

# AV-COURANT ALGEBROIDS AND GENERALIZED CR STRUCTURES

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ABSTRACT. We construct a generalization of Courant algebroids which are classified by the third cohomology group  $H^3(A, V)$ , where  $A$  is a Lie Algebroid, and  $V$  is an  $A$ -module. We see that both Courant algebroids and  $\mathcal{E}^1(M)$  structures are examples of them. Finally we introduce generalized CR structures on a manifold, which are a generalization of generalized complex structures, and show that every CR structure and contact structure is an example of a generalized CR structure.

## 1. INTRODUCTION

Courant algebroids and the Dirac structures associated to them were first introduced by Courant and Weinstein (see [6] and [7]) to provide a unifying framework for studying such objects as Poisson and symplectic manifolds. Aïssa Wade later introduced the related  $\mathcal{E}^1(M)$ -Dirac structures in [27] to describe Jacobi structures.

In [14], Hitchin defined generalized complex structures which are further described by Gualtieri in [13]. Generalized complex structures unify both symplectic and complex structures, interpolating between the two, and have appeared in the context of string theory [18]. In [15] Iglesias and Wade describe generalized contact structures, an odd-dimensional analog to generalized complex structures, using the language of  $\mathcal{E}^1(M)$ -Dirac structures.

In this paper, we shall define *AV-Courant Algebroids*, a generalization of Courant algebroids which also allows one to describe  $\mathcal{E}^1(M)$ -Dirac structures. We will show that these have a classification similar to Ševera's classification of exact Courant algebroids in [24].

To be more explicit, let  $M$  be a smooth manifold,  $A \rightarrow M$  be a Lie algebroid with anchor map  $a : A \rightarrow TM$ , and  $V \rightarrow M$  a vector bundle which is an  $A$ -module. If we endow  $V$  with the structure of a trivial Lie algebroid (that is: trivial bracket and anchor), then it is well known that the extensions of  $A$  by  $V$  are a geometric realization of  $H^2(A, V)$  (see [19], for instance). In this paper, we introduce *AV-Courant algebroids* and describe how they are a geometric realization of  $H^3(A, V)$ .

We then go on to show how to simplify the structure of certain *AV-Courant algebroids* by pulling them back to certain principal bundles. Indeed, in the most interesting cases, the pullbacks will simply be exact Courant algebroids.

We then introduce *AV-Dirac structures*, a special class of subbundles of an *AV-Courant algebroid* which generalize Dirac structures. Finally, we will introduce a special class of *AV-Dirac structures*, called *generalized CR structures*, which allow us to describe any complex, symplectic, CR or contact structure on a manifold, as well as many interpolations of those structures. Poisson and Jacobi structures are central to this description; and while Jacobi brackets do not generally satisfy the Leibniz rule, as a consequence of the ideas in this paper we show how one can think of a Jacobi structure on a manifold as a Lie bracket on the sections of a certain line bundle which does satisfy the Leibniz rule.

It is important to note that there are other constructions related to *AV-Courant algebroids*. For instance, recently Z. Chen, Z. Liu and Y.-H. Sheng introduced the notion of *E-Courant algebroids* [5] in order to unify the concepts of *omni-Lie algebroids* (introduced in [3], see also [4]) and *generalized Courant algebroids* or *Courant-Jacobi algebroids* (introduced in [23] and [11] respectively; they are

equivalent concepts see [23]). The key property that both  $E$ -Courant algebroids and  $AV$ -Courant algebroids share is that they replace the  $\mathbb{R}$ -valued bilinear form of Courant algebroids with one taking values in an arbitrary vector bundle ( $E$  or  $V$  respectively). Nevertheless, while there is some overlap between  $E$ -Courant algebroids and  $AV$ -Courant algebroids in terms of examples, these constructions are not equivalent. Moreover, this paper is distinguished from [5] by having the definition of generalized CR manifolds as one of its main goals.

Meanwhile generalized CRF structures, introduced and studied in great detail by Izu Vaisman in [26], and generalized CR structures describe similar objects. To summarize, a complex structure on a manifold  $M$  is a subbundle  $H \subset TM \otimes \mathbb{C}$  such that

$$(1.1) \quad H \oplus \bar{H} = TM \otimes \mathbb{C}$$

and  $[H, H] \subset H$ . The definition of a CR structure simply relaxes (1.1) to  $H \cap \bar{H} = 0$ . On the other hand, the definition of a generalized complex structure replaces  $TM$  with the standard Courant algebroid  $\mathbb{T}M = T^*M \oplus TM$  in the definition of a complex structure, and in addition, requires  $H \subset \mathbb{T}M \otimes \mathbb{C}$  to be isotropic.

The definition of a generalized CRF structure parallels the definition of a generalized complex structure, but relaxes the requirement that  $H \oplus \bar{H} = \mathbb{T}M \otimes \mathbb{C}$  to  $H \cap \bar{H} = 0$ . Among numerous interesting examples of generalized CRF structures are normal contact structures and normalized CR structures (namely those CR structures  $H \subset \mathbb{T}M \otimes \mathbb{C}$  for which there is a splitting  $\mathbb{T}M \otimes \mathbb{C} = H \oplus \bar{H} \oplus Q_c$  and  $[H, Q_c] \subset H \oplus Q_c$ ).

Generalized CR structures differ from generalized CRF structures in multiple ways, in particular they replace the standard Courant algebroid with an  $AV$ -Courant algebroid  $\mathbb{A}$ , and furthermore they take a different approach to describe contact and CR structures, using only maximal isotropic subbundles but allowing  $H \cap \bar{H}$  to contain ‘infinitesimal’ elements.

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## 2. $AV$ -COURANT ALGEBROIDS

Let  $M$  be a smooth manifold,  $A \rightarrow M$  a Lie algebroid, and  $V \rightarrow M$  a vector bundle which is an  $A$ -module, that is, there is a  $C^\infty(M)$ -linear Lie algebra homomorphism

$$(2.1) \quad \mathcal{L} : \Gamma(A) \rightarrow \text{End}(\Gamma(V))$$

satisfying the Leibniz rule. (See [19] for more details.)

For any  $A$ -module  $V$ , the sections of  $V \otimes \wedge^* A^*$  have the structure of a graded right  $\wedge^* \Gamma(A^*)$ -module, and there are several important derivations of its module structure which we shall use throughout this paper. The first is the interior product with a section  $X \in \Gamma(A)$ ,

$$\iota_X : \Gamma(V \otimes \wedge^i A^*) \rightarrow \Gamma(V \otimes \wedge^{i-1} A^*),$$

a derivation of degree -1.

The second is the Lie derivative, a derivation of degree 0, defined to be the unique derivation of  $V \otimes \wedge^* A^*$  whose restriction to  $V$  is given by (2.1), and such that the graded commutator with  $\iota$  satisfies

$$[\mathcal{L}_X, \iota_Y] = \iota_{[X, Y]}.$$

Finally the differential  $d$ , a derivation of degree 1, is defined inductively by the graded commutator  $\mathcal{L}_X = [d, \iota_X]$  (for all  $X \in \Gamma(A)$ ).

It is easy to check that  $d^2 = 0$ , and the cohomology groups of the complex  $(\Gamma(V \otimes \wedge^\bullet A^*), d)$  are denoted  $H^\bullet(A, V)$ .

**2.1. Definition of AV-Courant Algebroids.** Let  $A$  be a Lie algebroid, and  $V$  an  $A$ -module.

**Definition 1** (*AV-Courant Algebroid*). Let  $\mathbb{A}$  be a vector bundle over  $M$ , with a  $V$ -valued symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on the fibres of  $\mathbb{A}$ , and a bracket  $[[\cdot, \cdot]]$  on sections of  $\mathbb{A}$ . Suppose further that there is a short exact sequence of bundle maps

$$(2.2) \quad 0 \rightarrow V \otimes A^* \xrightarrow{j} \mathbb{A} \xrightarrow{\pi} A \rightarrow 0.$$

such that for any  $e \in \Gamma(\mathbb{A})$  and  $\xi \in \Gamma(V \otimes A^*)$ ,

$$(2.3) \quad \langle e, j(\xi) \rangle = \iota_{\pi(e)} \xi.$$

The bundle  $\mathbb{A}$  with these structures is called an *AV-Courant algebroid* if, for  $f \in C^\infty(M)$ , and  $e, e_i \in \Gamma(\mathbb{A})$ , the following axioms are satisfied:

$$\text{AV-1 } [[e_1, [e_2, e_3]]] = [[[e_1, e_2]], e_3] + [e_2, [[e_1, e_3]]]$$

$$\text{AV-2 } \pi([[e_1, e_2]]) = [\pi(e_1), \pi(e_2)]$$

$$\text{AV-3 } [[e, e]] = \frac{1}{2} D \langle e, e \rangle \text{ where } D = j \circ d$$

$$\text{AV-4 } \mathcal{L}_{\pi(e_1)} \langle e_2, e_3 \rangle = \langle [[e_1, e_2]], e_3 \rangle + \langle e_2, [[e_1, e_3]] \rangle$$

we will often refer to  $[[\cdot, \cdot]]$  as the Courant bracket.

*Remark 1.* Axioms (AV-1) and (AV-4) state that  $[[e, \cdot]]$  is a derivation of both the Courant bracket and the bilinear form, while Axiom (AV-2) describes the relation of the Courant bracket to the Lie algebroid bracket of  $A$ . One should interpret Axiom (AV-3) as saying that the failure of  $[[\cdot, \cdot]]$  to be skew symmetric is only an ‘infinitesimal’  $D(\cdot)$ .

The bracket is also derivation of  $\mathbb{A}$  as a  $C^\infty(M)$ -module in the sense that

$$[[e_1, f e_2]] = f [[e_1, e_2]] + a \circ \pi(e_1)(f) \cdot e_2$$

for any  $e_1, e_2 \in \Gamma(\mathbb{A})$  and  $f \in C^\infty(M)$ . In fact if  $e_3 \in \Gamma(\mathbb{A})$ ,

$$\begin{aligned} & \langle a \circ \pi(e_1)(f) \cdot e_2 + f [[e_1, e_2]] - [[e_1, f e_2]], e_3 \rangle \\ \text{(by (AV-4)) } &= \langle a \circ \pi(e_1)(f) \cdot e_2 + f [[e_1, e_2]], e_3 \rangle - \pi(e_1) \langle f e_2, e_3 \rangle + \langle f e_2, [[e_1, e_3]] \rangle \\ &= a \circ \pi(e_1)(f) \langle e_2, e_3 \rangle - \pi(e_1) \langle f e_2, e_3 \rangle + f (\langle [[e_1, e_2]], e_3 \rangle + \langle e_2, [[e_1, e_3]] \rangle) \\ \text{(by (AV-4)) } &= a \circ \pi(e_1)(f) \langle e_2, e_3 \rangle - \pi(e_1) \langle f e_2, e_3 \rangle + f \pi(e_1) \langle e_2, e_3 \rangle \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that  $V$  is an  $A$  module. Since this holds for all  $e_3 \in \Gamma(\mathbb{A})$ , and  $\langle \cdot, \cdot \rangle$  is non-degenerate, the statement follows.

