H(X(s^-),s,u))\tilde{N}(ds,du), \end{aligned}$$

where

$$H(x,t,u) = \left(\sum_{i=1}^{n} \gamma_i(t,u)x_i\right) \Big/ V(x).$$

Here we used the fact that $1 + H > 0$ and the inequality (2.3). Note from the inequality (2.2) and assumption (**A**) that

$$\begin{aligned} & X^T(s)(a(s) - B(s)X(s))/V(X(s)) - (X^T(s)\sigma(s))^2/(2V^2(X(s)) \\ & \leq \frac{\sum_{i=1}^{n} a_i(s)X_i(s)}{\sum_{i=1}^{n} X_i(s)} - \frac{\sum_{i=1}^{n} X_i(s) \sum_{j=1}^{n} b_{ij}(s)X_j(s)}{\sum_{i=1}^{n} X_i(s)} \\ & \leq \max_{1\leq i\leq n} \check{a}_i - \frac{\min_{1\leq i\leq n} \hat{b}_{ii}}{\sqrt{n}}|X(s)|. \end{aligned}$$

Let

$$M(t) := \int_0^t X^T(s)\sigma(s)/V(X(s))dW(s) \text{ and } \tilde{M}(t) := \int_0^t \int_{\mathbb{Y}} \ln(1+H(X(s^-),s,u))\tilde{N}(ds,du).$$

Compute by the boundedness of $\sigma$ that

$$\langle M\rangle(t) = \int_0^t (X^T(s)\sigma(s))^2/V^2(X(s))ds \leq \int_0^t |\sigma(s)|^2 ds \leq Kt.$$

On the other hand, by assumption (3.6) and the definition of $H$, for $x \in \mathbb{R}^n_+$ we obtain

$$H(x,t,u) \geq \delta$$

and, in addition to (2.3), for $-1 < \delta \leq 0$

$$\begin{aligned} |\ln(1+H(x,t,u))| &\leq |\ln(1+H(x,y,u))I_{\{\delta\leq H(x,t,u)\leq 0\}}| + |\ln(1+H(x,y,u))I_{\{0\leq H(x,t,u)\}}| \\ &\leq -\ln(1+\delta) + |H(x,t,u)|. \end{aligned}$$

This, together with (3.7), gives that

$$\begin{aligned} \langle\tilde{M}\rangle(t) &= \int_0^t \int_{\mathbb{Y}} (\ln(1+H(X(s),s,u)))^2 \lambda(du)ds \\ &\leq 2(-\ln(1+\delta))^2\lambda(\mathbb{Y})t + 2\int_0^t \int_{\mathbb{Y}} H^2(X(s),s,u))\lambda(du)ds \\ &\leq 2(-\ln(1+\delta))^2\lambda(\mathbb{Y})t + 2\int_0^t \int_{\mathbb{Y}} |\gamma(t,u)|^2\lambda(du)ds \\ &\leq (2(-\ln(1+\delta))^2\lambda(\mathbb{Y}) + K)t. \end{aligned}$$

Then the strong law of large numbers, Lemma 3.1, yields

$$\frac{1}{t}M(t) \to 0 \text{ a.s. and } \frac{1}{t}\tilde{M}(t) \to 0 \text{ as } t \to \infty,$$

and the conclusion follows. □

# 4 Variation-of-Constants Formula and the Sample Lyapunov Exponents

In this part we further discuss the long-term behaviors of model (1.5). To begin, we obtain the following variation-of-constant formula for 1-dimensional diffusion with jumps, which is interesting in its own right.

## 4.1 Variation-of-Constants Formula

t formula **Lemma 4.1.** Let $F, G, f, g : \mathbb{R}_+ \to \mathbb{R}$ and $H, h : \mathbb{R}_+ \times \mathbb{Y} \to \mathbb{R}$ be Borel-measurable and bounded functions with property $H > -1$, and $Y(t)$ satisfy

$$\begin{aligned} dY(t) &= [F(t)Y(t) + f(t)]dt + [G(t)Y(t) + g(t)]dW(t) \\ &\quad + \int_{\mathbb{Y}} [Y(t^-)H(t,u) + h(t,u)]\tilde{N}(dt,du), \\ Y(0) &= Y_0. \end{aligned} \tag{4.1}$$ eq3

Then the solution can be explicitly expressed as:

$$\begin{aligned} Y(t) = \Phi(t)\Big(Y_0 + \int_0^t \Phi^{-1}(s)\Big[\Big(f(s) - G(s)g(s) - \int_{\mathbb{Y}} \frac{H(s,u)h(s,u)}{1+H(s,u)}\lambda(du)\Big)ds \\ + g(s)dW(s) + \int_{\mathbb{Y}} \frac{h(s,u)}{1+H(s,u)}\tilde{N}(ds,du)\Big]\Big), \end{aligned}$$

where

$$\begin{aligned} \Phi(t) := \exp\Big[\int_0^t \Big(F(s) - \frac{1}{2}G^2(s) + \int_{\mathbb{Y}} [\ln(1+H(s,u)) - H(s,u)]\lambda(du)\Big)ds \\ + \int_0^t G(s)dW(s) + \int_0^t \int_{\mathbb{Y}} \ln(1+H(s,u))\tilde{N}(ds,du)\Big] \end{aligned}$$

is the fundamental solution of corresponding homogeneous linear equation

$$dZ(t) = F(t)Z(t)dt + G(t)Z(t)dW(t) + Z(t^-)\int_{\mathbb{Y}} H(t,u)\tilde{N}(dt,du). \tag{4.2}$$ eq8

*Proof.* Noting that

$$\begin{aligned} \Phi(t) = \exp\Big[\int_0^t \Big(F(s) - \frac{1}{2}G^2(s) + \int_{\mathbb{Y}} [\ln(1+H(s,u)) - H(s,u)]\lambda(du)\Big)ds \\ + \int_0^t G(s)dW(s) + \int_0^t \int_{\mathbb{Y}} \ln(1+H(s,u))\tilde{N}(ds,du)\Big] \end{aligned}$$

is the fundamental solution to Eq. (4.2), we then have

$$d\Phi(t) = F(t)\Phi(t)dt + G(t)\Phi(t)dW(t) + \Phi(t^-)\int_{\mathbb{Y}} H(t,u)\tilde{N}(dt,du). \tag{4.3}$$ eq4

