

A note on the improved (G'/G) – expansion method

M. S. Abdel Latif

*Mathematics and Engineering Physics Dept., Faculty of Engineering,
Mansoura University, Mansoura, Egypt.*

Abstract

In this paper, we show that the improved (G'/G) – expansion method is equivalent to the tanh method and gives the same exact solutions of nonlinear partial differential equations.

Keywords: Exact solution, Improved (G'/G) – expansion method, tanh method

1. Introduction

Recently, Many methods have been proposed for obtaining exact traveling wave solutions of partial differential equations. An example of these methods is the (G'/G) – expansion method [1] which is used to obtain traveling wave solutions of many models (see for example [2, 3]). Also, many improved and extended versions of this method have been proposed to get more exact solutions of partial differential equations (see for example [4, 5, 6]).

Many papers are published to comment on the classical version of the (G'/G) – expansion method. For example, the equivalence between the (G'/G) – expansion method and the tanh method is proved in [7, 8, 9]. Moreover, in [10], it is shown that the (G'/G) – expansion method is a specific form of the simplest equation method [11].

The improved (G'/G) – expansion method [5] is used to obtain new exact solutions of some models [12, 13]. In this paper, we show that this improved (G'/G) – expansion method is equivalent to the tanh method and doesn't give any new exact solutions of nonlinear partial differential equations.

2. The tanh method [8]

In this section, we give the detailed description of the tanh method. Suppose that a nonlinear evolution equation (NLEE) with independent variable u and two independent variables x and t is given by

$$H(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0, \quad (1)$$

where, H is a polynomial in $u(x, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. To determine u explicitly, one can follow the following steps:

Step 1: Use the traveling wave transformation:

$$u = u(\xi), \quad \xi = x - \nu t, \quad (2)$$

where, ν is a constant to be determined latter. Then, the NLEE (1) is reduced to a nonlinear ordinary differential equation (NLODE) for $u = u(\xi)$:

$$H(u, u', u'', u''', \dots) = 0. \quad (3)$$

step 2: Suppose that the NLODE (3) has the following solution:

$$u = \sum_{i=0}^n b_i (\tanh(k(\xi - \xi_0)))^i, \quad (4)$$

where, $k, b_i (i = 0, \dots, n)$ are constants to be determined later, ξ_0 is an arbitrary constant and n is a positive integer to be determined in step 3.

Step 3: Determine the positive integer n by balancing the highest order derivatives and nonlinear terms in Eq. (3).

Step 4: Substituting Eq. (4) into Eq. (3) and equating expressions of different power of $(\tanh(k(\xi - \xi_0)))^i$ to zero, we obtain coefficients b_i and the parameter k .

Step 5: Substituting b_i and k into Eq. (4), we can obtain the explicit solutions of Eq. (1) immediately.

3. The improved (G'/G) - expansion method [5]

In this section, we give the detailed description of the improved (G'/G) -expansion method. To determine u in Eq. (1) explicitly using the improved (G'/G) - expansion method, one can follow the following five steps:

Step 1: Use the traveling wave transformation (2) to reduce the NLEE (1) to the NLODE (3)

Step 2: Suppose that the NLODE (3) has the following solution:

$$u = \sum_{i=-n}^n \frac{a_i (G'/G)^i}{(1 + \sigma (G'/G))^i} = \sum_{i=-n}^n a_i \left(\frac{(G'/G)}{1 + \sigma (G'/G)} \right)^i, \quad (5)$$

where, σ and $a_i (i = -n, \dots, n)$ are constants to be determined later, n is a positive integer, and $G = G(\xi)$ satisfies the following second order linear ordinary differential equation(LODE):

$$G'' + \mu G = 0, \quad (6)$$

where, μ is a real constant. The general solutions of Eq. (6) can be listed as follows. When $\mu < 0$, we obtain the hyperbolic function solution of Eq. (6)

$$G = A_1 \cosh(\sqrt{-\mu}\xi) + A_2 \sinh(\sqrt{-\mu}\xi), \quad (7)$$

where, A_1 and A_2 are arbitrary constants. When $\mu > 0$, we obtain the trigonometric function solution of Eq. (6)

$$G = A_1 \cos(\sqrt{\mu}\xi) + A_2 \sin(\sqrt{\mu}\xi), \quad (8)$$

where, A_1 and A_2 are arbitrary constants. When $\mu = 0$, we obtain the linear solution of Eq. (6)

$$G = A_1 + A_2 \xi, \quad (9)$$

where, A_1 and A_2 are arbitrary constants

Step 3: Determine the positive integer n by balancing the highest order derivatives and nonlinear terms in Eq. (3).