*Remark 2.* One notices that (2.3) and exactness of (2.2) implies that the map

$$(2.4) \quad e \rightarrow \langle e, \cdot \rangle : \mathbb{A} \rightarrow V \otimes \mathbb{A}^*$$

is an injection. Consequently, if  $V$  is a line bundle, it follows that  $\mathbb{A} \simeq V \otimes \mathbb{A}^*$ , and  $j$  must be the composition

$$j : V \otimes A^* \xrightarrow{\text{id} \otimes \pi^*} V \otimes \mathbb{A}^* \simeq \mathbb{A}.$$

**2.2. Splitting.** We call  $\phi : A \rightarrow \mathbb{A}$  an isotropic splitting, if it splits the exact sequence (2.2) and  $\phi(A)$  is an isotropic subspace of  $\mathbb{A}$  with respect to the inner product.

*Remark 3.* Such splittings exist. In fact we may choose a splitting  $\lambda : A \rightarrow \mathbb{A}$ , which is not necessarily isotropic.

Then we have a map  $\gamma : A \rightarrow V \otimes A^*$  given by the composition

$$\gamma : A \xrightarrow{\lambda} \mathbb{A} \xrightarrow{e \rightarrow \langle e, \cdot \rangle} V \otimes \mathbb{A}^* \xrightarrow{\text{id} \otimes \lambda^*} V \otimes A^*.$$

We let  $\phi = \lambda - \frac{1}{2}j \circ \gamma$ . It is easy to check that  $\phi$  is an isotropic splitting.

If  $\phi : A \rightarrow \mathbb{A}$  is an isotropic splitting, then we have an isomorphism  $\phi \oplus j : A \oplus (V \otimes A^*) \rightarrow \mathbb{A}$ .

**Proposition 1.** *Let  $\phi : A \rightarrow \mathbb{A}$  be an isotropic splitting. Then under the above isomorphism, the bracket on  $A \oplus (V \otimes A^*)$  is given by*

$$(2.5) \quad \llbracket X + \xi, Y + \eta \rrbracket_\phi = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_X \iota_Y H_\phi,$$

where  $X, Y \in \Gamma(A)$ ,  $\xi, \eta \in \Gamma(V \otimes A^*)$ , and  $H_\phi \in \Gamma(V \otimes \wedge^3 A^*)$ , with  $dH_\phi = 0$ .

Furthermore, if  $\psi : A \rightarrow \mathbb{A}$  is a different choice of isotropic splitting, then  $\psi(X) = \phi(X) + j(\iota_X \beta)$ , and  $H_\psi = H_\phi - d\beta$ , where  $\beta \in \Gamma(V \otimes \wedge^2 A^*)$ .

The proof is relegated to the appendix, since it is parallel to the proof for ordinary Courant algebroids (see [1] or [24]).

**Theorem 1.** *Let  $A$  be a Lie algebroid and  $V$  an  $A$ -module. Then the isomorphism classes of  $AV$ -Courant algebroids are in bijective correspondence with  $H^3(A, V)$ .*

*Proof.* If  $H \in \Gamma(V \otimes \wedge^3 A^*)$ , and  $dH = 0$ , then let  $\mathbb{A} = A \oplus (V \otimes A^*)$ . We define  $\langle \cdot, \cdot \rangle$  by

$$(2.6) \quad \langle X + \xi, Y + \eta \rangle = \iota_X \eta + \iota_Y \xi,$$

where  $\xi, \eta \in \Gamma(V \otimes A^*)$  and  $X, Y \in \Gamma(A)$ . We define the bracket to be given by Equation (2.5). It is not difficult to check that this satisfies the axioms of an  $AV$ -Courant algebroid.

Conversely, by the above proposition, every  $AV$ -Courant algebroid defines a unique element of  $H^3(A, V)$ .  $\square$

### 3. EXAMPLES

*Example 1.* Let  $M$  be a point, then a Lie algebroid  $A$  is simply a Lie algebra, and an  $A$ -module  $V$  is a finite dimensional representation of  $A$  as a Lie algebra.  $H^i(A, V)$  is simply the  $V$ -valued Lie algebra cohomology, and  $H^3(A, V)$  classifies the  $AV$ -Courant algebroids over a point. Note that an  $AV$ -Courant algebroid over a point is a Lie algebra if and only if  $V$  is a trivial  $A$ -representation.

*Example 2 (Exact Courant Algebroids).* If we let  $A \simeq TM$  and  $V = M \times \mathbb{R}$  be the trivial line bundle with a trivial  $TM$ -module structure, then we have the class of exact Courant algebroids (see [6] or [7]) on  $M$ ,

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} \mathbb{A} \xrightarrow{\pi} TM \longrightarrow 0.$$

Theorem 1 then corresponds to Ševera's classification of exact Courant algebroids.

*Example 3 ( $\mathcal{E}^1(M)$  Structures).* The bundle  $\mathcal{E}^1(M)$  was introduced by A. Wade in [27], and is uniquely associated to a given manifold  $M$ . Within the context of our paper, it is easiest to define  $\mathcal{E}^1(M)$  by using the language of  $AV$ -Courant algebroids:

We let  $A = TM \oplus L$ , where  $L \simeq \mathbb{R}$  is spanned by the abstract symbol  $\frac{\partial}{\partial t}$ . The bracket is given by

$$[X \oplus f \frac{\partial}{\partial t}, Y \oplus g \frac{\partial}{\partial t}]_A = [X, Y]_{TM} \oplus (X(g) - Y(f)) \frac{\partial}{\partial t},$$

where  $X, Y \in \mathcal{X}(M)$ , and  $f, g \in C^\infty(M)$ .

Let  $V$  be the trivial line bundle spanned by the abstract symbol  $e^t$ , so that  $\Gamma(V) = \{e^t h | h \in C^\infty(M)\}$ .  $V$  has an  $A$ -module structure (as suggested by the choice of symbols) given by

$$(X \oplus f \frac{\partial}{\partial t})(e^t h) = e^t(X(h) + fh).$$

We let  $\mathbb{A} := (TM \oplus L) \oplus (V \otimes (T^*M \oplus L^*))$ , and define a bracket on sections by Equation (2.5). It is clear that this data defines an  $AV$ -Courant algebroid on  $M$ . If we set  $H = 0$  in Equation (2.5), then the pair  $(\mathbb{A}, \llbracket \cdot, \cdot \rrbracket)$  associated to  $M$  is the  $\mathcal{E}^1(M)$ -Structure, as introduced by A. Wade in [27].

*Example 4* (Equivariant  $AV$ -Courant Algebroids on Principal Bundles). Let  $\nu : P \rightarrow M$  be a  $G$ -principal bundle. Suppose that  $A$  is a Lie algebroid over  $P$  and  $V$  is an  $A$ -module; and that there is an  $AV$ -Courant algebroid on  $P$ ,

$$0 \rightarrow V \otimes A^* \rightarrow \mathbb{A} \rightarrow A \rightarrow 0.$$

If the action of  $G$  on  $P$  lifts to an action by bundle maps on  $V$ ,  $A$  and  $\mathbb{A}$ , such that all the structures involved are  $G$ -equivariant. Then the quotient,

$$0 \rightarrow (V \otimes A^*)/G \rightarrow \mathbb{A}/G \rightarrow A/G \rightarrow 0,$$

is an  $A/G$   $V/G$ -Courant algebroid.

*Example 5*. Let  $\nu : P \rightarrow M$  be a  $G$ -principal bundle, and  $W$  a  $k$ -dimensional vector space possessing a linear action of  $G$ . We regard  $W$  as a trivial bundle over  $P$ , and we consider the bundle  $\mathbb{T} := TP \oplus (W \otimes T^*P)$ , endowed with a  $W$ -valued symmetric bilinear form given by Equation (2.6). We also define a bracket on sections of  $\mathbb{T}$  by Equation (2.5) where  $H \in \Omega^3(P, W)^G$  is closed, then

$$0 \rightarrow W \otimes T^*P \xrightarrow{j} \mathbb{T} \xrightarrow{\pi} TP \rightarrow 0$$

is an equivariant  $TP$   $W$ -Courant algebroid on  $P$  (where  $j$  and  $\pi$  are the obvious inclusion and projection). Thus (as in Example 4), we have an  $AV$ -Courant algebroid on  $P/G$ , where  $A = TP/G$  is the Atiyah algebroid, and  $V = P \times_G W$

Note, if  $W$  is 1-dimensional, then the  $TP$   $W$ -Courant algebroid given above is simply an exact Courant algebroid.

As it turns out, this is quite a general example. Indeed if  $A$  is a transitive Lie algebroid, then locally all  $AV$ -Courant algebroids result from such a construction (See Section 5).

*Remark 4*. In the above example, one could replace  $P \times W$  with any flat bundle.

*Example 6*. As a special case of Example 5, if we take  $G = \mathbb{R}$ , then  $P = M \times \mathbb{R}$  is a  $G$ -principal bundle where the action is translation. We let  $W$  be the trivial line bundle over  $P$  and let  $t \in G$  act on  $W$  by scaling by  $e^t$ .

To describe the  $G$ -action explicitly, we make the identification  $\Gamma(W) = C^\infty(M \times \mathbb{R})$ ; and then the action is given by

$$(3.1) \quad \begin{aligned} \mathbb{R} \times C^\infty(M \times \mathbb{R}) &\rightarrow C^\infty(M \times \mathbb{R}) \\ (t, f(x, s)) &\rightarrow e^{-t} f(x, s + t) \end{aligned}$$

The quotient of the  $TP$   $W$ -Courant algebroid on  $P$  with  $H = 0$  under this action is precisely the  $\mathcal{E}^1(M)$ -Structure on  $M = P/\mathbb{R}$ .

*Example 7*. If  $A$  is a Lie algebroid over  $M$ ,  $V$  is an  $A$ -module, and  $\mathbb{A}$  is an  $AV$ -Courant algebroid on the manifold  $M$ , and if  $F \subset M$  is a leaf of the singular foliation defined by  $a(A)$ , then  $i^* \mathbb{A}$  is an  $i^* A$   $i^* V$ -Courant algebroid on  $F$ , where  $i : F \rightarrow M$  is the inclusion.