By [16, Theorem 1.19, p10], Eq. (4.1) has a unique solution $Y(t), t \geq 0$. We assume that

$$Y(t) = \Phi(t)\left(Y(0) + \int_0^t \Phi^{-1}(s)\left[\bar{f}(s)ds + \bar{g}(s)dW(s) + \int_{\mathbb{Y}} \bar{h}(s,u)\tilde{N}(ds,du)\right]\right),$$

where $\bar{f}$, $\bar{g}$, and $\bar{h}$ are functions to be determined. Let

$$\bar{Y}(t) = Y(0) + \int_0^t \Phi^{-1}(s)\left[\bar{f}(s)ds + \bar{g}(s)dW(s) + \int_{\mathbb{Y}} \bar{h}(s,u)\tilde{N}(ds,du)\right],$$

which means

$$d\bar{Y}(t) = \Phi^{-1}(t)\left[\bar{f}(t)dt + \bar{g}(t)dW(t) + \int_{\mathbb{Y}} \bar{h}(t,u)\tilde{N}(dt,du)\right]. \tag{4.4}$$ eq5

Observing that $\Phi$ and $\bar{Y}$ are real-valued Lévy type stochastic integrals, by Itô's product formula, e.g., [1, Theorem 4.4.13, p231], we can deduce that

$$dY(t) = \Phi(t^-)d\bar{Y}(t) + \bar{Y}(t^-)d\Phi(t) + d[\Phi,\bar{Y}](t), \tag{4.5}$$ eq6

where $[\Phi,\bar{Y}]$ is the cross quadratic variation of processes $\Phi$ and $\bar{Y}$, and by (4.14) in [1, p230]

$$d[\Phi,\bar{Y}](t) = G(t)\bar{g}(t)dt + \int_{\mathbb{Y}} H(t,u)\bar{h}(t,u)N(dt,du). \tag{4.6}$$ eq7

Putting (4.3), (4.4), and (4.6) into (4.5), we deduce that

$$\begin{aligned}
dY(t) &= \left[\bar{f}(t)dt + \bar{g}(t)dW(t) + \int_{\mathbb{Y}} \bar{h}(t,u)\tilde{N}(dt,du)\right] \\
&\quad + F(t)Y(t)dt + G(t)Y(t)dW(t) + Y(t^-)\int_{\mathbb{Y}} H(t,u)\tilde{N}(dt,du) \\
&\quad + G(t)\bar{g}(t)dt + \int_{\mathbb{Y}} H(t,u)\bar{h}(t,u)N(dt,du) \\
&= \left[\bar{f}(t) + F(t)Y(t) + G(t)\bar{g}(t) + \int_{\mathbb{Y}} H(t,u)\bar{h}(t,u)\lambda(du)\right]dt + [\bar{g}(t) + G(t)Y(t)]dW(t) \\
&\quad + \int_{\mathbb{Y}} \left[\bar{h}(t,u) + Y(t^-)H(t,u) + H(t,u)\bar{h}(t,u)\right]\tilde{N}(dt,du).
\end{aligned}$$

Setting

$$\bar{f}(t) + G(t)\bar{g}(t) + \int_{\mathbb{Y}} H(t,u)\bar{h}(t,u)\lambda(du) = f(t)$$

and

$$\bar{g}(t) = g(t) \text{ and } \bar{h}(t,u) + H(t,u)\bar{h}(t,u) = h(t,u),$$

hence we derive that

$$\bar{f}(t) = f(t) - G(t)g(t) - \int_{\mathbb{Y}} \frac{H(t,u)h(t,u)}{1+H(t,u)}\lambda(du), \bar{g}(t) = g(t) \text{ and } \bar{h}(t,u) = \frac{h(t,u)}{1+H(t,u)}$$

and the required expression follows. □

## 4.2 One Dimensional Competitive Model

In what follows, we shall study some properties of processes $Y_i(t)$ defined by (2.4), which is actually one dimensional competitive model.

solution **Lemma 4.2.** Under assumption (**A**), Eq. (2.4) admits a unique positive solution $Y_i(t), t \geq 0$, which admits the explicit formula

$$Y_i(t) = \frac{\Phi_i(t)}{\frac{1}{X_i(0)} + \int_0^t \Phi_i(s) b_{ii}(s) ds}, \tag{4.7}$$ eq9

where

$$\Phi_i(t) := \exp\Big( \int_0^t \Big[ a_i(s) - \frac{1}{2}\sigma_i^2(s) + \int_{\mathbb{Y}} (\ln(1+\gamma_i(s,u)) - \gamma_i(s,u))\lambda(du) \Big] ds + \int_0^t \sigma_i(s) dW(s) + \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(s,u)) \tilde{N}(ds,du) \Big).$$

*Proof.* It is easy to see that $\Phi_i(t)$ is integrable in any finite interval, hence $Y_i(t)$ will never reach 0. Letting $\bar{Y}_i(t) := \frac{1}{Y_i(t)}$ and applying the Itô formula we have

$$\begin{aligned} d\bar{Y}_i(t) = & -\frac{1}{Y_i^2(t)} Y_i(t)[(a_i(t) - b_{ii}(t)Y_i(t))dt + \sigma_i(t)dW(t)] + \frac{1}{2}\frac{2}{Y_i^3(t)}\sigma_i^2(t)Y_i^2(t)dt \\ & + \int_{\mathbb{Y}} \left[ \frac{1}{(1+\gamma_i(t,u))Y_i(t)} - \frac{1}{Y_i(t)} + \frac{1}{Y_i^2(t)} Y_i(t)\gamma_i(t,u) \right] \lambda(du)dt \\ & + \int_{\mathbb{Y}} \left[ \frac{1}{(1+\gamma_i(t,u))Y_i(t^-)} - \frac{1}{Y_i(t^-)} \right] \tilde{N}(dt,du), \end{aligned}$$

that is,

$$\begin{aligned} d\bar{Y}(t) = \bar{Y}(t^-) \Big[ \Big( \sigma_i^2(t) - a_i(t) + \int_{\mathbb{Y}} \Big( \frac{1}{1+\gamma_i(t,u)} - 1 + \gamma_i(t,u) \Big) \lambda(du) \Big) dt - \sigma_i(t) dW(t) \\ + \int_{\mathbb{Y}} \Big( \frac{1}{1+\gamma_i(t,u)} - 1 \Big) \tilde{N}(dt,du) \Big] + b_{ii}(t)dt. \end{aligned} \tag{4.8}$$ eq2

By Lemma 4.1, Eq. (4.8) has an explicit solution and the conclusion (4.7) follows. □

**Definition 4.1.** The solution of Eq. (2.4) is said to be stochastically permanent if for any $\epsilon \in (0,1)$ there exit positive constants $H_1 := H_1(\epsilon)$ and $H_2 := H_2(\epsilon)$ such that

$$\liminf_{t\to\infty} \mathbb{P}\{Y_i(t) \leq H_1\} \geq 1-\epsilon \text{ and } \liminf_{t\to\infty} \mathbb{P}\{Y_i(t) \geq H_2\} \geq 1-\epsilon.$$

permanent **Theorem 4.1.** Let assumption (**A**) hold. Assume further that there exists constant $c_1 > 0$ such that, for any $t \geq 0$ and $i = 1, \cdots, n$,

$$a_i(t) - \sigma_i^2(t) - \int_{\mathbb{Y}} \frac{\gamma_i^2(t,u)}{1+\gamma_i(t,u)} \lambda(du) \geq c_1, \tag{4.9}$$ eq03

then the solution $Y_i(t), t \geq 0$ of Eq. (2.4) is stochastically permanent.

