A PRESENTATION FOR THE AUTOMORPHISMS OF THE 3-SPHERE THAT PRESERVE A GENUS TWO HEEGAARD SPLITTING

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ABSTRACT. *Scharlemann constructed a connected simplicial 2-complex Γ with an action by the group $\mathcal{H}_2$ of isotopy classes of orientation preserving homeomorphisms of S^3 that preserve the isotopy class of an unknotted genus 2 handlebody V. In this paper we prove that the 2-complex Γ is contractible. Therefore we get a finite presentation of $\mathcal{H}_2$.*

1. Introduction

Let $\mathcal{H}_g$ be the group of isotopy classes of orientation preserving homeomorphisms of S^3 that preserve the isotopy class of an unknotted genus g handlebody V. In 1933 Goeritz [Go] proved that $\mathcal{H}_2$ is finitely generated. In 1977 Goeritz's theorem was generalized to arbitrary genus $g \geq 2$ by Jerome Powell [Po]. In 2003 Martin Scharlemann noticed that Powell's proof contains a serious gap. Scharlemann [Sc] gave a modern proof of Goeritz's theorem by introducing a simplicial 2-complex Γ, with an action by $\mathcal{H}_2$, that deformation retracts onto a graph $\tilde{\Gamma}$. Given any two distinct vertices v, $\tilde{v}$ of Γ, Scharlemann constructed a vertex u in Γ that is adjacent to v and "closer" to $\tilde{v}$ (by "closer" we mean the intersection number of u and $\tilde{v}$, see Definition 1). Hence $\mathcal{H}_2$ acts on the connected graph $\tilde{\Gamma}$ and is generated by the isotopy classes of elements denoted by α, β, γ and δ (see Section 2 for a complete description). In this paper we study the geometry of Γ by showing that u is essentially unique (for a precise statement see Proposition 2). We derive the following theorem.

Theorem 1. *The graph $\tilde{\Gamma}$ is a tree, and shortest paths can be calculated algorithmically.*

Note that $\tilde{\Gamma}$ is locally infinite. So calculating paths is not trivial. We also get

Theorem 2. **i:** $\mathcal{H}_2 =< [\alpha],[\beta],[\gamma],[\delta] \mid [\alpha]^2 = [\gamma]^2 = [\delta]^3 = [\alpha\gamma]^2 = [\alpha\delta\alpha\delta^{-1}] = [\alpha\beta\alpha\beta^{-1}] = 1, [\gamma\beta\gamma] = [\alpha\beta], [\delta] = [\gamma\delta^2\gamma] >$

ii: $\mathcal{H}_2 \cong (\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \underset{\mathbb{Z}_2 \oplus \mathbb{Z}_2}{*} (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2$

Acknowledgement

This paper is a result of my thesis. I am grateful to my advisor, Professor Marc Culler, for his kind supervision and support. I would like to thank Ian Agol for his suggestions for the proofs of Lemma 1 and Lemma 2. I would also like to thank Yo'av Rieck for his helpful conversations.

2. Preliminaries

We give a description of the 2-complex Γ introduced by Scharlemann in [Sc]. For details about Γ we refer the reader to [Sc].

Let V be an unknotted handlebody of genus two in S^3 and let W be the closure of its complement. Let T be the boundary of V. Then T is a genus two Heegaard surface for S^3. Let $\mathcal{H}_2$ denote the group of isotopy classes of orientation preserving homeomorphisms of S^3 that leave the genus two handlebody V invariant. A sphere P in S^3 is called a *reducing sphere* for T if P intersects T transversely in a simple closed curve which is homotopically non-trivial on T. For any reducing sphere P for T let c_P denote $P \cap T$ and let v_P denote the isotopy class of c_P on T.

Definition 1. *For any two reducing spheres R, Q for T, define the intersection number of v_R and v_Q as*

$$v_R \cdot v_Q = \min_{R' \in v_R \quad Q' \in v_Q} |c_{R'} \cap c_{Q'}|$$

where $|c_{R'} \cap c_{Q'}|$ is the geometric intersection number of $c_{R'}$ with $c_{Q'}$.