## 4. AV-DIRAC STRUCTURES

**Definition 2** (*AV-Dirac Structure*). Let  $M$  be a manifold,  $A \rightarrow M$  be a Lie algebroid over  $M$ ,  $V \rightarrow M$  an  $A$ -module, and  $\mathbb{A}$  an  $AV$ -Courant algebroid. Suppose that  $L \subset \mathbb{A}$  is a subbundle, since  $\mathbb{A}$  has a non-degenerate inner product, we can define  $L^\perp = \{v \in \mathbb{A} \mid \langle v, u \rangle = 0 \forall u \in L\}$ .

We call  $L$  an *almost AV-Dirac structure* if  $L^\perp = L$ . An *AV-Dirac structure* is an almost  $AV$ -Dirac structure,  $L \subset \mathbb{A}$  which is involutive with respect to the bracket  $\llbracket, \rrbracket$ .

*Remark 5.* If  $L \subset \mathbb{A}$  is an  $AV$ -Dirac structure, then  $\llbracket e, e \rrbracket = \frac{1}{2}D\langle e, e \rangle = 0$  for any section  $e \in \Gamma(L)$ , so  $\llbracket, \rrbracket$  is skew-symmetric when restricted to  $L$ , and then by the other properties of the bracket, it follows that  $a \circ \pi : L \rightarrow TM$  is a Lie algebroid, and  $\pi : L \rightarrow A$  is a Lie algebroid morphism.

*Example 8* (*Invariant Dirac Structure on a Principal Bundle*). Using the notation of Example 4, suppose that the  $A/G$   $V/G$ -Courant algebroid  $\mathbb{A}/G$  on  $M$  is the quotient of a  $AV$ -Courant algebroid  $\mathbb{A}$  on  $P$ . If  $L \subset \mathbb{A}$  is an  $AV$ -Dirac structure which is  $G$  invariant, then it is clear that  $L/G \subset \mathbb{A}/G$  is an  $A/G$   $V/G$ -Dirac structure (see Example 4).

*Example 9* ( $\mathcal{E}^1(M)$ -Dirac Structures). Using Example 3, we can describe  $\mathcal{E}^1(M)$ , the bundle introduced by A. Wade in [27], as an  $AV$ -Courant algebroid. In this context, the  $\mathcal{E}^1(M)$ -Dirac structures (also introduced by A. Wade in [27]) correspond directly to the  $AV$ -Dirac structures.

## 5. TRANSITIVE LIE ALGEBROIDS

**5.1. Simplifying  $AV$ -Courant Algebroids.** Suppose that  $A$  is a Lie algebroid,  $V$  is an  $A$ -module, and  $\mathbb{A}$  is an  $AV$ -Courant algebroid over  $M$  (where we use the notation given in the definition of  $AV$ -Courant algebroids). We will assume for the duration of this section that  $M$  is connected, and we require that  $A$  be a transitive Lie algebroid, namely the anchor map  $a : A \rightarrow TM$  is surjective (see [19] for more details).

Since  $\mathbb{A}$  may be quite complicated, we wish to examine whether this  $AV$ -Courant algebroid is the quotient of a much simpler  $A'V'$ -Courant algebroid on a principal bundle over  $M$ , where  $A'$  is a very simple Lie algebroid and  $V'$  is a very simple  $A'$ -module. To be more explicit, we wish to examine whether  $\mathbb{A}$  results from the construction in Example 5. For this to be true, it is clearly necessary that  $A$  be the Atiyah algebroid of that principal bundle; namely if  $P$  is the principal bundle, then  $A = TP/G$ . The existence of such a principal bundle is equivalent to the integrability of  $A$  as a Lie algebroid:

**Proposition 2.** *Suppose that  $A \rightarrow M$  is an integrable transitive Lie algebroid, that is to say, there exists a source-simply connected Lie groupoid  $\Gamma \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} M$  with Lie algebroid  $A$  (see [19] for more details). Then  $A$  is the Atiyah algebroid of a principal bundle.*

*Conversely, if  $A$  is the Atiyah algebroid of a principal bundle, then  $A$  is an integrable Lie algebroid.*

*Proof.* Suppose first that  $A$  is integrable, then using the notation in the statement of the proposition, where  $s : \Gamma \rightarrow M$  is the source map and  $t : \Gamma \rightarrow M$  is the target map, let  $x \in M$ , let  $P = \Gamma_x := s^{-1}(x)$ , and let  $G = \Gamma_x^x := s^{-1}(x) \cap t^{-1}(x)$ .

Since  $A$  is transitive,  $t : P \rightarrow M$  is a surjective submersion, for clarity, we define  $p := t|_P$ . Furthermore, if  $y \in M$ , and  $g \in \Gamma_x^y$ , then  $g : p^{-1}(x) \rightarrow p^{-1}(y)$  is a diffeomorphism, so  $p : P \rightarrow M$  is a fibre bundle, with its fibre diffeomorphic to  $G$ . In addition,  $G$  has a right action on  $P$ , given by right multiplication in the Lie groupoid. If  $p^{-1}(y) = \Gamma_x^y$  is a fibre, and  $g \in \Gamma_x^y$  then the diffeomorphism  $g : p^{-1}(x) \rightarrow p^{-1}(y)$  is given by left groupoid multiplication while the action of  $G$  on  $P$  is given by right groupoid multiplication, so it is clear that the two operations commute, from which it follows that  $G$  preserves the fibres of  $P$ , acting transitively and freely on them. Thus  $P$  is a principal  $G$  bundle.

Since  $A$  is the Lie algebroid of  $\Gamma$ , it can be identified with the right invariant vector-fields on  $\Gamma$  tangent to the source fibres. However, since  $A$  is transitive, any two source fibres are diffeomorphic by right multiplication by some element. Thus  $A$  can be identified with the  $G$  invariant vector fields on  $P$ .

Conversely, if  $A$  is the Atiyah algebroid of some principal bundle, it obviously integrates to the gauge groupoid associated to that principal bundle (see [9] or Remark 6), and we may take  $\Gamma$  to be the source-simply connected cover of the gauge groupoid.  $\square$

We now examine whether  $V$  is an associated vector bundle.

**Proposition 3.** *Suppose that  $A$  is an integrable transitive Lie algebroid, and  $V \rightarrow M$  is an  $A$ -module. Then there exists a (possibly disconnected) Lie group  $G$ , and a simply connected principal  $G$ -bundle  $P \rightarrow M$  such that  $V$  is the quotient bundle of  $P \times \mathbb{R}^k$ , for some  $G$  action on  $\mathbb{R}^k$ . In this setting, the standard action of  $\mathcal{X}(P)$  on  $C^\infty(P, \mathbb{R}^k)$  induces the module structure on  $V$ .*

*Proof.* Using the notation and the Lie groupoid described in the previous proposition, we consider  $\Gamma_x \times V_x$ , where  $V_x$  is the fibre of  $V$  at  $x$ . We may assume that  $\Gamma$  is source simply connected and consequently since  $V$  is an  $A$ -module, by Lie's second theorem there exists a Lie groupoid morphism  $\Gamma \rightarrow \mathbf{GL}(V)$ .<sup>1</sup> Thus  $\Gamma$  acts on  $V$  and we have a map  $\tilde{p} : \Gamma_x \times V_x \rightarrow V$  given by  $(g, v) \rightarrow gv$ , this is clearly a surjective submersion<sup>2</sup>. Furthermore,

$$\tilde{p}(g, v) = \tilde{p}(g', v') \Leftrightarrow g^{-1}g' \in \Gamma_x^x \text{ and } v = (g^{-1}g')v'.$$

Thus, letting  $G = \Gamma_x^x$  and  $P = \Gamma_x$ , we have  $V \simeq (\Gamma_x \times V_x)/G \simeq (P \times V_x)/G$ .

Furthermore, identifying  $V_x$  with  $\mathbb{R}^k$ , if  $X \in \mathcal{X}(P) \simeq \mathcal{X}(\Gamma_x)$ , and  $\sigma \in C^\infty(P, \mathbb{R}^k)$ , then the standard action of  $X$  on  $\sigma$  is given by  $X(\sigma)_z = \frac{\partial}{\partial t}|_{t=0} \sigma(e^{tX}z)$  for any  $z \in P \simeq \Gamma_x$ . If we suppose that  $X$  and  $\sigma$  are  $G$  invariant, then

$$\tilde{p}\left(\frac{\partial}{\partial t}\Big|_{t=0} \sigma(e^{tX}(z))\right) = \frac{\partial}{\partial t}\Big|_{t=0} (e^{-tX} \tilde{p}(\sigma))_{p(z)} = (\mathcal{L}_X \tilde{p}(\sigma))_{p(z)},$$

since we defined the action of  $\Gamma$  on  $V$  in terms of the  $A$ -module structure of  $V$ .  $\square$

**Proposition 4.** *Suppose that  $A$  is an integrable Lie algebroid, and  $V \rightarrow M$  is an  $A$ -module. Then  $\mathbb{A}$  results from the construction given in Example 5. Namely there exists a Lie group  $G$ , and a principal  $G$ -bundle  $P \rightarrow M$  such that  $\mathbb{A}$  is the quotient of a TP  $\mathbb{R}^k$ -Courant algebroid. Furthermore, if  $L \subset \mathbb{A}$  is an AV-Dirac structure, then it is also the quotient of a corresponding TP  $\mathbb{R}^k$ -Dirac structure on  $P$ .*

*Consequently, if  $V$  is a line-bundle, then  $\mathbb{A}$  is simply the quotient of an exact Courant algebroid on  $P$ .*

*Proof.* We choose some isotropic splitting of  $\mathbb{A}$ , so that

$$\mathbb{A} \simeq A \oplus (V \otimes A^*),$$

the bracket is given by Equation (2.5), and the symmetric bilinear form by Equation (2.6). Then we can use the previous propositions to lift the right hand side to a principal bundle:

By the above propositions, there exists a (possibly disconnected) Lie group  $G$ , and a simply connected  $G$ -principal bundle,  $\nu : P \rightarrow M$ , such that  $A \simeq TP/G$ , and in addition to this there is a  $G$ -action on  $W := \mathbb{R}^{\dim(V)}$ , say  $\lambda : G \rightarrow \mathbf{GL}(W)$  such that  $V = P \times_G W$ . In this setting,

<sup>1</sup>See, for instance, [8], [20], or [22], for more details. Here  $\mathbf{GL}(V)$  is the Lie groupoid of linear isomorphisms of the fibres of  $V$ , namely  $\mathbf{GL}(V)_x^y = \text{Hom}(V_x, V_y)$ .

<sup>2</sup>Since  $A$  is transitive and  $M$  is connected,  $t : \Gamma_x \rightarrow M$  is a surjective submersion. Let  $y \in M$ , and  $\sigma : U \rightarrow \Gamma_x$  be a section (so that  $t \circ \sigma = \text{id}$ ). Then  $(z, v) \rightarrow \sigma(z)(v) : U \times V_x \rightarrow V_U$  is a diffeomorphism.