Step 4: Substituting (5) along with (6) into Eq. (6) and then setting all the coefficients of $(G'/G)^k$, ($k = 1, 2, 3, \dots$) of the resulting systems numerator to zero, yields a set of over-determined nonlinear algebraic equations for ν, σ and $a_i (i = -n, \dots, n)$.

Step 5: Assuming that the constants ν, σ and $a_i (i = -n, \dots, n)$ can be obtained by solving the algebraic equations in Step 4, then substituting these constants and the known general solutions of Eq. (6) into (5), we can obtain the explicit solutions of Eq. (1) immediately.

4. Equivalence of the two methods

In the second step of the improved (G'/G) - expansion method let $y = \frac{(G'/G)}{1+\sigma(G'/G)}$, Eqs. (5), (6) are transformed into

$$u = \sum_{i=-n}^n a_i y^i, \quad (10)$$

$$y' + (1 + \mu\sigma^2)y^2 - 2\sigma\mu y + \mu = 0. \quad (11)$$

The general solution of Eq. (11) is given by [8]

$$y = \alpha + \beta \tanh(\sqrt{-\mu}(\xi - \xi_0)), \quad (12)$$

where, $\alpha = \frac{\sigma\mu}{1+\mu\sigma^2}$ and $\beta = \frac{\sqrt{-\mu}}{1+\mu\sigma^2}$.

Substituting solution (12) into expansion (10) we have

$$u = \sum_{i=-n}^n a_i (\alpha + \beta \tanh(\sqrt{-\mu}(\xi - \xi_0)))^i = u_1 + u_2, \quad (13)$$

where,

$$u_1 = \sum_{i=0}^n a_i (\alpha + \beta \tanh(\sqrt{-\mu}(\xi - \xi_0)))^i = \sum_{i=0}^n b_i (\tanh(-\mu(\xi - \xi_0)))^i, \quad (14)$$

$$\begin{aligned} u_2 &= \sum_{i=-n}^{-1} a_i (\alpha + \beta \tanh(\sqrt{-\mu}(\xi - \xi_0)))^i = \\ &= \sum_{i=1}^n a_{-i} \left(\frac{1}{\alpha + \beta \tanh(\sqrt{-\mu}(\xi - \xi_0))} \right)^i = \\ &= \sum_{i=1}^n a_{-i} \left(\frac{\alpha}{\alpha^2 - \beta^2} - \frac{\alpha}{\alpha^2 - \beta^2} + \frac{1}{\alpha + \beta \tanh(\sqrt{-\mu}(\xi - \xi_0))} \right)^i = \\ &= \sum_{i=1}^n a_{-i} \left(\frac{\alpha}{\alpha^2 - \beta^2} + \frac{\beta}{\beta^2 - \alpha^2} \frac{\frac{\beta}{\alpha} + \tanh(\sqrt{-\mu}(\xi - \xi_0))}{1 + \frac{\beta}{\alpha} \tanh(\sqrt{-\mu}(\xi - \xi_0))} \right)^i = \\ &= \sum_{i=1}^n a_{-i} \left(\frac{\alpha}{\alpha^2 - \beta^2} + \frac{\beta}{\beta^2 - \alpha^2} \tanh(\sqrt{-\mu}(\xi - \xi_0) + z) \right)^i, \quad z = \tanh^{-1} \frac{\beta}{\alpha}, \end{aligned} \quad (15)$$

therefore, u_2 may be rewritten as

$$u_2 = \sum_{i=0}^n c_i (\tanh(\sqrt{-\mu}(\xi - \xi_0) + z))^i, \quad (16)$$

hence,

$$u = u_1 + u_2 = \sum_{i=0}^n b_i (\tanh(-\mu(\xi - \xi_0)))^i + \sum_{i=0}^n c_i (\tanh(\sqrt{-\mu}(\xi - \xi_0) + z))^i, \quad (17)$$

since ξ_0 is an arbitrary constant, then we can write

$$u = \sum_{i=0}^n d_i (\tanh(-\mu(\xi - \xi_1)))^i, \quad (18)$$

where, ξ_1 is an arbitrary constant. It is now clear that the two methods will give the same solutions expressed in terms of the tanh function.