*Proof.* The first part of the proof follows by the Chebyshev inequality and Corollary 3.2. Observe that (4.7) can be rewritten in the form

$$\begin{aligned}\frac{1}{Y_i(t)} = \frac{1}{X_i(0)} \exp\Big( \int_0^t -\Big[a_i(s) - \frac{1}{2}\sigma_i^2(s) + \int_{\mathbb{Y}} (\ln(1+\gamma_i(s,u)) - \gamma_i(s,u))\lambda(du)\Big]ds \\ - \int_0^t \sigma_i(s)dW(s) - \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(s,u))\tilde{N}(ds,du)\Big) \\ + \int_0^t b_{ii}(s) \exp\Big( \int_s^t -\Big[a(r) - \frac{1}{2}\sigma_i^2(r) + \int_{\mathbb{Y}} (\ln(1+\gamma_i(r,u)) - \gamma_i(r,u))\lambda(du)\Big]dr \\ - \int_s^t \sigma_i(r)dW(r) - \int_s^t \int_{\mathbb{Y}} \ln(1+\gamma_i(r,u))\tilde{N}(dr,du)\Big)ds.\end{aligned} \tag{4.10}$$

eq02

By, e.g., [1, Corollary 5.2.2, p253], we notice that

$$\begin{aligned}\exp\Big( -\frac{1}{2}\int_0^t \sigma_i^2(s)ds - \int_0^t \int_{\mathbb{Y}} \Big(\frac{1}{1+\gamma_i(s,u)} - 1 + \ln(1+\gamma_i(s,u))\Big)\lambda(du)ds \\ - \int_0^t \sigma_i(s)dW(s) - \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(s,u))\tilde{N}(ds,du)\Big)\end{aligned}$$

is a local martingale. Hence letting $\bar{M}_i(t) := \frac{1}{Y_i(t)}$ and taking expectations on both sides of (4.10) leads to

$$\begin{aligned}\mathbb{E}\bar{M}_i(t) = \frac{1}{X_i(0)} \exp\Big( -\int_0^t \Big[a_i(s) - \sigma_i^2(s) - \int_{\mathbb{Y}} \frac{\gamma_i^2(s,u)}{1+\gamma_i(s,u)}\lambda(du)\Big]ds \\ + \int_0^t b_{ii}(s) \exp\Big( -\int_s^t \Big[a_i(r) - \sigma_i^2(r) - \int_{\mathbb{Y}} \frac{\gamma_i^2(r,u)}{1+\gamma_i(r,u)}\lambda(du)\Big]drds,\end{aligned}$$

which, combining (4.9), yields

$$\mathbb{E}\bar{M}_i(t) \le \frac{1}{X_i(0)}e^{-c_1 t} + \int_0^t b_{ii}(s)e^{-c_2(t-s)}ds \le \frac{\check{b}}{c_1} + \left(\frac{1}{X_i(0)} - \frac{\check{b}}{c_1}\right)e^{-c_1 t}. \tag{4.11}$$

eq05

Hence there exists a constant $K > 0$ such that

$$\mathbb{E}\bar{M}_i(t) \le K. \tag{4.12}$$

eq04

Furthermore, for any $\epsilon > 0$ and constant $H_2(\epsilon) > 0$, thanks to the Chebyshev inequality and (4.12)

$$\mathbb{P}\{Y_i(t) \ge H_2\} = \mathbb{P}\left\{\bar{M}_i(t) \le 1/H_2\right\} = 1 - \mathbb{P}\left\{\bar{M}_i(t) > 1/H_2\right\} \ge 1 - H_2\mathbb{E}\bar{M}_i(t) \ge 1 - \epsilon$$

whenever $H_2 = \epsilon/K$, as required.

□

symptotic **Theorem 4.2.** Let the conditions of Theorem 4.1 hold. Then Eq. (2.4) has the property

$$\lim_{t\to\infty} \mathbb{E}|Y_i(t,x) - Y_i(t,y)|^{\frac{1}{2}} = 0 \text{ uniformly in } (x,y) \in \mathbb{K}\times\mathbb{K}, \tag{4.13}$$ eq06

where $\mathbb{K}$ is any compact subset of $(0,\infty)$.

*Proof.* By the Hölder inequality

$$\begin{aligned}\mathbb{E}|Y_i(t,x) - Y_i(t,y)|^{\frac{1}{2}} &= \mathbb{E}\left(Y_i(t,x)Y_i(t,y)\left|\frac{1}{Y_i(t,y)} - \frac{1}{Y_i(t,x)}\right|\right)^{\frac{1}{2}}\\ &\le (\mathbb{E}(Y_i(t,x)Y_i(t,y)))^{\frac{1}{2}}\left(\mathbb{E}\left|\frac{1}{Y_i(t,y)} - \frac{1}{Y_i(t,x)}\right|\right)^{\frac{1}{2}}.\end{aligned}$$

To show the desired assertion it is sufficient to estimate the two terms on the right-hand side of the last step. By virtue of the Itô formula,

$$\begin{aligned}d(Y_i(t,x)Y_i(t,y)) &= Y_i(t^-,x)dY_i(t,y) + Y_i(t^-,y)dY_i(t,x) + d[Y_i(t,x),Y_i(t,y)]\\ &= Y_i(t^-,x)Y_i(t^-,y)\left[(a_i(t) - b_{ii}(t)Y_i(t,y))dt + \sigma_i(t)dW(t) + \int_{\mathbb{Y}}\gamma_i(t,u)\tilde{N}(dt,du)\right]\\ &\quad + Y_i(t^-,x)Y_i(t^-,y)\left[(a_i(t) - b_{ii}(t)Y_i(t,x))dt + \sigma_i(t)dW(t) + \int_{\mathbb{Y}}\gamma_i(t,u)\tilde{N}(dt,du)\right]\\ &\quad + \sigma_i^2(t)Y_i(t,x)Y_i(t,y)dt + \int_{\mathbb{Y}}\gamma_i^2(t,u)Y_i(t^-,x)Y_i(t^-,y)N(dt,du)\\ &= (2a_i(t) + \sigma_i^2(t))Y_i(t,x)Y_i(t,y)dt - b_{ii}(t)Y_i(t,x)Y_i(t,y)(Y_i(t,x) + Y_i(t,y))dt\\ &\quad + 2\sigma_i(t)Y_i(t,x)Y_i(t,y)dW(t) + 2\int_{\mathbb{Y}}\gamma_i(t,u)Y_i(t^-,x)Y_i(t^-,y)\tilde{N}(dt,du)\\ &\quad + \int_{\mathbb{Y}}\gamma_i^2(t,u)Y_i(t^-,x)Y_i(t^-,y)N(dt,du).\end{aligned}$$