$\Gamma(V \otimes \wedge^i A^*) \simeq \Omega^i(P, W)^G$ , and  $d : \Gamma(V \otimes \wedge^i A^*) \rightarrow \Gamma(V \otimes \wedge^{i+1} A^*)$  is the restriction of the exterior derivative  $d$  to  $\Omega^*(P, W)^G$ .

Thus since  $H \in \Gamma(V \otimes \wedge^3 A^*) \simeq \Omega^3(P, W)^G$ , it is clear that we may view  $H$  as a  $G$ -invariant element of  $\Omega^3(P, W)$ , and define the  $TP$   $W$ -Courant algebroid  $W \otimes T^*P \rightarrow \mathbb{T} \rightarrow TP$  in terms of it: Namely,  $\mathbb{T} \simeq TP \oplus (W \otimes T^*P)$  endowed with a  $W$ -valued symmetric bilinear form given by Equation (2.6), and the bracket given by Equation (2.5). (See Example 5 for more details on this construction.)

It is clear that  $\mathbb{A}$  is the quotient of this  $TP$   $W$ -Courant algebroid.

Equivalently, it is easy to see that  $TP = \nu^*A$ ,  $W = \nu^*V$  and  $\mathbb{T} = \nu^*\mathbb{A}$ . The  $W$ -valued symmetric bilinear form on  $\mathbb{T}$  is simply the pullback of the  $V$ -valued symmetric bilinear form on  $\mathbb{A}$ , and if  $e_1, e_2 \in \Gamma(\mathbb{A})$ , then  $[\nu^*e_1, \nu^*e_2] = \nu^*[[e_1, e_2]]$ , and the bracket on  $\mathbb{T}$  is then extended to arbitrary sections of  $\mathbb{T}$  by Axioms (AV-3) and Remark 1.

Next, let  $\tilde{L} = \nu^*(L) \subset \mathbb{T}$ . It is obvious that  $L^\perp = L \Rightarrow \tilde{L}^\perp = \tilde{L}$ ; and similarly since  $L$  is involutive, so is  $\tilde{L}$ .

Thus  $\tilde{L} \subset \mathbb{T}$  is a  $TP$   $W$ -Dirac structure, and  $\tilde{L}/G = L$ .  $\square$

*Example 10.* If  $A = TM$  and  $V$  is a flat vector bundle over  $M$ , then following the proof of Proposition 4 we see that  $G = \pi_1(M)$  is the fundamental group, and  $P = \tilde{M}$  is the simply connected covering space of  $M$  over which the pullback of  $V$  is a trivial vector bundle.

*Remark 6.* The above propositions construct the principal bundle  $P$ , and the Lie group  $G$ . Suppose however, that we already have a Lie group  $G'$ , and a connected  $G'$ -principal bundle  $\nu' : P' \rightarrow M$  such that  $A \simeq TP'/G'$ . It will not be difficult to see that  $\mathbb{A}$  is the quotient of a  $AV$ -Courant algebroid on  $P'$ .

Let  $\mathcal{G} = (P' \times P')/G'$ , where we take the quotient by the diagonal action. Then

$$\mathcal{G} \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} M$$

is a Lie groupoid with Lie algebroid  $A$ , where the source map is  $s : [u, v] \rightarrow \nu'(v)$ , the target map is  $t : [u, v] \rightarrow \nu'(u)$ , and the multiplication is  $[u, v] \cdot [v, w] = [u, w]$ .<sup>3</sup> Hence by Lie's second theorem, (see [8], [20], or [22], for instance for more details) since  $\Gamma$ , the Lie groupoid used in the proof of Proposition 4, is source simply connected, there is a unique Lie groupoid morphism  $\Phi : \Gamma \rightarrow \mathcal{G}$  which restricts to the identity map on the Lie algebroid  $A$ .

It follows that  $\Phi|_P : P \rightarrow P'$  is a covering map,<sup>4</sup> and  $\Phi|_G : G \rightarrow G'$  is a covering morphism of Lie groups.<sup>5</sup> It is easy to see that  $H = \ker(\Phi|_G) \simeq \pi(P')$ , and  $P' = P/H$ .

Thus, we may take the quotient of the  $TP$   $W$ -Courant algebroid on  $P$  (constructed in Proposition 4) by  $H$ , to form a  $TP'$   $W/H$ -Courant algebroid on  $P'$  whose quotient by  $G'$  is  $\mathbb{A}$ . It is important to note that while  $W$  is a trivial vector bundle,  $W/H$  is a flat vector bundle.

**Corollary 1.** *Suppose that  $V$  is an  $A$ -module, and  $M$  is contractible, then  $\mathbb{A}$  is the quotient of a  $TP$   $\mathbb{R}^k$ -Courant algebroid  $\mathbb{R}^k \otimes T^*P \rightarrow \mathbb{T} \rightarrow TP$  on some principal  $G$ -bundle,  $P$ . (See Example 4). Furthermore, if  $L \subset \mathbb{A}$  is an  $AV$ -Dirac structure, then it is also the quotient of a  $TP$   $\mathbb{R}^k$ -Dirac structure  $\tilde{L} \subset \mathbb{T}$ .*

*Proof.* Every transitive Lie algebroid is integrable over a contractible space, see [19] for details.  $\square$

<sup>3</sup>An element of  $\mathcal{G}$  is an equivalence class, which we may view as a subset of  $\nu'^{-1}(y) \times \nu'^{-1}(z)$  which is  $G$  invariant. As such, we may view it as the graph of an equivariant diffeomorphism  $\nu'^{-1}(y) \rightarrow \nu'^{-1}(z)$ . The multiplication in  $\mathcal{G}$  is simply the composition of these diffeomorphisms. See [9] for details

<sup>4</sup>Here we use the identifications  $P = \Gamma_x$  and  $P' = \mathcal{G}_x$ . It is a covering map since the right invariant vector fields, which are identified with the sections of  $A$ , span the tangent space of the source fibres.

<sup>5</sup>Here we use the identifications  $G = \Gamma_x^x$  and  $G' = \mathcal{G}_x^x$ .



**5.2. Contact Manifolds.** Iglesias and Wade show how to describe contact manifolds as  $\mathcal{E}^1(M)$ -Dirac Structures in [15]. Thus in light of Example 9, we can describe them as AV-Dirac structures. We will now describe this same construction from a more geometric perspective, similar to their description in [16]:

To simplify things, we assume that  $(M, \xi)$  is a co-oriented contact manifold, and we use the fact that there is a one-to-one correspondence between co-oriented contact manifolds and symplectic cones; namely,  $(N, \omega_N)$  is a symplectic cone, where  $N = \text{Ann}^+(\xi) \subset T^*M$ ,  $\omega_N$  is the two form induced from the standard symplectic form on  $T^*M$  and the action is scalar multiplication along the fibres.

Now  $X \rightarrow \iota_X \omega_N : TN \rightarrow T^*N$  defines an isomorphism. We let  $L \subset TN \oplus T^*N$  be the graph of this morphism; and it is easy to check that  $L$  is a maximal isotropic subbundle of  $TN \oplus T^*N$ , and since  $\omega_N$  is closed,  $L$  is a Dirac subbundle of the standard Courant algebroid on  $N$ . To summarize,  $(M, \xi)$  is a contact manifold if and only if  $L$  is the graph of an isomorphism, or simply  $L \cap TN = 0$ .

We may make the identifications  $N \simeq M \times \mathbb{R}^+$  and  $x, t \rightarrow x, \ln(t) : M \times \mathbb{R}^+ \simeq M \times \mathbb{R}$ ; consequently, as described in Example 6, the quotient of the standard Courant algebroid on  $N = M \times \mathbb{R}$  by the  $\mathbb{R}$  action (3.1) yields an  $\mathcal{E}^1(M)$  bundle on  $M$ ; or an AV-Courant algebroid where  $A = TN/\mathbb{R}$  and  $V$  is the trivial line bundle on  $M$ .

Since  $\omega_N$  is preserved by this action, it follows that its graph,  $L$ , is  $\mathbb{R}$ -invariant and defines an  $\mathcal{E}^1(M)$ -Dirac structure which we denote by  $\tilde{L}_\xi$ . It is perhaps important to note that  $\tilde{L}_\xi$  is defined intrinsically. We may conclude that:

**Proposition 5.**  *$(M, \xi)$  is a contact manifold if and only if  $\tilde{L}_\xi \cap A = 0$  (under the canonical splitting).*

## 6. CR-STRUCTURES AND COURANT ALGEBROIDS

Suppose  $M$  is a smooth manifold, let  $H \subset TM$  be a subbundle and suppose  $J \in \Gamma(\text{Hom}(H, H))$  is such that  $J^2 = -\text{id}$ . Then  $(H, J)$  is called an almost CR structure. We let  $H_{1,0} \subset \mathbb{C} \otimes H \subset \mathbb{C} \otimes TM$  denote the  $+i$ -eigenbundle of  $J$ , if  $H_{1,0}$  is involutive, then it is called a CR-structure. It is possible to describe this as a Courant algebroid:

We consider the bundle  $H^* \oplus H \simeq T^*M \oplus H / \text{Ann}(H)$ , and the bundle map  $\mathbb{J} := -J^* \oplus J \in \Gamma(\text{Hom}(H^* \oplus H, H^* \oplus H^*))$ . It is clear that  $\mathbb{J}^2 = -\text{id}$ . Let  $L = \ker(\mathbb{J} - i) \oplus \text{Ann}(H) \subset \mathbb{C} \otimes (TM \oplus T^*M)$ .

**Proposition 6.**  *$L$  is involutive under the standard Courant bracket if and only if  $J$  defines a CR structure.*

*Proof.* We notice that  $L = H_{1,0} \oplus \text{Ann}(H_{1,0})$ . Therefore  $L$  is involutive under the Courant bracket only if  $\pi(L) = H_{1,0}$  is involutive, where  $\pi : TM \oplus T^*M \rightarrow TM$  is the projection. Thus  $J$  defines a CR structure.

Conversely, suppose that  $H_{1,0}$  is involutive. Then if  $I$  is the ideal generated by  $\text{Ann}(H_{1,0})$  in  $\Gamma(\mathbb{C} \otimes \wedge T^*M)$ , then  $I$  is closed under the differential:  $dI \subset I$ .

In particular, if we restrict our attention to a local neighborhood on  $M$ , and  $\alpha_i$  is a local basis for  $\text{Ann}(H_{1,0})$  and  $\xi \in \Gamma(\text{Ann}(H_{1,0}))$ , then  $d\xi = \sum_i \beta_i \wedge \alpha_i$  for some  $\beta_i \in \Omega^1(M, \mathbb{C})$ . Thus, for any  $X \in \Gamma(H_{1,0})$ , we have,

$$\iota_X d\xi = \sum_i \beta_i(X) \alpha_i \in \Gamma(\text{Ann}(H_{1,0})),$$

and

$$\mathcal{L}_X \xi = d\iota_X \xi + \iota_X d\xi = \iota_X d\xi \in \Gamma(\text{Ann}(H_{1,0})).$$

It follows that  $L$  is involutive under the standard Courant bracket. □

In the next section we shall generalize this construction.