5. Another proof of equivalence

In this section we will proof that The solution formula (5) will give solutions in the form of the tanh function and the rational function only.

Case 1: When $\mu < 0$, we have

$$\begin{aligned} \frac{G'}{G} &= \sqrt{-\mu} \frac{A_2 \cosh(\sqrt{-\mu}\xi) + A_1 \sinh(\sqrt{-\mu}\xi)}{A_1 \cosh(\sqrt{-\mu}\xi) + A_2 \sinh(\sqrt{-\mu}\xi)} = \sqrt{-\mu} \frac{1 + \frac{A_1}{A_2} \tanh(\sqrt{-\mu}\xi)}{\frac{A_1}{A_2} + \tanh(\sqrt{-\mu}\xi)} \\ &= \sqrt{-\mu} \coth(\sqrt{-\mu}\xi + d), \quad d = \tanh^{-1} \frac{A_1}{A_2}, \end{aligned} \quad (19)$$

$$\frac{1 + \sigma(G'/G)}{(G'/G)} = \frac{G + \sigma G'}{G'} = \frac{G}{G'} + \sigma = \frac{1}{\sqrt{-\mu}} \tanh(\sqrt{-\mu}\xi + d) + \sigma, \quad (20)$$

$$\begin{aligned} \sum_{j=-n}^{-1} a_j \left(\frac{(G'/G)}{1 + \sigma(G'/G)} \right)^j &= \sum_{j=1}^n a_{-j} \left(\frac{1 + \sigma(G'/G)}{(G'/G)} \right)^j \\ &= \sum_{j=1}^n a_{-j} \left(\frac{1}{\sqrt{-\mu}} \tanh(\sqrt{-\mu}\xi + d) + \sigma \right)^j = \sum_{j=0}^n b_j (\tanh(\sqrt{-\mu}\xi + d))^j, \end{aligned} \quad (21)$$

$$\begin{aligned}
\frac{(G'/G)}{1 + \sigma(G'/G)} &= \frac{G'}{G + \sigma G'} = \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}\xi + d) + \sigma\sqrt{-\mu}} \\
&= \left(\frac{\mu\sigma}{1 + \mu\sigma^2} - \frac{\mu\sigma}{1 + \mu\sigma^2} + \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}\xi + d) + \sigma\sqrt{-\mu}} \right) \\
&= \left(\frac{\mu\sigma}{1 + \mu\sigma^2} + \frac{\sqrt{-\mu}}{1 + \mu\sigma^2} \frac{1}{\sigma\sqrt{-\mu}} + \tanh(\sqrt{-\mu}\xi + d) \right) \\
&= \frac{\mu\sigma}{1 + \mu\sigma^2} + \frac{\sqrt{-\mu}}{1 + \mu\sigma^2} \tanh(\sqrt{-\mu}\xi + d + k), \quad k = \tanh^{-1} \frac{1}{\sigma\sqrt{\mu}}, \quad (22)
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^n a_j \left(\frac{(G'/G)}{1 + \sigma(G'/G)} \right)^j &= \sum_{j=1}^n a_j \left(\frac{\mu\sigma}{1 + \mu\sigma^2} + \frac{\sqrt{-\mu}}{1 + \mu\sigma^2} \tanh(\sqrt{-\mu}\xi + d + k) \right)^j \\
&= \sum_{j=0}^n c_j (\tanh(\sqrt{-\mu}\xi + d + k))^j, \quad (23)
\end{aligned}$$

So, in this case (when $\mu < 0$) Eq. (5) can be rewritten as

$$\begin{aligned}
u &= \sum_{j=-n}^n a_j \left(\frac{(G'/G)}{1 + \sigma(G'/G)} \right)^j = \sum_{j=0}^n b_j (\tanh(\sqrt{-\mu}\xi + d))^j + \\
&\quad \sum_{j=0}^n c_j (\tanh(\sqrt{-\mu}\xi + d + k))^j = \sum_{j=0}^n e_j (\tanh(\sqrt{-\mu}(\xi - \xi_0)))^j, \quad (24)
\end{aligned}$$

where, ξ_0 is an arbitrary constant.