Let Γ be a complex whose vertices are isotopy classes of reducing spheres for T. A collection $P_0, ..., P_n$ of reducing spheres bounds an n-simplex in Γ if and only if $v_{P_i} \cdot v_{P_j} = 4$ for all $0 \leq i \neq j \leq n$. In fact $n \leq 2$ [ST, Lemma 2.5]. So Γ is a simplicial 2-complex. Let $\triangle$ be any 2-simplex of Γ. We denote by $S_\triangle$ the "spine" of $\triangle$, which is the subcomplex of the barycentric subdivision consisting of all closed 1-simplices that contain the barycenter and a vertex of $\triangle$. Clearly $\triangle$ deformation retracts onto $S_\triangle$. Let $\tilde{\Gamma}$ be

$$\bigcup_{\triangle} S_\triangle.$$

So $\tilde{\Gamma}$ is a graph. Since no two 2-simplicies of Γ share an edge [ST, Lemma 2.5], the simplicial 2-complex Γ deformation retracts onto the graph $\tilde{\Gamma}$.

A *belt curve* on a genus two surface is a homotopically nontrivial separating simple closed curve. Let P denote a reducing sphere whose intersection with T is a belt curve, which we denote c_P. The reducing sphere P divides S^3 into two 3-balls $B^{\pm}$ whose intersections with the genus two surface T are two genus one surfaces $T^{\pm} = T \cap B^{\pm}$, each having one boundary component. The surface T^- (resp. T^+) contains two simple closed curves B, Z (resp. C, Y) meeting at one point. The curve B (resp. C) bounds a non-separating disc in W, homotopically non-trivial in V. The curve Z (resp. Y) bounds a non-separating disc in V, homotopically non-trivial in W. The genus two surface T contains two disjoint simple closed curves A and X. The curve A is homotopically non-trivial in V, disjoint from B and C, bounds a non-separating disc in W, and intersects Z and Y at one point. The curve X is homotopically non-trivial in W, disjoint from Z, Y and A, bounds a non-separating disc in V, and intersects B and C at one point. See figure 1.

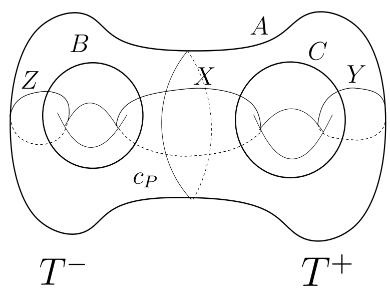


FIGURE 1. The curves c_P, A, B, C, X, Y and Z

Throughout this paper, unless otherwise stated, whenever we choose a reducing sphere R for T such that $v_R \neq v_P$ we will assume that the curve c_R intersects c_P, B, C, Y, Z transversely and minimally, and intersects A transversely.

There exist three automorphisms α, β, γ of S^3 with the following properties. The automorphism α is an orientation preserving homeomorphism of S^3 that preserves V and P, and that maps the curves A, B, C to A, B, C respectively by an orientation reversing map. The homeomorphism α is the hyperelliptic involution which preserves every simple closed curve (upto isotopy). The automorphism β is an orientation preserving homeomorphism of S^3 that preserves V and P, fixes T^- pointwise, and maps C to C and Y to Y by an orientation reversing map. Also $|A \cap \beta(X)| = 2$. The automorphism γ preserves V and P, and maps the curves c_P to c_P and A to A by an orientation reversing map. See figure 2. Scharlemann [Sc] showed that $\mathcal{H}_2$ is generated by the isotopy classes $[\alpha]$, $[\beta]$, $[\gamma]$ and $[\delta]$ where δ is any orientation preserving homeomorphism of S^3 such that $\delta(V) = V$ and $v_P \cdot v_{\delta(P)} = 4$. In this paper we will take δ as the following homeomorphism. Consider the genus two handlebody V as a regular neighborhood of a sphere, centered at the origin, with three holes. The homeomorphism δ is $\frac{2\pi}{3}$ rotation of V about the vertical z-axis. See figure 2.