## 7. GENERALIZED CR STRUCTURES

Suppose that  $M$  is a manifold,  $A$  is a Lie algebroid over  $M$ ,  $V$  an  $A$ -module of rank one over  $M$ , and  $\mathbb{A}$  an  $AV$ -Courant algebroid over  $M$ . Suppose further that  $A$  has some distinguished subbundle  $H \subset A$ , and consider the bundle given by

$$\mathbb{H} = q(\pi^{-1}(H)), \text{ where } q : \pi^{-1}(H) \rightarrow \pi^{-1}(H)/j(V \otimes \text{Ann}(H)).$$

Then the pairing on  $\mathbb{A}$  restricts non-degenerately to  $\mathbb{H}$ , and we have an exact sequence

$$0 \rightarrow V \otimes H^* \xrightarrow{j} \mathbb{H} \xrightarrow{\pi} H \rightarrow 0.$$

**Definition 3.**  $\mathbb{J} \in \Gamma(\text{Hom}(\mathbb{H}, \mathbb{H}))$  is called a generalized CR structure if:

- (1)  $\mathbb{J}$  is orthogonal (preserves the pairing on  $\mathbb{H}$ )
- (2)  $\mathbb{J}^2 = -1$
- (3)  $L := q^{-1}(\ker(\mathbb{J} - i)) \subset \mathbb{C} \otimes \mathbb{A}$  is involutive.

*Remark 7.*  $L := q^{-1}(\ker(\mathbb{J} - i)) \subset \mathbb{C} \otimes \mathbb{A}$  is a maximal isotropic subspace of  $\mathbb{A}$  since  $\ker(\mathbb{J} - i)$  is a maximal isotropic subspace of  $\mathbb{H}$ . In particular, since we assume that  $L$  is involutive, it is an  $AV$ -Dirac structure.

*Remark 8.* Here we have relaxed the requirement  $L \cap \bar{L} = 0$  in the definition of a generalized complex structure. While we have allowed  $L \cap \bar{L}$  to be non-trivial, it must lie in  $j(V \otimes \text{Ann}(H)) \subset V \otimes A^*$ . As pointed out in Remark 1, this can be interpreted as saying that  $L \cap \bar{L}$  only fails to be trivial up to an ‘infinitesimal’. On the other hand, we still require  $L$  to be an  $AV$ -Dirac structure.

This is in contrast to the approach taken by generalized CRF structures, introduced by Izu Vaisman in [26], which requires  $L \cap \bar{L} = 0$ , but does not require  $L$  to be a Dirac structure.

It is well known that one can canonically associate a Poisson structure to every generalized complex structure. The analogue for generalized CR structures is to endow  $V \otimes A^*$  with a non-trivial Lie algebroid structure, which we shall do in a canonical fashion following the corresponding argument given for generalized complex structures in [13].

We have an inclusion  $i : H \rightarrow A$ , and consequently a map  $\mathbb{J} \circ j \circ (\text{id} \otimes i^*) : V \otimes A^* \rightarrow \mathbb{H}$ , which (abusing notation), we shall simply call  $\mathbb{J}$ . We consider the family of subspaces of  $\mathbb{A}$  given by

$$D_t := e^{t\mathbb{J}}(V \otimes A^*) + V \otimes \text{Ann}(H) = q^{-1}(e^{t\mathbb{J}}(V \otimes H^*)).$$

Since  $e^{t\mathbb{J}} = \cos(t) + \sin(t)\mathbb{J} : \mathbb{H} \rightarrow \mathbb{H}$  is orthogonal, and  $j(V \otimes H^*)$  is a lagrangian subspace of  $\mathbb{H}$ , it follows that  $D_t$  is lagrangian for each  $t$ .

The following proposition is a slight generalization of a result of Gualtieri [13].

**Proposition 7.** (*Gualtieri*) *The family  $D_t$  of almost  $AV$ -Dirac structures is integrable for all  $t$ .*

*Proof.* Let  $\xi_1, \xi_2 \in \Gamma(V \otimes A^*)$ , then since  $V \otimes A^* \subset L \oplus \bar{L}$ , we may choose  $X_j \in \Gamma(L)$ , and  $Y_j \in \Gamma(\bar{L})$ , such that  $\xi_j = X_j + Y_j$ . It follows that  $\mathbb{J}\xi_j = iX_j - iY_j + V \otimes \text{Ann}(H)$ . In fact, since  $L \cap \bar{L} = V \otimes \text{Ann}(H)$ , by choosing  $X_j$  and  $Y_j$  appropriately, we may suppose that  $iX_j - iY_j$  is any given representative of  $\mathbb{J} \circ i^*(\xi_j)$  in  $\pi^{-1}(H)$ . Abusing notation will use the term  $\mathbb{J}(\xi_j)$  and our particular choice of representative  $iX_j - iY_j$  interchangeably. Then,

$$\begin{aligned} & [[\mathbb{J}\xi_1, \mathbb{J}\xi_2] - [\xi_1, \xi_2]] \\ &= [[iX_1 - iY_1, iX_2 - iY_2] - [X_1 + Y_1, X_2 + Y_2]] \\ &= -2[[X_1, X_2]] - 2[[Y_1, Y_2]] \end{aligned}$$

and

$$\begin{aligned} & [[\mathbb{J}\xi_1, \xi_2] - [\xi_1, \mathbb{J}\xi_2]] \\ &= [[iX_1 - iY_1, X_2 + Y_2] - [X_1 + Y_1, iX_2 - iY_2]] \\ &= 2i[[X_1, X_2]] - 2i[[Y_1, Y_2]] \end{aligned}$$

Thus, since  $L$  and hence  $\bar{L}$  are involutive, we have  $[\mathbb{J}\xi_1, \mathbb{J}\xi_2] - [\xi_1, \xi_2] + V \otimes \text{Ann}(H) = \mathbb{J}([\mathbb{J}\xi_1, \xi_2] - [\xi_1, \mathbb{J}\xi_2]) + V \otimes \text{Ann}(H)$

We let  $a = \cos(t)$  and  $b = \sin(t)$ , and we have,

$$\begin{aligned} & [(a + b\mathbb{J})\xi_1, (a + b\mathbb{J})\xi_2] \\ &= ab([\xi_1, \mathbb{J}\xi_2] + [\mathbb{J}\xi_1, \xi_2]) + b^2[\mathbb{J}\xi_1, \mathbb{J}\xi_2] \\ &= ab([\xi_1, \mathbb{J}\xi_2] + [\mathbb{J}\xi_1, \xi_2]) + b^2([\mathbb{J}\xi_1, \mathbb{J}\xi_2] - [\xi_1, \xi_2]) \end{aligned}$$

So modulo  $V \otimes \text{Ann}(H)$ , we see that

$$\begin{aligned} & [(a + b\mathbb{J})\xi_1, (a + b\mathbb{J})\xi_2] + V \otimes \text{Ann}(H) \\ &= b(a + b\mathbb{J})([\xi_1, \mathbb{J}\xi_2] + [\mathbb{J}\xi_1, \xi_2]) + V \otimes \text{Ann}(H) \end{aligned}$$

Since  $[\xi_1, \mathbb{J}\xi_2] + [\mathbb{J}\xi_1, \xi_2] \in V \otimes A^*$  it follows that  $(\cos(t) + \sin(t)\mathbb{J})(V \otimes A^*) + V \otimes \text{Ann}(H)$  is involutive.  $\square$

We next consider the map  $P : V \otimes A^* \rightarrow H \xrightarrow{i} A$ , which for  $\xi, \eta \in V \otimes A^*$ , is given by

$$\langle P(\xi), \eta \rangle = \left\langle \frac{\partial}{\partial t} \Big|_{t=0} e^{t\mathbb{J}}(\xi), \eta \right\rangle = \langle \mathbb{J}\xi, \eta \rangle \quad (= \langle i \circ \pi \circ \mathbb{J} \circ j \circ i^*(\xi), \eta \rangle).$$

Clearly, since  $\mathbb{J}$  is an orthogonal almost complex structure on  $\mathbb{H}$ ,  $P$  will be given by an element of  $\Gamma(V^* \otimes \wedge^2 A)$ , which we will also denote by  $P$ . Adapting a proposition given in [13], we have:

**Proposition 8.** (*Gualtieri*) *The bivector field  $P = i \circ \pi \circ \mathbb{J} \circ j \circ i^* : V \otimes A^* \rightarrow A$  defines a Lie algebroid structure on  $V \otimes A^*$ : The bracket is given by*

$$[\xi, \eta] = \iota_{P(\cdot, \xi)} d\eta - \iota_{P(\cdot, \eta)} d\xi + d(P(\xi, \eta)),$$

where  $\xi, \eta \in V \otimes A^*$ , and the anchor map by  $\xi \rightarrow a \circ P(\xi, \cdot) : V \otimes A^* \rightarrow TM$ , where  $a : A \rightarrow TM$  is the anchor map of  $A$ . Furthermore, the map  $\xi \rightarrow a \circ P(\xi, \cdot) : V \otimes A^* \rightarrow A$  is a Lie-algebroid morphism.

The proof is an adaptation of one found in [13].

*Proof.* We choose a splitting of the AV-Courant algebroid, and use the isomorphism and notation described in Proposition 1. Then if we choose  $t$  sufficiently small, the AV-Dirac structures  $D_t$  can be described as the graphs of  $\beta_t \in \Gamma(V^* \otimes \wedge^2 A)$ .

In [24], it is shown that the integrability condition of a twisted Poisson structure  $\beta$  over a 3-form background  $\gamma$ , is  $[\beta, \beta] = \wedge^3 \tilde{\beta}(\gamma)$ , where  $\tilde{\beta} : T^*M \rightarrow TM$  is given by  $\tilde{\beta}(\xi)(\eta) = \beta(\xi, \eta)$ . We would like to derive a similar equation for  $\beta_t$ , but we have not defined a bracket for sections of  $V^* \otimes \wedge^2 A$ . In order to define such a bracket, we first define a sheaf of rings over  $M$ :

We let  $\mathcal{F} := (S(V) \otimes S(V^*)) / I$ , where  $S(V)$  denotes the symmetric algebra generated by  $V$ , and  $I$  is the ideal generated by  $u \otimes f - f(u)$  for  $f \in \Gamma(V^*)$  and  $u \in \Gamma(V)$ . Since  $V$  is one dimensional, if  $t \in \Gamma(V)$  is a local basis then  $\mathcal{F}$  is locally isomorphic to  $C^\infty(M)[t, t^{-1}]$  as a ring. It is clear that it has a well defined  $\mathbb{Z}$  grading, which for a homogeneous  $v \in \mathcal{F}$ , we denote by  $\tilde{v}$

$\Gamma(S(V) \otimes S(V^*))$  is a  $\Gamma(A)$  module, where sections of  $\Gamma(A)$  act as derivations, and it is easy to check that  $\Gamma(I)$  is a sub-module. Thus it is clear that  $\Gamma(A)$  acts on  $\Gamma(\mathcal{F})$  by derivations satisfying the Leibniz rule with respect to the ring structure on  $\mathcal{F}$ .