Thus, in view of Jensen's inequality and the familiar inequality $a+b \ge 2\sqrt{ab}$ for any $a,b\ge 0$, we deduce that

$$\begin{aligned}\mathbb{E}(Y_i(t,x)Y_i(t,y)) &\le xy + \int_0^t \delta_i(s)\mathbb{E}(Y_i(s,x)Y_i(s,y))ds\\ &\quad - \mathbb{E}\int_0^t b_{ii}(s)(Y_i(s,x)Y_i(s,y)(Y_i(s,x) + Y_i(s,y)))ds\\ &\le xy + \int_0^t \delta_i(s)\mathbb{E}(Y_i(s,x)Y_i(s,y))ds - \int_0^t b_{ii}(s)(\mathbb{E}(Y_i(s,x)Y_i(s,y)))^{\frac{3}{2}}ds,\end{aligned}$$

where $\delta_i(t) := 2a_i(t) + \sigma_i^2(t) + \int_{\mathbb{Y}}\gamma_i^2(t,u)\lambda(du)$. By the comparison theorem,

$$\begin{aligned}\mathbb{E}(Y_i(t,x)Y_i(t,y)) &\le \left(1/\sqrt{xy}e^{-\frac{1}{2}\int_0^t\delta_i(s)ds} + \frac{1}{2}\int_0^t b_{ii}(s)e^{-\frac{1}{2}\int_s^t\delta_i(\tau)d\tau}ds\right)^{-2}\\ &\le \left(\hat{b}/\check{\delta}_i + (1/\sqrt{xy} - \hat{b}_{ii}/\check{\delta}_i)e^{-\frac{\check{\delta}_i t}{2}}\right)^{-2}.\end{aligned} \tag{4.14}$$ eq07

On the other hand, thanks to (4.7) we have

$$\begin{aligned}
&\frac{1}{Y_i(t,x)} - \frac{1}{Y_i(t,y)} \\
&= \left(\frac{1}{x} - \frac{1}{y}\right) \exp\Big( - \int_0^t \Big[a_i(s) - \frac{1}{2}\sigma_i^2(s) + \int_{\mathbb{Y}} (\ln(1+\gamma_i(s,u)) - \gamma_i(s,u))\lambda(du)\Big] ds \\
&\quad - \int_0^t \sigma_i(s) dW(s) - \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(s,u))\tilde{N}(ds,du)\Big).
\end{aligned}$$

In the same way as (4.11) was done, it follows from (4.9) that

$$\mathbb{E}\left|\frac{1}{Y_i(t,x)} - \frac{1}{Y_i(t,y)}\right| \le \left|\frac{1}{x} - \frac{1}{y}\right| e^{-c_2 t}. \tag{4.15}$$ eq19

Thus (4.13) follows by combining (4.14) and (4.15). □

If $a_i, b_{ii}, \sigma_i, \gamma_i$ are time-independent, Eq. (2.4) reduces to

$$dY_i(t) = Y_i(t^-)\left[(a_i - b_{ii}Y_i(t))dt + \sigma_i dW(t) + \int_{\mathbb{Y}} \gamma_i(u)\tilde{N}(dt,du)\right], \tag{4.16}$$ eq21

with original value $x > 0$. Let $p(t,x,dy)$ denote the transition probability of solution process $Y_i(t,x)$ and $\mathbb{P}(t,x,A)$ denote the probability of event $\{Y_i(t,x) \in A\}$, where $A$ is a Borel measurable subset of $(0,\infty)$. It is similar to that of Corollary 3.1, under the conditions of Theorem 4.1 there exists an invariant measure for $Y_i(t,x)$. Moreover by the standard procedure [15, p213-216], we know that Theorem 4.2 implies the uniqueness of invariant measure. That is:

tribution **Theorem 4.3.** Under the conditions of Theorem 4.1 and 4.2, the solution $Y_i(t,x)$ of Eq. (4.16) has a unique invariant measure.

We further need the following exponential martingale inequality with jumps, e.g., [1, Theorem 5.2.9, p291].

artingale **Lemma 4.3.** Assume that $g : [0,\infty) \to \mathbb{R}$ and $h : [0,\infty) \times \mathbb{Y} \to \mathbb{R}$ are both predictable $\mathcal{F}_t$-adapted processes such that for any $T > 0$

$$\int_0^T |g(t)|^2 dt < \infty \text{ a.s. and } \int_0^T \int_{\mathbb{Y}} |h(t,u)|^2 \lambda(du)dt < \infty \text{ a.s.}$$

Then for any constants $\alpha, \beta > 0$

$$\begin{aligned}
\mathbb{P}\Big\{ \sup_{0\le t\le T} \Big[ &\int_0^t g(s)dW(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2 ds + \int_0^t \int_{\mathbb{Y}} h(s,u)\tilde{N}(ds,du) \\
&- \frac{1}{\alpha}\int_0^t \int_{\mathbb{Y}} [e^{\alpha h(s,u)} - 1 - \alpha h(s,u)]\lambda(du)ds\Big] > \beta\Big\} \le e^{-\alpha\beta}.
\end{aligned}$$

property **Lemma 4.4.** Let assumption (**A**) hold. Assume further that for any $t \geq 0$ and $i = 1, \cdots, n$

$$\sup_{t\geq 0} \int_0^t \int_{\mathbb{Y}} e^{s-t}[\gamma_i(s,u) - \ln(1+\gamma_i(s,u))]\lambda(du)ds < \infty. \tag{4.17}$$ eq32

Then

$$\limsup_{t\to\infty} \frac{\ln Y_i(t)}{\ln t} \leq 1, \text{ a.s. for each } i = 1, \cdots, n,$$

*Proof.* For any $t \geq 0$ and $i = 1, \cdots, n$, applying the Itô formula

$$\begin{aligned} e^t \ln Y_i(t) = \ln X_i(0) &+ \int_0^t e^s \Big[ \ln Y_i(s) + a_i(s) - b_{ii}(s)Y_i(s) - \frac{1}{2}\sigma_i^2(s) \\ &+ \int_{\mathbb{Y}} [\ln(1+\gamma_i(s,u)) - \gamma_i(s,u)]\lambda(du)\Big]ds \\ &+ \int_0^t e^s \sigma_i(s)dW(s) + \int_0^t \int_{\mathbb{Y}} e^s \ln(1+\gamma_i(s,u))\tilde{N}(ds,du). \end{aligned}$$