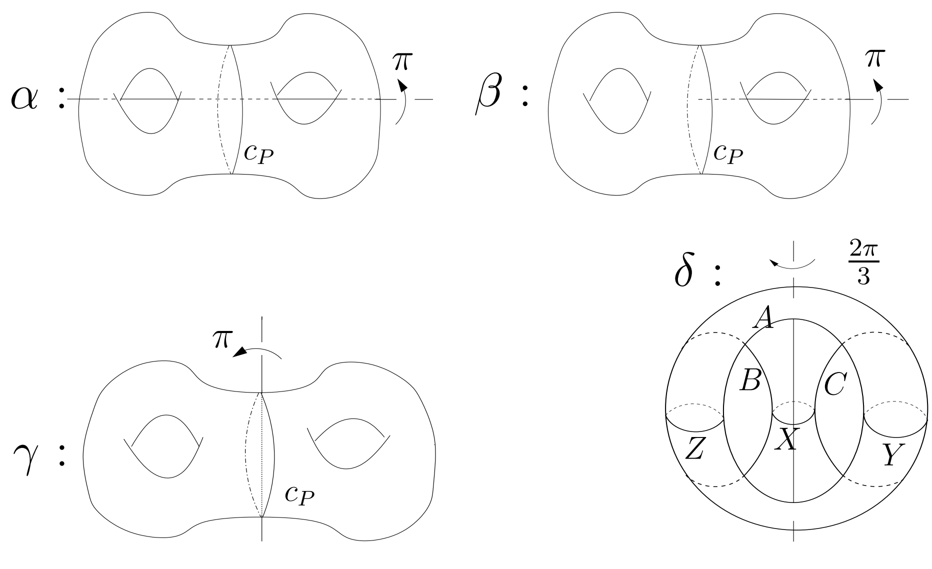


FIGURE 2. Homeomorphisms α, β, γ and δ

3. ARC FAMILIES OF REDUCING SPHERES ON $T^{\pm}$

Definition 2. *For any properly embedded arc $\nu \subset T^{\pm}$ we may write $[\nu] \in H_1(T^{\pm}, \partial T^{\pm}; \mathbb{Z})$ as $a[\mu] + b[\lambda]$ where $\mu = Z$ and $\lambda = B$ if $\nu \subset T^-$, and $\mu = Y$ and $\lambda = C$ if $\nu \subset T^+$. The slope of ν is defined to be $|\frac{a}{b}| \in \mathbb{Q}^+ \cup \infty$.*

Definition 3. *For any reducing sphere Q such that $v_Q \neq v_P$, let $N(Q, T^{\pm}, a)$ denote the number of arcs in $Q \cap T^{\pm}$ of slope a.*

Definition 4. *Denote any oriented curve D on T by $\overrightarrow{D}$ and the curve oriented in the direction opposite to $\overrightarrow{D}$ by $\overleftarrow{D}$.*

Definition 5. *Orient the curves A, B, C, X, Y, Z in such a way that $\delta^2(\overrightarrow{A}) = \delta(\overrightarrow{B}) = \overrightarrow{C}$ and $\delta^2(\overrightarrow{X}) = \delta(\overrightarrow{Y}) = \overrightarrow{Z}$. Up to isotopy there are natural homeomorphisms Ω, $\Psi : S^3 \longrightarrow S^3$ where Ω maps V to W and $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{X}$, $\overrightarrow{Y}$, $\overrightarrow{Z}$ to $\overleftarrow{X}$, $\overleftarrow{Y}$, $\overleftarrow{Z}$, $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ respectively, and Ψ maps W*

to W *and* $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{X}$, $\overrightarrow{Y}$, $\overrightarrow{Z}$ *to* $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, $\overleftarrow{X}$, $\overleftarrow{Y}$, $\overleftarrow{Z}$ *respectively (see figure 3). Let* $\Theta = \Psi\Omega$.

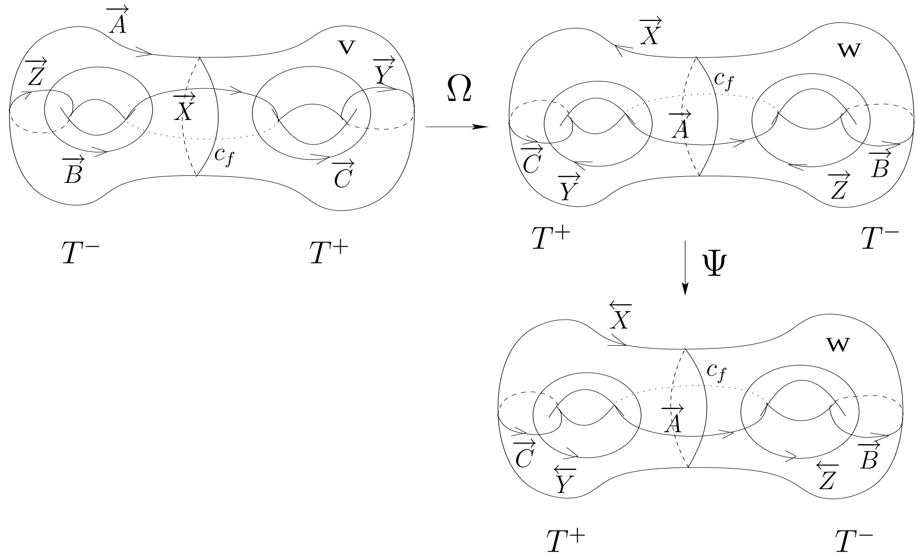


FIGURE 3. Homeomorphism $\Theta = \Psi\Omega$

Proposition 1. *Let* Q *be a reducing sphere for* T *such that* $v_Q \neq v_P$. *Then* $N(Q, T^-, a) = N(Q, T^+, \frac{1}{a})$.