We define a bracket on  $\mathcal{F} \otimes \wedge^* A$ , as follows (for  $v, w \in \Gamma(\mathcal{F})$  and  $P, Q \in \Gamma(\wedge^* A)$ ):

- $[X, v] = Xv$  for any  $X \in \Gamma(A)$ , and  $[v, w] = 0$
- $[P \wedge Q, v] = P \wedge [Q, v] + (-1)^{|Q|} [P, v] \wedge Q$
- $[P, Q]$  is given by the Schouten-Nijenhuis bracket.
- $[vP, wQ] = (v[P, w])Q - (-1)^{(|P|-1)(|Q|-1)} (w[Q, v])P + vw[P, Q]$ .

If we write  $|vP| = i$  for  $P \in \wedge^i A$ , and  $\deg(vP) = (\tilde{v}, |vP|)$ , then it is clear that our bracket satisfies the following identities (for homogeneous  $a, b, c \in \Gamma(\mathcal{F} \otimes \wedge^* A)$ ):

- $\deg(ab) = \deg(a) + \deg(b)$  and  $\deg([a, b]) = \deg(a) + \deg(b) - (0, 1)$
- $(ab)c = a(bc)$  and  $ab = (-1)^{|a||b|}ba$
- $[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|}b[a, c]$
- $[a, b] = -(-1)^{(|a|-1)(|b|-1)}[b, a]$
- $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)}[b, [a, c]]$

We next extend  $d$  to a map  $d : \mathcal{F} \otimes \wedge^i A^* \rightarrow \mathcal{F} \otimes \wedge^{i+1} A^*$  in the obvious way. We also have a natural  $\mathcal{F}$ -bilinear pairing on  $\Gamma(\mathcal{F} \otimes \wedge^* A^*) \times \Gamma(\mathcal{F} \otimes \wedge^* A)$ , which for  $v_i, w_j \in \mathcal{F}$ ,  $\alpha_i \in \Gamma(A^*)$ , and  $X_j \in \Gamma(A)$ , is given by

$$\langle (v_1 \otimes \alpha_1) \cdots (v_p \otimes \alpha_p), (w_1 \otimes X_1) \cdots (w_q \otimes X_q) \rangle = \begin{cases} 0 & \text{if } p \neq q \\ \det(v_i w_j \otimes \alpha_i(X_j)) & \text{if } p = q \end{cases}.$$

We define a morphism  $\iota : \mathcal{F} \otimes \wedge^* A \rightarrow \text{End}(\mathcal{F} \otimes \wedge^* A^*)$  by  $\langle \xi, PQ \rangle = \langle \iota_P \xi, Q \rangle$ . For  $P \in \mathcal{F} \otimes A$ ,  $\iota_P$  is a derivation.

We also define a morphism  $\check{\iota} : \mathcal{F} \otimes \wedge^* A^* \rightarrow \text{End}(\mathcal{F} \otimes \wedge^* A)$  by  $\langle \xi \eta, P \rangle = \langle \xi, \check{\iota}(\eta)P \rangle$ . For  $\alpha \in \mathcal{F} \otimes A^*$ ,  $\check{\iota}(\alpha)$  is a derivation on the right. Namely,  $\check{\iota}(\alpha)(PQ) = P\check{\iota}(\alpha)Q + (-1)^{|\alpha|}(\check{\iota}(\alpha)P)Q$  (where  $P, Q \in \mathcal{F} \otimes \wedge^* A$  are homogeneous).

Next, we notice that  $\iota_{[P, Q]} = -[[\iota_Q, d], \iota_P]$ . This is easy to check, following the argument given in [21]. Also following an argument in [21] one can verify that, for  $\eta \in \Gamma(\mathcal{F} \otimes A^*)$ ,

$$\check{\iota}(\eta)[P, Q] - [P, \check{\iota}(\eta)Q] - (-1)^{|\alpha|-1}[\check{\iota}(\eta)P, Q] = (-1)^{|\alpha|-2}(\check{\iota}(d\eta)(PQ) - P\check{\iota}(d\eta)Q - (\check{\iota}(d\eta)P)Q).$$

From this, we calculate, for any  $\beta \in \Gamma(\mathcal{F} \otimes \wedge^2 A)$  and  $\xi, \eta \in \Gamma(\mathcal{F} \otimes A^*)$ ,

$$[\check{\iota}(\xi)\beta, \check{\iota}(\eta)\beta] = \frac{1}{2}\check{\iota}(\xi\eta)[\beta, \beta] + [\beta, \langle \eta\xi, \beta \rangle] + \frac{1}{2}(\check{\iota}(\eta d\xi)\beta^2 - \check{\iota}(\xi d\eta)\beta^2) - \langle d\xi, \beta \rangle \check{\iota}(\eta)\beta + \langle d\eta, \beta \rangle \check{\iota}(\xi)\beta.$$

Furthermore, it is not difficult to verify that  $[\beta, \langle \eta\xi, \beta \rangle] = \check{\iota}(d\beta(\eta, \xi))\beta$ , while

$$\frac{1}{2}(\check{\iota}(\eta d\xi)\beta^2 - \check{\iota}(\xi d\eta)\beta^2) - \langle d\xi, \beta \rangle \check{\iota}(\eta)\beta + \langle d\eta, \beta \rangle \check{\iota}(\xi)\beta = \check{\iota}(\iota_{\check{\iota}(\xi)\beta} d\eta - \iota_{\check{\iota}(\eta)\beta} d\xi)\beta.$$

Thus, we have, for  $\beta \in \Gamma(V^* \otimes \wedge^2 A)$ ,

$$\begin{aligned} & \llbracket -\check{\iota}(\xi)\beta + \xi, -\check{\iota}(\eta)\beta + \eta \rrbracket_\phi \\ &= [\check{\iota}(\xi)\beta, \check{\iota}(\eta)\beta] - \iota_{\check{\iota}(\xi)\beta} d\eta + \iota_{\check{\iota}(\eta)\beta} d\xi + d(\beta(\xi, \eta)) + \iota_{\check{\iota}(\xi)\beta} \iota_{\check{\iota}(\eta)\beta} H \\ &= \check{\iota}(\iota_{\check{\iota}(\xi)\beta} d\eta - \iota_{\check{\iota}(\eta)\beta} d\xi - d(\beta(\xi, \eta)))\beta - \iota_{\check{\iota}(\xi)\beta} d\eta + \iota_{\check{\iota}(\eta)\beta} d\xi + d(\beta(\xi, \eta)) \\ &+ \frac{1}{2}\check{\iota}(\xi\eta)[\beta, \beta] + \iota_{\check{\iota}(\xi)\beta} \iota_{\check{\iota}(\eta)\beta} H. \end{aligned}$$

It follows that  $\beta_t$  defines an  $AV$ -Dirac structure under our chosen splitting if and only if  $\frac{1}{2}\check{\iota}(\eta\xi)[\beta_t, \beta_t] = \check{\iota}(\iota_{\check{\iota}(\xi)\beta_t} \iota_{\check{\iota}(\eta)\beta_t} H)\beta_t$ . To rewrite this, we let  $\tilde{\beta} : \mathcal{F} \otimes A^* \rightarrow \mathcal{F} \otimes A$  be the map  $\alpha \rightarrow -\check{\iota}(\alpha)\beta$ . The condition is then

$$[\beta_t, \beta_t] = 2 \wedge^3 \tilde{\beta}_t(H).$$

We differentiate both sides by  $t$ , and evaluate at 0. Since we have  $P = \frac{\partial}{\partial t}|_0 \beta_t$  and  $\beta_0 = 0$ , the cubic term vanishes, and we see that the condition is

$$[P, P] = 0.$$

The result follows immediately from this. □

We also have a bracket  $\{, \}$  on  $\Gamma(V)$ , which for  $v, w \in \Gamma(V)$  is given by  $\{v, w\} = P(dv, dw)$ .

It satisfies the following properties (for  $f \in C^\infty(M)$ ):

- $\{\cdot, \cdot\}$  is bilinear.
- $\{v, w\} = -\{w, v\}$
- $\{v, fw\} = f\{v, w\} + (a \circ P(dv)(f))w$
- $\{u, \{v, w\}\} = \{\{u, v\}, w\} + \{v, \{u, w\}\}$  (for any  $u, v, w \in \Gamma(V)$ )

Since  $V$  is a line-bundle, this is quite similar to a Poisson structure. In particular, if  $U \subset M$  is an open set on which  $\sigma \in \Gamma(V|_U)$  is a local basis such that  $P(\sigma) = 0$ , then we have a morphism

$$\rho : C^\infty(U) \xrightarrow{f \mapsto f\sigma} \Gamma(V|_U),$$

which allows us to define a Poisson structure on  $U$ , by

$$\{f, g\} = \rho^{-1}\{\rho(f), \rho(g)\}.$$

In particular, if in some neighborhood  $U \subset M$ ,  $V$  admits a non-zero  $A$ -parallel section  $\sigma \in \Gamma(V|_U)$ , then  $P(\sigma) = 0$ , and thus  $U$  is endowed with a Poisson structure. In fact, the Poisson structure associated to  $U$  in this way is unique up to a constant multiple. Furthermore, if it exists at one point on a leaf of  $A$ , then it exists for any neighborhood of any point in that leaf:

*Remark 9* (Poisson Structure on a Leaf of  $A$ ). Suppose that  $F \subset M$  is a connected leaf of the foliation given by  $A$ , then  $a : A|_F \rightarrow TF$  is a Lie algebroid, and we have an exact sequence of Lie algebroids given by  $0 \rightarrow L = \ker(a) \rightarrow A|_F \rightarrow TF \rightarrow 0$ , where  $L$  is actually a bundle of Lie algebras. The following are equivalent:

- $V$  admits an  $A|_F$ -parallel section for any neighborhood  $U \subset F$ .
- $L$  acts trivially on  $V|_F$ .
- $L_x$  acts trivially on  $V_x$ , for some point  $x \in F$ .<sup>6</sup>

Note that, up to a constant multiple, there is a unique  $A$ -parallel section of  $V|_F$ . Thus, if  $\sigma \in \Gamma(V|_F)$  is a non-zero  $A$ -parallel section we can associate a Poisson structure to  $F$ , unique up to a constant multiple.