Case 2: When $\mu > 0$, we have

$$\begin{aligned}
\frac{G'}{G} &= \sqrt{\mu} \frac{A_2 \cos(\sqrt{\mu}\xi) - A_1 \sin(\sqrt{\mu}\xi)}{A_1 \cos(\sqrt{\mu}\xi) + A_2 \sin(\sqrt{\mu}\xi)} = \sqrt{\mu} \frac{1 - \frac{A_1}{A_2} \tan(\sqrt{\mu}\xi)}{\frac{A_1}{A_2} + \tan(\sqrt{\mu}\xi)} \\
&= \sqrt{\mu} \cot(\sqrt{\mu}\xi + d), \quad d = \tan^{-1} \frac{A_1}{A_2}, \quad (25)
\end{aligned}$$

$$\frac{1 + \sigma(G'/G)}{(G'/G)} = \frac{G + \sigma G'}{G'} = \frac{G}{G'} + \sigma = \frac{1}{\sqrt{\mu}} \tan(\sqrt{\mu}\xi + d) + \sigma, \quad (26)$$

$$\begin{aligned}
\sum_{j=-n}^{-1} a_j \left(\frac{(G'/G)}{1 + \sigma(G'/G)} \right)^j &= \sum_{j=1}^n a_{-j} \left(\frac{1 + \sigma(G'/G)}{(G'/G)} \right)^j \\
&= \sum_{j=1}^n a_{-j} \left(\frac{1}{\sqrt{\mu}} \tan(\sqrt{\mu}\xi + d) + \sigma \right)^j = \sum_{j=0}^n b_j (\tan(\sqrt{\mu}\xi + d))^j, \quad (27)
\end{aligned}$$

$$\begin{aligned}
\frac{(G'/G)}{1 + \sigma(G'/G)} &= \frac{G'}{G + \sigma G'} = \frac{\sqrt{\mu}}{\tan(\sqrt{\mu}\xi + d) + \sigma\sqrt{\mu}} \\
&= \left(-\frac{\mu\sigma}{1 - \mu\sigma^2} + \frac{\mu\sigma}{1 - \mu\sigma^2} + \frac{\sqrt{\mu}}{\tan(\sqrt{\mu}\xi + d) + \sigma\sqrt{\mu}} \right) \\
&= \left(-\frac{\mu\sigma}{1 - \mu\sigma^2} + \frac{\sqrt{\mu}}{1 - \mu\sigma^2} \frac{\frac{1}{\sigma\sqrt{\mu}} + \tan(\sqrt{\mu}\xi + d)}{1 + \frac{1}{\sigma\sqrt{\mu}} \tan(\sqrt{\mu}\xi + d)} \right) \\
&= -\frac{\mu\sigma}{1 + \mu\sigma^2} + \frac{\sqrt{\mu}}{1 - \mu\sigma^2} \tan(\sqrt{\mu}\xi + d - k), \quad k = \tan^{-1} \frac{1}{\sigma\sqrt{\mu}}, \quad (28)
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^n a_j \left(\frac{(G'/G)}{1 + \sigma(G'/G)} \right)^j &= \sum_{j=1}^n a_j \left(-\frac{\mu\sigma}{1 - \mu\sigma^2} + \frac{\sqrt{\mu}}{1 - \mu\sigma^2} \tan(\sqrt{\mu}\xi + d - k) \right)^j \\
&= \sum_{j=0}^n c_j (\tan(\sqrt{\mu}\xi + d - k))^j, \quad (29)
\end{aligned}$$