Note that, for $c, x > 0$, $\ln x - cx$ attains its maximum value $-1 - \ln c$ at $x = \frac{1}{c}$. Thus it follows from the inequality (2.3) that

$$\begin{aligned} e^t \ln Y_i(t) \leq \ln X_i(0) &+ \int_0^t e^s \Big[ -1 - \ln b_{ii}(s) + a_i(s) - \frac{1}{2}\sigma_i^2(s)\Big]ds \\ &+ \int_0^t e^s \sigma_i(s)dW(s) + \int_0^t \int_{\mathbb{Y}} e^s \ln(1+\gamma_i(s,u))\tilde{N}(ds,du). \end{aligned} \tag{4.18}$$ eq109

In the light of Lemma 4.3, for any $\alpha, \beta, T > 0$,

$$\begin{aligned} \mathbb{P}\Big\{ \sup_{0\leq t\leq T} \Big[ &\int_0^t e^s \sigma_i(s)dW(s) - \frac{\alpha}{2}\int_0^t e^{2s}\sigma_i^2(s)ds + \int_0^t \int_{\mathbb{Y}} e^s \ln(1+\gamma_i(s,u))\tilde{N}(ds,du) \\ &- \frac{1}{\alpha}\int_0^t \int_{\mathbb{Y}} \Big[ e^{\alpha e^s \ln(1+\gamma_i(s,u))} - 1 - \alpha e^s \ln(1+\gamma_i(s,u))\Big]\lambda(du)ds\Big] \geq \beta \Big\} \leq e^{-\alpha\beta}. \end{aligned}$$

Choose $T = k\gamma, \alpha = \epsilon e^{-k\gamma}$, and $\beta = \frac{\theta e^{k\gamma}\ln k}{\epsilon}$, where $k \in \mathbb{N}, 0 < \epsilon < 1, \gamma > 0$, and $\theta > 1$ in the above equation. Since $\sum_{k=1}^{\infty} k^{-\theta} < \infty$, we can deduce from the Borel-Cantalli Lemma that there exists an $\Omega_i \subseteq \Omega$ with $\mathbb{P}(\Omega_i) = 1$ such that for any $\epsilon \in \Omega_i$ an integer $k_i = k_i(\omega, \epsilon)$ can be found such that

$$\begin{aligned} &\int_0^t e^s \sigma_i(s)dW(s) + \int_0^t \int_{\mathbb{Y}} e^s \ln(1+\gamma_i(s,u))\tilde{N}(ds,du) \\ &\leq \frac{\theta e^{k\gamma}\ln k}{\epsilon} + \frac{\epsilon e^{-k\gamma}}{2}\int_0^t e^{2s}\sigma_i^2(s)ds \\ &\quad + \frac{1}{\epsilon e^{-k\gamma}}\int_0^t \int_{\mathbb{Y}} \Big[ (1+\gamma_i(s,u))^{\epsilon e^{s-k\gamma}} - 1 - \epsilon e^{s-k\gamma}\ln(1+\gamma_i(s,u))\Big]\lambda(du)ds \end{aligned}$$

whenever $k \geq k_i, 0 \leq t \leq k\gamma$. Next, note from the inequality (2.1) that, for any $\omega \in \Omega_i$ and $0 < \epsilon < 1, 0 \leq t \leq k\gamma$ with $k \geq k_i$,

$$\begin{aligned}
&\frac{1}{\epsilon e^{t-k\gamma}} \int_0^t \int_{\mathbb{Y}} \Big[ (1+\gamma_i(s,u))^{\epsilon e^{s-k\gamma}} - 1 - \epsilon e^{s-k\gamma} \ln(1+\gamma_i(s,u)) \Big] \lambda(du)ds \\
&\leq \quad \int_0^t \int_{\mathbb{Y}} e^{s-t} (\gamma_i(s,u) - \ln(1+\gamma_i(s,u))) \lambda(du)ds.
\end{aligned}$$

Thus, for $\omega \in \Omega_i$ and $(k-1)\gamma \leq t \leq k\gamma$ with $k \geq k_i + 1$, we have

$$\begin{aligned}
\frac{\ln Y_i(t)}{\ln t} \leq \frac{\ln X_i(0)}{e^t \ln t} &+ \frac{\theta e^{k\gamma} \ln k}{\epsilon e^{(k-1)\gamma} \ln((k-1)\gamma)} \\
&+ \frac{1}{\ln t} \int_0^t e^{s-t} \Big[ -1 - \ln b_{ii}(s) + a_i(s) - \frac{1}{2}(1 - \epsilon e^{s-k\gamma}) \sigma_i^2(s) \Big] ds \\
&+ \frac{1}{\ln t} \int_0^t \int_{\mathbb{Y}} e^{s-t} [\gamma_i(s,u) - \ln(1+\gamma_i(s,u))] \lambda(du)ds.
\end{aligned}$$

Letting $k \uparrow \infty$,together with assumption (**A**) and (4.17), leads to

$$\limsup_{t\to\infty} \frac{\ln Y_i(t)}{\ln t} \leq \frac{\theta e^{\gamma}}{\epsilon},$$

and the conclusion follows by setting $\gamma \downarrow 0, \epsilon \uparrow 1$, and $\theta \downarrow 1$. □

Noting the limit $\lim_{t\to\infty} \frac{\ln t}{t} = 0$, we have the following corollary.

corollary **Corollary 4.1.** Under the conditions of Lemma 4.4

$$\limsup_{t\to\infty} \frac{\ln Y_i(t)}{t} \leq 0, \text{ a.s. for each } i = 1, \cdots, n,$$

and therefore

$$\limsup_{t\to\infty} \frac{\ln\Big(\prod_{i=1}^n Y_i(t)\Big)}{t} \leq 0, \text{ a.s.}$$

**Corollary 4.2.** Under the conditions of Lemma 4.4

$$\limsup_{t\to\infty} \frac{\ln(X_i(t))}{t} \leq 0, \text{ a.s. for each } i = 1, \cdots, n,$$

and therefore

$$\limsup_{t\to\infty} \frac{\ln\Big(\prod_{i=1}^n X_i(t)\Big)}{t} \leq 0, \text{ a.s.}$$

*Proof.* Recalling

$$Z_i(t) \le X_i(t) \le Y_i(t), t \ge 0, i = 1, \cdots, n$$

and combining Corollary 4.1, we complete the proof.