*Remark 10* (Jacobi Structure). If  $V$  does not admit  $A$ -parallel sections, but for some  $U \subset M$  there is a canonical choice of a local basis  $\sigma \in \Gamma(V|_U)$ , we may still consider the isomorphism

$$\rho : C^\infty(U) \xrightarrow{f \mapsto f\sigma} \Gamma(V|_U),$$

which allows us to define a bracket on  $C^\infty(U)$  by

$$[f, g]_\sigma = \rho^{-1}\{\rho(f), \rho(g)\}.$$

One notices that this bracket endows  $C^\infty(U)$  with a Lie algebra structure which is local in the sense that the linear operator

$$D_f : C^\infty(U) \xrightarrow{g \mapsto [f, g]_\sigma} C^\infty(U)$$

is local for all  $f \in C^\infty(U)$ . It is an important result (See [25],[17] or [12]) that for any local Lie algebra structure, there exists unique  $\Lambda \in \Gamma(\wedge^2 TM)$ , and  $E \in \Gamma(TM)$  with  $[\Lambda, \Lambda] = -2\Lambda \wedge E$  and  $[\Lambda, E] = 0$  such that

$$[f, g]_\sigma = \{f, g\}_\Lambda + f\mathcal{L}_X g - g\mathcal{L}_X f,$$

where  $\{f, g\}_\Lambda = \check{\nu}_{df}\check{\nu}_{dg}\Lambda$ .

The triple  $(U, \Lambda, E)$  is then called a Jacobi manifold. Note however the dependence of  $\Lambda$  and  $E$  on  $\sigma$ ; this is unlike the local Poisson structure which (if it exists) is unique up to a constant multiple.

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<sup>6</sup>This follows from the fact that for any  $x, y \in F$  there is a Lie algebroid morphism of  $A$  covering a diffeomorphism of  $M$  which takes  $x$  to  $y$ . In addition these morphisms can be assumed to come from flowing along a section of  $A$ , and hence extend to  $V$ .

*Example 11 (CR Structures).* As described in Section 6, a CR-structure on a manifold  $M$  can be described by a generalized CR structure. In this case,  $V$  can be taken to be the trivial bundle, and  $A$  can be taken to be  $TM$ . It follows from the above discussion that there is a Poisson structure  $P \in \Gamma(\wedge^2 TM)$  associated to the CR structure.

If  $L \subset \mathbb{C} \otimes TM$  is the CR-structure, and  $H = \mathbf{Re}(L \oplus \bar{L}) \subset TM$ , then  $P(T^*M) \subset H$ . So the symplectic foliation associated to  $P$  is everywhere tangent to  $H$ .

*Example 12 (Quotients of Generalized Complex Structures).* If the procedures described in Example 8 and Example 4 are applied to a generalized complex structure, then one obtains a generalized CR structure.

*Example 13 (Contact Structures and Generalized Contact Structures).* Suppose that  $M$  is a contact manifold, then there is a canonical way to associate a generalized CR structure to  $M$ . In particular, if  $N = M \times \mathbb{R}$  is its symplectization, then  $N$  admits a generalized complex structure corresponding to its symplectic structure.  $\mathbb{R}$  acts on  $N$ , and the quotient is a generalized CR structure on  $M$  (In the sense of Example 6 and Example 9)

This procedure is also described in [15] and [16], where they describe it as a generalized contact structure. In fact any generalized contact structure results from the quotient of generalized complex structure, and as such can also be described as a generalized CR structure.

Since the Lie algebroid  $A$  and the vector bundle  $V$  describe an  $\mathcal{E}^1(M)$  structure, as given in Example 3, it can be checked that  $V$  does not admit parallel sections, and thus, in general,  $P \in \Gamma(V^* \otimes \wedge^2 A)$  does not describe a Poisson structure, but rather a Jacobi structure. When the generalized contact structure is simply a contact structure, then  $P$  corresponds to a Jacobi structure describing the contact structure.

To be more explicit, we let  $M$  be a contact manifold with contact distribution  $\xi \subset TM$ , and  $N = M \times \mathbb{R}$  its symplectization, where we let  $t : M \times \mathbb{R} \rightarrow \mathbb{R}$  be the projection to the second factor, and  $\omega \in \Omega^2(N)$  denote the corresponding symplectic form. (That is,  $\omega = e^t(d\eta + dt \wedge \eta)$ , where  $\eta \in \text{Ann}(\xi)$  is nowhere vanishing.) We note that  $\mathcal{L}_{\frac{\partial}{\partial t}} \omega = \omega$ .

Since  $N$  is a symplectic manifold, we can associate to it a canonical generalized complex structure  $\mathcal{J} : TN \oplus T^*N \rightarrow TN \oplus T^*N$  on the standard Courant algebroid

$$0 \rightarrow T^*N \rightarrow TN \oplus T^*N \rightarrow TN \rightarrow 0$$

(see [13] for details).

The Poisson bivector  $\pi \in \Gamma(\wedge^2 TN)$  associated to this generalized complex structure has the property that  $\mathcal{L}_{\frac{\partial}{\partial t}} \pi = -\pi$  (since it is the Poisson bivector corresponding to  $\omega$ ). It follows that we can write  $\pi = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$  for  $E \in \Gamma(M)$ , and  $\Lambda \in \Gamma(\wedge^2 M)$ . Then  $[\pi, \pi] = 0$  implies that

$$0 = [\pi, \pi] = [e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E), e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)] = e^{-2t}[\Lambda, \Lambda] - 2e^{-2t}\Lambda \wedge E + 2e^{-2t}\frac{\partial}{\partial t} \wedge [\Lambda, E].$$

From this it follows that  $[\Lambda, \Lambda] = -2\Lambda \wedge E$  and  $[\Lambda, E] = 0$ , which are the defining conditions for a Jacobi structure  $(\Lambda, E)$  on  $M$ .

Now, we consider the  $TM \oplus \mathbb{R} - \mathbb{R}$  Courant algebroid structure on  $M$ , given by taking the quotient by the  $G = \mathbb{R}$  action on  $N = M \times \mathbb{R}$ ,

$$0 \rightarrow T^*N/G \rightarrow (TN \oplus T^*N)/G \rightarrow TN/G \rightarrow 0,$$

and the generalized CR structure on  $M$  given by quotient homomorphism

$$\mathbb{J} := \mathcal{J}/G : (TN \oplus T^*N)/G \rightarrow (TN \oplus T^*N)/G.$$

They define an  $AV$ -Courant algebroid, where  $A = TN/G$ , and the bundle  $V \rightarrow M$  is trivial, with  $\Gamma(V) \simeq C^\infty(N)^G$  (this is in fact an  $\mathcal{E}^1(M)$  structure, see [15]). Abusing notation, we denote by  $e^t \in \Gamma(V)$  the section associated to the  $G$ -invariant function  $e^t \in C^\infty(N)$ .

Then the bivector  $P \in \Gamma(V^* \otimes \wedge^2 A)$  associated to the generalized CR structure on  $M$  is simply  $e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$ , and it defines a Jacobi structure on  $M$ , with bivector field  $\Lambda$  and vector field  $E$ . Since  $\Lambda^n \wedge E \neq 0$  (where  $\dim(M) = 2n+1$ ) this Jacobi structure corresponds to a contact structure. In fact, the contact distribution is given by  $\text{span}\{\check{\iota}_\alpha \Lambda \mid \alpha \in T^*M\}$ ; and if  $\theta \in \Omega^1(M)$  satisfies  $\check{\iota}_\theta \Lambda = 0$  and  $\check{\iota}_\theta E = 1$ , then  $\theta$  is a contact form. It is not difficult to see that this is the original contact structure,  $\xi$ , defined on  $M$ . (In fact, if  $\omega = e^t(d\eta + dt \wedge \eta)$  is the symplectic form on  $N$  (where  $\eta \in \text{Ann}(\xi)$  is nowhere vanishing), then  $E$  is a reeb vector field for  $\eta$  and  $\theta = \eta$ .)

We must note that, if instead of trivializing  $V$  by the section  $e^t \in \Gamma(V)$ , we made the transformation  $e^t \rightarrow fe^t$ , for some nowhere vanishing  $f \in C^\infty(M)$ , then the appropriate changes to the Jacobi structure would be  $\Lambda \rightarrow f\Lambda$ ,  $E \rightarrow fE - \check{\iota}_{df}\Lambda$ , and the transformation for the contact form would be  $\theta \rightarrow \frac{1}{f}\theta$ . Thus it is clear that the freedom to modify the trivializing section of  $V$  by a scalar multiple does not change the contact distribution, and fully accounts for the freedom to change the contact form by a scalar multiple. Indeed the generalized CR structure is defined intrinsically.

## 8. APPENDIX: PROOF OF PROPOSITION 1

Suppose that  $M$  is a manifold,  $A$  is a Lie algebroid over  $M$ ,  $V$  is an  $A$ -module over  $M$ , and  $\mathbb{A}$  is an AV-Courant algebroid over  $M$ .

For  $X, Y \in \Gamma(A)$ , we have the following identities:

- $[\iota_X, \iota_Y] = 0$
- $[d, \iota_X] = \mathcal{L}_X$
- $[\mathcal{L}_X, \iota_Y] = \iota_{[X, Y]}$
- $[d, d] = 0$
- $[\mathcal{L}_X, d] = 0$
- $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]}$

We will provide the proof we promised for Proposition 1, which we restate here:

**Proposition 9.** *Let  $\phi : A \rightarrow \mathbb{A}$  be an isotropic splitting. Then under the isomorphism  $\phi \oplus j : A \oplus (V \otimes A^*) \rightarrow \mathbb{A}$ , the bracket is given by*

$$\llbracket X + \xi, Y + \eta \rrbracket_\phi = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_X \iota_Y H_\phi,$$

where  $X, Y \in \Gamma(A)$ ,  $\xi, \eta \in \Gamma(V \otimes A^*)$ , and  $H_\phi \in \Gamma(V \otimes \wedge^3 A^*)$ , with  $dH_\phi = 0$ .

Furthermore, if  $\psi : A \rightarrow \mathbb{A}$  is a different choice of isotropic splitting, then  $\psi(X) = \phi(X) + j(\iota_X \beta)$ , and  $H_\psi = H_\phi - d\beta$ , where  $\beta \in \Gamma(V \otimes \wedge^2 A^*)$ .