So, in this case (when $\mu > 0$) Eq. (5) can be rewritten as

$$\begin{aligned}
u &= \sum_{j=-n}^n a_j \left(\frac{(G'/G)}{1 + \sigma(G'/G)} \right)^j = \sum_{j=0}^n b_j (\tan(\sqrt{\mu}\xi + d))^j \\
&\quad + \sum_{j=0}^n c_j (\tan(\sqrt{\mu}\xi + d - k))^j = \sum_{j=0}^n e_j (\tan(\sqrt{\mu}(\xi - \xi_0)))^j, \quad (30)
\end{aligned}$$

where, ξ_0 is an arbitrary constant. By considering the formula [14]

$$\tan(i\alpha) = i \tanh(\alpha), \quad i = \sqrt{-1}, \quad (31)$$

Eq. (30) can be reformulated as

$$\begin{aligned}
 u &= \sum_{j=0}^n e_j (\tan(\sqrt{\mu}(\xi - \xi_0)))^j = \sum_{j=0}^n e_j (-\tan(i\sqrt{-\mu}(\xi - \xi_0)))^j = \\
 &\sum_{j=0}^n e_j (-i \tanh(\sqrt{-\mu}(\xi - \xi_0)))^j = \sum_{j=0}^n f_j (\tanh(\sqrt{-\mu}(\xi - \xi_0)))^j, \quad \mu < 0,
 \end{aligned} \tag{32}$$

which is equivalent to the solution (24) in case 1.

Case 3. When $\mu = 0$, in this case we will simply obtain the rational solution.

6. Conclusion

It is shown that the improved (G'/G) - expansion method is equivalent to the tanh method and doesn't give any new exact solutions of nonlinear partial differential equations.

References

- [1] M. L. Wang, X. Z. Li, and J. L. Zhang. The (G'/G) - expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A* 372 (2008) 417-423.
- [2] I. Aslan, T. Ozis, Analytic study on two nonlinear evolution equations by using the (G'/G) - expansion method. *Appl. Math. Comput.* 209 (2009) 425-429.
- [3] M. K. Elboree. Hyperbolic and trigonometric solutions for some nonlinear evolution equations. *Commun. Nonlinear Sci. Numer. Simulat.* 17 (2012) 4085-4096.
- [4] S. Guo and Y. Zhou . The extende (G'/G) - expansion method and its applications to the Witham-BroerKaup-Like equations and coupled Hirota-Satsuma KdV equations. *Appl. Math. Comput.* 215 (2010) 3214-3221.
- [5] S. Guo, Y. Zhou and C. Zhao. The improved (G'/G) - expansion method and its applications to the BroerKaup equations and approximate long water wave equations. *Appl. Math. Comput.* 216 (2010) 1965-1971.

- [6] M. N. Alama, M. A. Akbarb and S. T. Mohyud-Din. A novel (G'/G) – expansion method and its application to the Boussinesq equation. Chin. Phys. B. 23 No. 2 (2014) 020203.
- [7] L. C. Ping. (G'/G) – expansion method equivalent to extended tanh function method. Commun. Theor. Phys. 51 (2009) 985-988.
- [8] N. A. Kudryashov. A note on the (G'/G) – expansion method. Appl. Math. Comput. 217 (2010) 1755-1758.
- [9] E. J. Parkes. Observations on the basic (G'/G) – expansion method for finding solutions to nonlinear evolution equations. Appl. Math. Comput. 217 (2010) 1759-1763.
- [10] I. Aslan. A note on the (G'/G) – expansion method again. Appl. Math. Comput. 217 (2010) 937-938.
- [11] N. A. Kudryashov. Simplest equation method to look for exact solutions of nonlinear differential equations. Chaos Solitons Fractals. 24 (2005) 1217-1231.
- [12] Y. He, S. Li and Y. Long. Exact solutions to the sharma-tasso-olver equation by using improved (G'/G) - expansion method. J. Appl. Math. (2013) 247234.
- [13] Y. He. New exact solutions for a higher order wave equation of kdv type using multiple (G'/G) - expansion methods. Adv. in Math. Phys. (2014) 148132.
- [14] N.A. Kudryashov and D. I. Sinelshchikov. A note on Abundant new exact solutions for the $(3+1)$ -dimensional JimboMiwa equation. J. Math. Anal. Appl. 24 (2005) 1217-1231.