□

YT **Theorem 4.4.** Let the conditions of Lemma 4.4 hold. Assume further that for any $t \ge 0$ and $i = 1, \cdots, n$

$$R_i(t) := a_i(t) - \frac{1}{2}\sigma^2(t) + \int_{\mathbb{Y}} (\ln(1 + \gamma_i(t,u)) - \gamma_i(t,u))\lambda(du) \ge 0, \tag{4.19}$$ eq105

and there exists constant $c_2 > 0$ such that

$$\int_{\mathbb{Y}} (\ln(1 + \gamma_i(t,u)))^2 \lambda(du) \le c_2. \tag{4.20}$$ eq113

Then for each $i = 1, \cdots, n$

$$\lim_{t\to\infty} \frac{\ln Y_i(t)}{t} = 0 \text{ a.s.} \tag{4.21}$$

*Proof.* According to Corollary 4.1, it suffices to show $\liminf_{t\to\infty} \frac{\ln Y_i(t)}{t} \ge 0$. Denote for $t \ge 0$

$$M_i(t) := \int_0^t \sigma_i(s) dW(s) \text{ and } \bar{M}_i(t) := \int_0^t \int_{\mathbb{Y}} \ln(1 + \gamma_i(s,u))\tilde{N}(ds, du).$$

Note that

$$[M_i](t) = \langle M_i \rangle(t) = \int_0^t \sigma_i^2(s) ds \le \check{\sigma}_i^2 t,$$

and by (4.20)

$$\langle \bar{M}_i \rangle(t) = \int_0^t \int_{\mathbb{Y}} (\ln(1 + \gamma_i(s,u)))^2 \lambda(du) ds \le c_2 t.$$

Since

$$\int_0^t \frac{1}{(1+s)^2} ds = -\frac{1}{1+s}\Big|_0^t = \frac{t}{1+t} < \infty,$$

together with Lemma 3.1, we then obtain

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sigma_i(s) dW(s) = 0 \text{ a.s. and } \lim_{t\to\infty} \frac{1}{t} \int_0^t \int_{\mathbb{Y}} \ln(1 + \gamma_i(s,u))\tilde{N}(ds, du) = 0 \text{ a.s.} \tag{4.22}$$ eq121

Moreover, it is easy to see that for any $t > s$

$$\int_s^t \sigma_i(r) dW(r) = \int_0^t \sigma_i(r) dW(r) - \int_0^s \sigma_i(r) dW(r)$$

and

$$\int_s^t \int_{\mathbb{Y}} \ln(1+\gamma_i(r,u))\tilde{N}(dr, du) = \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(r,u))\tilde{N}(dr, du) - \int_0^s \int_{\mathbb{Y}} \ln(1+\gamma_i(r,u))\tilde{N}(dr, du).$$

Consequently, for any $\epsilon > 0$ we can deduce that there exists constant $T > 0$ such that

$$\left|\int_s^t \sigma_i(r)dW(r)\right| \le \epsilon(s+t) \text{ a.s. and } \left|\int_s^t \int_{\mathbb{Y}} \ln(1+\gamma_i(r,u))\tilde{N}(dr,du)\right| \le \epsilon(s+t) \text{ a.s.} \tag{4.23}$$

eq106

whenever $t > s \ge T$. Furthermore, by Lemma 4.2, together with (4.23), we have for $t \ge T$

$$\begin{aligned}\frac{1}{Y_i(t)} \le \frac{1}{Y_i(T)} \exp\Big(\int_T^t -\Big[a_i(s) - \frac{1}{2}\sigma_i^2(s) + \int_{\mathbb{Y}}(\ln(1+\gamma_i(s,u)) - \gamma_i(s,u))\lambda(du)\Big]ds \\ + 2\epsilon(t+T)\Big) \\ + \int_T^t b_{ii}(s) \exp\Big(-\int_s^t \Big[a_i(r) - \frac{1}{2}\sigma_i^2(r) + \int_{\mathbb{Y}}(\ln(1+\gamma_i(r,u)) - \gamma_i(r,u))\lambda(du)\Big]dr \\ + 2\epsilon(s+t)\Big)ds, \text{ a.s.}\end{aligned}$$

This further gives that for any $t \ge T$

$$\begin{aligned}e^{-4\epsilon(t+T)}\frac{1}{Y_i(t)} \le \frac{1}{Y_i(T)} \exp\Big(\int_T^t -\Big[a_i(s) - \frac{1}{2}\sigma^2(s) + \int_{\mathbb{Y}}(\ln(1+\gamma_i(s,u)) - \gamma_i(s,u))\lambda(du)\Big]ds \\ + \int_T^t b_{ii}(s) \exp\Big(-\int_s^t \Big[a_i(r) - \frac{1}{2}\sigma_i^2(r) + \int_{\mathbb{Y}}(\ln(1+\gamma_i(r,u)) - \gamma_i(r,u))\lambda(du)\Big]dr \\ - 2\epsilon(t-s) - 2\epsilon T\Big)ds, \text{ a.s.}\end{aligned}$$

Thus in view of (4.19) there exists constant $K > 0$ such that for any $t \ge T$

$$e^{-4\epsilon(t+T)}\frac{1}{Y_i(t)} \le K, \text{ a.s.}$$

Hence for any $t \ge T$

$$\frac{1}{t}\ln\frac{1}{Y_i(t)} \le 4\epsilon\Big(1 + \frac{T}{t}\Big) + \frac{1}{t}\ln K, \text{ a.s.}$$

and the conclusion follows by letting $t \to \infty$ and the arbitrariness of $\epsilon > 0$. □

## 4.3 Further Properties of $n-$Dimensional Competitive Models

We need the following lemma.

lemma1 **Lemma 4.5.** Let the conditions of Theorem 4.4 hold. Assume further that for $i, j = 1, \cdots, n$

$$R_{ij} := \sup\left\{\frac{b_{ij}(t)}{b_{jj}(t)}, t \ge 0, i \ne j\right\} \tag{4.24}$$

eq120

satisfy

$$R_i(t) - \sum_{i \ne j} R_{ij}R_j(t) > 0,\ t \ge 0. \tag{4.25}$$

eq116

Then

$$\liminf_{t\to\infty}\frac{\ln Z_i(t)}{t}\geq 0,\ \text{a.s.} \tag{4.26}$$

where $Z_i(t), i=1,\cdots,n$ are solutions of (2.5).

**Remark 4.1.** For $i,j=1,\cdots,n$ and $t\geq 0$, if $b_{ij}(t)$ takes finite-number values, then condition (4.24) must hold.