*Proof.* The proof will follow immediately from the following lemmas: □

**Lemma 1.** *If  $\xi \in \Gamma(V \otimes A^*)$ , and  $e \in \Gamma(\mathbb{A})$ , then  $\llbracket e, j(\xi) \rrbracket = j(\mathcal{L}_{\pi(e)} \xi)$*

*Proof.* Let  $e_1, e_2 \in \Gamma(\mathbb{A})$ ,  $\xi \in \Gamma(V \otimes A^*)$ .

$$\begin{aligned} \langle \llbracket e_1, j(\xi) \rrbracket, e_2 \rangle &= \mathcal{L}_{\pi(e_1)} \langle j(\xi), e_2 \rangle - \langle j(\xi), \llbracket e_1, e_2 \rrbracket \rangle \\ &= \mathcal{L}_{\pi(e_1)} \iota_{\pi(e_2)} \xi - \iota_{\pi([e_1, e_2])} \xi \\ &= \mathcal{L}_{\pi(e_1)} \iota_{\pi(e_2)} \xi - \iota_{[\pi(e_1), \pi(e_2)]} \xi \\ &= \mathcal{L}_{\pi(e_1)} \iota_{\pi(e_2)} \xi - [\mathcal{L}_{\pi(e_1)}, \iota_{\pi(e_2)}] \xi \\ &= \iota_{\pi(e_2)} \mathcal{L}_{\pi(e_1)} \xi \\ &= \langle j(\mathcal{L}_{\pi(e_1)} \xi), e_2 \rangle \end{aligned}$$

□

**Lemma 2.** *If  $\xi \in \Gamma(V \otimes A^*)$ , and  $e \in \Gamma(\mathbb{A})$ , then  $\llbracket j(\xi), e \rrbracket = -j(\iota_{\pi(e)} d\xi)$ .*

*Proof.*

$$\begin{aligned}
\llbracket j(\xi), e \rrbracket &= D\langle j(\xi), e \rangle - \llbracket e, j(\xi) \rrbracket \\
&= j(d\iota_{\pi(e)}\xi) - j(\mathcal{L}_{\pi(e)}\xi) \\
&= j(d\iota_{\pi(e)}\xi - (\iota_{\pi(e)}d\xi + d\iota_{\pi(e)}\xi)) \\
&= -j(\iota_{\pi(e)}d\xi)
\end{aligned}$$

□

**Lemma 3.** *If  $\phi : A \rightarrow \mathbb{A}$  is an isotropic splitting, and if  $X, Y \in \Gamma(A)$  then*

$$\llbracket \phi(X), \phi(Y) \rrbracket - \phi(\llbracket X, Y \rrbracket) = j(\iota_X \iota_Y H),$$

where  $H \in \Gamma(V \otimes \wedge^3 A^*)$ .

*Proof.* Let  $\phi$  be an isotropic splitting, and  $X, Y, Z \in \Gamma(A)$ . Then

$$\pi(\llbracket \phi(X), \phi(Y) \rrbracket - \phi(\llbracket X, Y \rrbracket)) = 0,$$

so by exactness of the sequence (2.2),  $\llbracket \phi(X), \phi(Y) \rrbracket - \phi(\llbracket X, Y \rrbracket) \in j(\Gamma(V \otimes A^*))$ . We define  $H$  by

$$\begin{aligned}
H(X, Y, Z) &= \langle \phi(Z), \llbracket \phi(X), \phi(Y) \rrbracket - \phi(\llbracket X, Y \rrbracket) \rangle \\
&= \langle \phi(Z), \llbracket \phi(X), \phi(Y) \rrbracket \rangle,
\end{aligned}$$

where the second equality follows since  $\phi$  is an isotropic splitting. It is obvious that  $H$  is tensorial in  $Z$ . Furthermore, making repeated use of the fact that  $\phi$  is an isotropic splitting, we check that  $H$  is skew-symmetric:

$$\begin{aligned}
\langle \phi(Z), \llbracket \phi(X), \phi(Y) \rrbracket \rangle &= \langle \phi(Z), -\llbracket \phi(Y), \phi(X) \rrbracket + D\langle \phi(X), \phi(Y) \rangle \rangle \\
&= -\langle \phi(Z), \llbracket \phi(Y), \phi(X) \rrbracket \rangle
\end{aligned}$$

and

$$\begin{aligned}
0 &= \mathcal{L}_X \langle \phi(Z), \phi(Y) \rangle \\
&= \langle \llbracket \phi(X), \phi(Z) \rrbracket, \phi(Y) \rangle + \langle \phi(Z), \llbracket \phi(X), \phi(Y) \rrbracket \rangle
\end{aligned}$$

It follows that  $H \in \Gamma(V \otimes \wedge^3 A^*)$ . □

**Lemma 4.** *Using the notation of the previous lemmas,  $dH = 0$ .*

*Proof.* Using the fact that  $[\mathcal{L}_X, \iota_Y] = \iota_{[X, Y]}$ , it is easy to show that

$$d\iota_Z \iota_Y \iota_X + \iota_Z \iota_Y \iota_X d = \mathcal{L}_Z \iota_Y \iota_X + \mathcal{L}_Y \iota_X \iota_Z + \mathcal{L}_X \iota_Z \iota_Y + \iota_Z \iota_{[Y, X]} + \iota_Y \iota_{[X, Z]} + \iota_X \iota_{[Z, Y]}.$$

Let  $\phi : A \rightarrow \mathbb{A}$  be an isotropic splitting. We shall use the identification

$$A \oplus (V \otimes A^*) \xrightarrow{\phi \oplus j} \mathbb{A}$$

explicitly throughout this section. We have, for  $X, Y, Z \in \Gamma(A)$ ,

$$\llbracket X, Y \rrbracket_\phi = [X, Y] + \iota_X \iota_Y H.$$

Then using Axiom (AV-1) from the definition of an AV-Courant algebroid, we see that

$$\begin{aligned}
0 &= \llbracket Z, \llbracket Y, X \rrbracket_\phi \rrbracket_\phi - \llbracket \llbracket Z, Y \rrbracket_\phi, X \rrbracket_\phi - \llbracket Y, \llbracket Z, X \rrbracket_\phi \rrbracket_\phi \\
&= \llbracket Z, [Y, X] + \iota_Y \iota_X H \rrbracket_\phi - \llbracket [Z, Y] + \iota_Z \iota_Y H, X \rrbracket_\phi - \llbracket Y, [Z, X] + \iota_Z \iota_X H \rrbracket_\phi \\
&= \llbracket Z, [Y, X] \rrbracket_\phi + \mathcal{L}_Z \iota_Y \iota_X H - \llbracket [Z, Y], X \rrbracket_\phi + \iota_X d\iota_Z \iota_Y H - \llbracket Y, [Z, X] \rrbracket_\phi - \mathcal{L}_Y \iota_Z \iota_X H \\
&= \llbracket Z, [Y, X] \rrbracket_\phi - \llbracket [Z, Y], X \rrbracket_\phi - \llbracket Y, [Z, X] \rrbracket_\phi \\
&\quad + \mathcal{L}_Z \iota_Y \iota_X H + \mathcal{L}_X \iota_Z \iota_Y H + \mathcal{L}_Y \iota_X \iota_Z H - d\iota_Z \iota_Y \iota_X H \\
&= [Z, [Y, X]] + \iota_Z \iota_{[Y, X]} H - \llbracket [Z, Y], X \rrbracket - \iota_{[Z, Y]} \iota_X H - [Y, [Z, X]] - \iota_Y \iota_{[Z, X]} H \\
&\quad + \mathcal{L}_Z \iota_Y \iota_X H + \mathcal{L}_X \iota_Z \iota_Y H + \mathcal{L}_Y \iota_X \iota_Z H - d\iota_Z \iota_Y \iota_X H \\
&= [Z, [Y, X]] - \llbracket [Z, Y], X \rrbracket - [Y, [Z, X]] + \iota_Z \iota_{[Y, X]} H + \iota_X \iota_{[Z, Y]} H + \iota_Y \iota_{[X, Z]} H \\
&\quad + \mathcal{L}_Z \iota_Y \iota_X H + \mathcal{L}_X \iota_Z \iota_Y H + \mathcal{L}_Y \iota_X \iota_Z H - d\iota_Z \iota_Y \iota_X H \\
&= \iota_Z \iota_Y \iota_X dH.
\end{aligned}$$



□

**Lemma 5.** *Let  $\phi : A \rightarrow \mathbb{A}$  and  $\psi : A \rightarrow \mathbb{A}$  be two isotropic splittings, and let  $H_\phi$  and  $H_\psi$  be the elements of  $\Gamma(V \otimes \wedge^3 A^*)$  associated to the corresponding splittings. Namely, if  $X, Y \in \Gamma(A)$ , then  $\llbracket \phi(X), \phi(Y) \rrbracket = \phi([X, Y]) + j\iota_X \iota_Y H_\phi$ , and similarly for  $H_\psi$ .*

*Then there exists  $\beta \in \Gamma(V \otimes \wedge^2 A^*)$  such that  $\psi(X) = \phi(X) + j(\iota_X \beta)$  and  $H_\psi = H_\phi - d\beta$ .*

*Proof.* Since  $\phi$  and  $\psi$  are splittings, we see that

$$\pi((\phi - \psi)(X)) = 0.$$

Thus, by the exactness of the sequence (2.2),  $(\phi - \psi)(X) = j \circ S(X)$  for some linear map  $S : A \rightarrow V \otimes A^*$ .

However since the splittings are isotropic,

$$\begin{aligned} 0 &= \langle \phi(X), \phi(Y) \rangle \\ &= \langle \psi(X) + j \circ S(X), \psi(Y) + j \circ S(Y) \rangle \\ &= S(X)(Y) + S(Y)(X), \end{aligned}$$

so we can define  $\beta \in \Gamma(V \otimes \wedge^2 A^*)$  by  $\iota_X \beta = S(X)$ . Then, we see that

$$\begin{aligned} \psi([X, Y]) + \iota_X \iota_Y H_\psi &= \llbracket \phi(X) + j(\iota_X \beta), \phi(Y) + j(\iota_Y \beta) \rrbracket \\ &= \phi([X, Y]) + j(\mathcal{L}_X \iota_Y \beta - \iota_Y d_X \beta + \iota_X \iota_Y H_\phi) \\ &= \phi([X, Y]) + j(\iota_X \iota_Y H_\phi) + j(\mathcal{L}_X \iota_Y \beta - \iota_Y \mathcal{L}_X \beta + \iota_Y \iota_X d\beta) \\ &= \phi([X, Y]) + j(\iota_X \iota_Y H_\phi) + j(\iota_{[X, Y]} \beta + \iota_Y \iota_X d\beta) \\ &= \phi([X, Y]) + j(\iota_{[X, Y]} \beta) + j(\iota_X \iota_Y H_\phi - \iota_X \iota_Y d\beta) \\ &= \psi([X, Y]) + j(\iota_X \iota_Y H_\phi - \iota_X \iota_Y d\beta), \end{aligned}$$

so we have

$$H_\psi = H_\phi - d\beta.$$

□

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