*Proof.* It is sufficient to show $\limsup_{t\to\infty}\frac{1}{t}\ln\frac{1}{Z_i(t)}\leq 0$. Note from Lemma 4.2 that for any $t>s\geq 0$

$$\begin{aligned}\frac{1}{Z_i(t)}=\frac{1}{Z_i(s)}\exp\Big(&\int_s^t-\Big[a_i(r)-\sum_{i\neq j}b_{ij}(r)Y_j(r)-\frac{1}{2}\sigma_i^2(r)+\int_{\mathbb{Y}}(\ln(1+\gamma_i(r,u))-\gamma_i(r,u))\lambda(du)\Big]dr\\&-\int_s^t\sigma_i(s)dW(s)-\int_s^t\int_{\mathbb{Y}}\ln(1+\gamma_i(s,u))\tilde{N}(ds,du)\Big)\\&+\int_s^t b_{ii}(r)\exp\Big(-\int_r^t\Big[a_i(\tau)-\sum_{i\neq j}b_{ij}(\tau)Y_j(\tau)-\frac{1}{2}\sigma_i^2(\tau)\\&+\int_{\mathbb{Y}}(\ln(1+\gamma_i(\tau,u))-\gamma_i(\tau,u))\lambda(du)\Big]d\tau\\&-\int_r^t\sigma_i(\tau)dW(\tau)-\int_r^t\int_{\mathbb{Y}}\ln(1+\gamma_i(\tau,u))\tilde{N}(d\tau,du)\Big)dr.\end{aligned} \tag{4.27}$$

eq115

Applying the Itô formula, for any $t>s\geq 0$

$$\begin{aligned}\int_s^t b_{ii}(r)Y_i(r)dr=&\ln Y_i(s)-\ln Y_i(t)\\&+\int_s^t\left[a_i(r)-\frac{1}{2}\sigma_i^2(r)+\int_{\mathbb{Y}}(\ln(1+\gamma_i(r,u))-\gamma_i(r,u))\lambda(du)\right]ds\\&+\int_s^t\sigma_i(r)dW(r)+\int_s^t\int_{\mathbb{Y}}\ln(1+\gamma_i(r,u))\tilde{N}(dr,du).\end{aligned} \tag{4.28}$$

eq107

This, together with Theorem 4.4 and (4.23), yields that for any $\epsilon>0$ there exists $\bar{T}>0$ such that

$$\begin{aligned}\int_s^t b_{ii}(r)Y_i(r)dr\leq&\int_s^t\left[a_i(r)-\frac{1}{2}\sigma_i^2(r)+\int_{\mathbb{Y}}(\ln(1+\gamma_i(r,u))-\gamma_i(r,u))\lambda(du)\right]ds\\&+3\epsilon(s+t)\end{aligned} \tag{4.29}$$

eq114

whenever $t \geq s \geq \bar{T}$. Moreover taking into account (4.28) and (4.29), we have for $t > s \geq \bar{T}$

$$\begin{aligned}\int_s^t b_{ij}(r)Y_j(r)dr &= \int_s^t \frac{b_{ij}(r)}{b_{jj}(r)} b_{jj}(r)Y_j(r)dr \\ &\leq R_{ij}\int_s^t b_{jj}(r)Y_j(r)dr \\ &\leq 3\epsilon(s+t)R_{ij} \\ &\quad + \int_s^t R_{ij}\left[a_i(r) - \frac{1}{2}\sigma_i^2(r) + \int_{\mathbb{Y}}(\ln(1+\gamma_i(r,u)) - \gamma_i(r,u))\lambda(du)\right]ds.\end{aligned}$$

Putting this into (4.27) leads to

$$\begin{aligned}\frac{1}{Z_i(t)} &= \frac{1}{Z_i(s)}\exp\Big(-\int_s^t \Big[R_i(r) - \sum_{i\neq j} R_{ij}R_j(r)\Big]dr + \epsilon(s+t)\Big(3\sum_{i\neq j}R_{ij} + 2\Big)\Big) \\ &\quad + \int_s^t b_{ii}(r)\exp\Big(-\int_r^t \Big[R_i(\tau) - \sum_{i\neq j} R_{ij}R_j(\tau)\Big]d\tau \\ &\quad + \epsilon(r+t)\Big(3\sum_{i\neq j}R_{ij} + 2\Big)\Big)dr,\end{aligned}$$

which, in addition to (4.25), implies

$$\frac{1}{Z_i(t)} = \frac{1}{Z_i(s)}\exp\Big(\epsilon(s+t)\Big(3\sum_{i\neq j}R_{ij} + 2\Big)\Big) + \int_s^t b_{ii}(r)\exp\Big(\epsilon(r+t)\Big(3\sum_{i\neq j}R_{ij} + 2\Big)\Big)dr.$$

Carrying out similar arguments to Theorem 4.4, we can deduce that there exists $K > 0$ such that for $t > s \geq \bar{T}$

$$\exp\Big(-2\epsilon(s+t)\Big(3\sum_{i\neq j}R_{ij} + 2\Big)\Big)\frac{1}{Z_i(t)} \leq K$$

and the conclusion follows. □

Now a combination of Theorem 4.4 and Lemma 4.5 gives the following theorem.

**Theorem 4.5.** Under the conditions of Lemma 4.5, for each $i = 1, \cdots, n$

$$\lim_{t\to\infty}\frac{\ln X_i(t)}{t} = 0, \text{ a.s.}$$

Another important property of a population dynamics is the extinction which means every species will become extinct. The most natural analogue for the stochastic population dynamics (1.5) is that every species will become extinct with probability 1. To be precise, let us give the definition.

**Definition 4.2.** Stochastic population dynamics (1.5) is said to be extinct with probability 1 if, for every initial data $x_0 \in \mathbb{R}^n_+$, the solution $X_i(t), t \geq 0$, has the property

$$\lim_{t\to\infty} X_i(t) \to 0 \quad \text{a.s. .}$$

**Theorem 4.6.** Let assumption (**A**) and (4.20) hold. Assume further that

$$\eta_i := \limsup_{t\to\infty} \frac{1}{t}\int_0^t \beta_i(s)ds < 0,$$

where, for $t \geq 0$ and $i = 1, \cdots, n$,

$$\beta_i(t) := a_i(t) - \frac{1}{2}\sigma_i^2(t) - \int_{\mathbb{Y}} (\gamma_i(t,u) - \ln(1+\gamma_i(t,u)))\lambda(du).$$

Then stochastic population dynamics (1.5) is extinct a.s.

*Proof.* Recalling by the comparison theorem that, for any $t \geq 0$ and $i = 1, \cdots, n$,

$$X_i(t) \leq Y_i(t),$$

we only need to verify $\limsup_{t\to\infty} Y_i(t) = 0$ a.s., due to

$$0 \leq \liminf_{t\to\infty} X_i(t) \leq \limsup_{t\to\infty} X_i(t) \leq \limsup_{t\to\infty} Y_i(t).$$

Since $b_i(t) \geq 0$, by (4.7) it is easy to deserve that

$$\begin{aligned} Y_i(t) &\leq X_i(0)\exp\Big(\int_0^t \beta_i(s)ds + \int_0^t \sigma_i(s)dW(s) + \int_0^t\int_{\mathbb{Y}} \ln(1+\gamma_i(s,u))\tilde{N}(ds,du)\Big) \\ &= X_i(0)\exp\Big(t\Big(\frac{1}{t}\int_0^t \beta_i(s)ds + \frac{1}{t}\int_0^t \sigma_i(s)dW(s) \\ &\quad + \frac{1}{t}\int_0^t\int_{\mathbb{Y}} \ln(1+\gamma_i(s,u))\tilde{N}(ds,du)\Big)\Big). \end{aligned}$$

Thanks to $\eta_i < 0$, in addition to (4.22), we deduce that $\limsup_{t\to\infty} Y_i(t) = 0$ a.s. and the conclusion follows.

□

**Remark 4.2.** In Theorem 4.3, we know that one dimensional our model has a unique invariant measure under some conditions, however we can not obtain the same result for $n$dimensional model ($n \geq 2$).

# 5 Conclusions and Further Remarks

In this paper, we discuss competitive Lotka-Volterra population dynamics with jumps. We show that the model admits a unique global positive solution, investigate uniformly finite $p$-th moment with $p > 0$, stochastic ultimate boundedness, invariant measure and long-term behaviors of solutions. Moreover, using a variation-of-constants formula for a class of SDEs with jumps, we provide explicit solution for the model, investigate precisely the sample Lyapunov exponent for each component and the extinction of our $n$-dimensional model.

As we mentioned in the introduction section, random perturbations of interspecific or intraspecific interactions by white noise is one of ways to perturb population dynamics. In [13], Mao, et al. investigate stochastic $n$-dimensional Lotka-Volterra systems

$$dX(t) = \text{diag}(X_1(t), \cdots, X_n(t)) \left[(a + BX(t))dt + \sigma X(t)dW(t)\right], \tag{5.1}$$ eq53

where $a = (a_1, \cdots, a_n)^T, B = (b_{ij})_{n\times n}, \sigma = (\sigma_{ij})_{n\times n}$. It is interesting to know what would happen if stochastic Lotka-Volterra systems (5.1) are further perturbed by jump diffusions, namely

$$\begin{aligned} dX(t) = \text{diag}(X_1(t^-), \cdots, X_n(t^-)) \Big[&(a + BX(t))dt + \sigma X(t)dW(t) \\ &+ \int_{\mathbb{Y}} \gamma(X(t^-), u)\tilde{N}(dt, du)\Big], \end{aligned} \tag{5.2}$$ eq54

where $\gamma = (\gamma_1, \cdots, \gamma_n)^T$. On the other hand, the hybrid systems driven by continuous-time Markov chains have been used to model many practical systems where they may experience abrupt changes in their structure and parameters caused by phenomena such as environmental disturbances [15]. As mentioned in Zhu and Yin [25, 26], interspecific or intraspecific interactions are often subject to environmental noise, and the qualitative changes cannot be described by the traditional (deterministic or stochastic) Lotka-Volterra models. For example, interspecific or intraspecific interactions often vary according to the changes in nutrition and/or food resources. We use the continuous-time Markov chain $r(t)$ with a finite state space $\mathcal{M} = \{1, \cdots, m\}$ to model these abrupt changes, and need to deal with stochastic hybrid population dynamics with jumps

$$\begin{aligned} dX(t) = \text{diag}(X_1(t^-), \cdots, X_n(t^-)) \Big[&(a(r(t)) + B(r(t))X(t))dt + \sigma(r(t))X(t)dW(t) \\ &+ \int_{\mathbb{Y}} \gamma(X(t^-), r(t), u)\tilde{N}(dt, du)\Big]. \end{aligned} \tag{5.3}$$ eq55

We will report these in our following papers.

# References


[1] Applebaum, D., *Lévy Processes and Stochastics Calculus*, Cambridge University Press, $2^{nd}$ Edition, 2009.

[2] Gard, T., Persistence in stochastic food web models, *Bull. Math. Biol.*, **46** (1984), 357-370.

[3] Gard, T., Stability for multispecies population models in random environments, *Nonlinear Anal.*, **10** (1986), 1411-1419.

[4] Gopalsamy, K., *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Dordrecht, 1992.

[5] Hu, G. and Wang, K., On stochastic Logistic Equation with Markovian Switching and White Noise, *Osaka J. Math.*, 2010, preprint.

[6] Jiang, D. and Shi, N., A note on nonautonomous logistic equation with random perturbation, *J. Math. Anal. Appl.*, **303** (2005), 164-172.

[7] Kuang, Y. , *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, 1993.

[8] Kunita, H., Itô's stochastic calculus: Its surprising power for applications, *Stochastic Process. Appl.*, **120** (2010), 622-652.

[9] Li, X., Tang, C., Ji, X, The criteria for globally stable equilibrium in $n$-dimensional LotkaVolterra systems, *J. Math. Anal. Appl.*, **240** (1999), 600-606.

[10] Lipster, R., A strong law of large numbers for local martingales, *Stochastics*, **3** (1980) 217-228.

[11] Liu, M. and Wang, K., Persistence and extinction in stochastic non-autonomous logistic systems, *J. Math. Anal. Appl.*, **375** (2011), 443-457.

[12] Mao, X., *Stochastic Differential Equations and Applications*, Horwood Publishing, $2^{nd}$ Edition, 2008.

[13] Mao, X., Marion, G. and Renshaw, E., Environmental noise suppresses explosion in population dynamics, *Stochastic Process. Appl.*, **97** (2002), 95-110.

[14] Mao, X., Yuan, C. and Zou, J., Stochastic differential delay equations of population dynamics, *J. Math. Anal. Appl.*, **304** (2005), 296-320.

[15] Mao, X. and Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[16] Øksendal, B. and Sulem, A., *Applied stochastic control of jump diffusions*, $2^{nd}$ Edition, Springer, Berlin, 2007.

[17] Peng, S. and Zhu, X., Necessary and sufficient condition for comparison theorem of 1-dimensional stochastic differential equations, *Stochastic Process. Appl.*, **116** (2006), 370-380.

[18] Protter, P. E., *Stochastic Integration and Differential Equations*, $2^{nd}$ Edition, Springer-Verlag, New York, 2004.

[19] Prato, D. and Zabczyk, J., *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, 1996.

[20] Roubik, D., Experimental community studies: Time-series tests of competition between African and neotropical bees, *Ecology*, **64** (1983), 971-978.

[21] Roughgarden, J., *Theory of Population Genetics and Evolutionary Ecology: An Introduction*, Macmillan, New York, 1979.

[22] Takeuchi, Y. and Adachi, N., The existence of globally stable equilibria of ecosystems of the generalized Volterra type, *J. Math. Biol.*, **10** (1980), 401-415.

[23] Takeuchi, Y. and Adachi, N., The stability of generalized Volterra equations, *J. Math. Anal. Appl.*, **62** (1978), 453-473.

[24] Xiao, D. and Li, W., Limit cycles for the competitive three dimensional LotkaVolterra system, *J. Differential Equations*, **164** (2000), 1-15.

[25] Zhu, C. and Yin, G., On hybrid competitive Lotka-Volterra ecosystems, *Nonlinear Anal.*, **71** (2009), e1370-e1379.

[26] Zhu, C. and Yin, G., On competitive Lotka-Volterra model in random enrironments, *J. Math. Anal. Appl.*, **357** (2009), 154-170.