Proof. Without loss of generality, we may assume that $Q = w(P)$ where w is a word in α, β, γ and δ.

Claim. $\Theta(c_Q) = c_Q$

Proof of Claim. The hyperelliptic involution α preserves the isotopy class of any simple closed curve on T. After an isotopy, we may assume that $\alpha(c_Q) = c_Q$. Let us write w as $a_1a_2...a_n$ where $a_i \in \{\alpha, \beta^{\pm1}, \gamma, \delta^{\pm1}\}$. The homeomorphism Θ satisfies $\Theta\alpha = \alpha\Theta$, $\Theta\beta = \alpha\beta\Theta$, $\Theta\gamma = \alpha\gamma\Theta$, $\Theta\delta = \delta\Theta$, and $\Theta(c_P) = c_P$. Then $\Theta(c_Q) = \Theta(w(c_P)) = \Theta(a_1a_2...a_n(c_P)) = b_1b_2...b_n\Theta(c_P)$ where b_i is α if $a_i = \alpha$, $\alpha\beta$ if $a_i = \beta$, $\alpha\gamma$ if $a_i = \gamma$, δ if $a_i = \delta$. So $b_1b_2...b_n\Theta(c_P) = b_1b_2...b_n(c_P) = a_1a_2...a_n(c_P) = w(c_P) = c_Q$.

Since Θ maps the curves A, B, C, X, Y, Z to X, Y, Z, A, B, C respectively, it takes the arcs of c_Q of slope a on T^- to the arcs of c_Q of slope $\frac{1}{a}$ on T^+. □

Lemma 1. *Let* Q *be any reducing sphere for* T *such that* $v_Q \neq v_P$. *Then* $N(Q, T^-, 0) \neq N(Q, T^-, \infty)$.

Proof. Suppose that $N(Q, T^-, 0) = N(Q, T^-, \infty) = m$. By Proposition 1, $N(Q, T^+, 0) = N(Q, T^+, \infty) = m$ and $N(Q, T^-, 1) = N(Q, T^+, 1)$.

The curve c_Q bounds a disc in V. So c_Q must have a "wave" τ [VKF] with respect to one of the curves Y, Z. Say with respect to Y. Then the arc τ of c_Q starts at Y, goes to T^- then comes back to Y on the same side without touching Z. So all the arcs of c_Q intersecting Z must intersect the arc on Y that is bounded by ends of τ. Then we get $N(Q,T^-,\infty)+N(Q,T^-,1)+2 \leq N(Q,T^+,\infty)+N(Q,T^+,1)$, a contradiction. □

Definition 6. *For any reducing sphere Q for T such that $v_Q \neq v_P$, let $F^{\pm}_{Q,a}$ denote the arc family of c_Q on $T^{\pm}$ of slope a.*

Notation 1. *We will fix the following notation: Let Q be a reducing sphere for T.*

- *If $N(Q,T^-,0)=n\neq 0$ then e_{01}, e_{02},..., e_{0n}, e_{1n}, e_{1n-1},..., e_{11} are going to denote consecutive end points, on c_P, of the arcs in $F^-_{Q,0}$ where e_{0j}, e_{1j} are end points of the same arc, and h_{01}, h_{02},..., h_{0n}, h_{1n}, h_{1n-1},..., h_{11} are going to denote consecutive end points, on c_P, of the arcs in $F^+_{Q,\infty}$ where h_{0j}, h_{1j} are end points of the same arc (existence of h_{ij} is guaranteed by Proposition 1).*
- *If $N(Q,T^-,\infty)=m\neq 0$ then g_{01}, g_{02},..., g_{0m}, g_{1m}, g_{1m-1},..., g_{11} are going to denote consecutive end points, on c_P, of the arcs in $F^-_{Q,\infty}$ where g_{0j}, g_{1j} are end points of the same arc, and f_{01}, f_{02},..., f_{0m}, f_{1m}, f_{1m-1},..., f_{11} are going to denote consecutive end points, on c_P, of the arcs in $F^+_{Q,0}$ where f_{0j}, f_{1j} are end points of the same arc.*
- *If $N(Q,T^-,1)=p\neq 0$ then k_{01}, k_{02},..., k_{0p}, k_{1p}, k_{1p-1},..., k_{11} are going to denote consecutive end points, on c_P, of the arcs in $F^-_{Q,1}$ where k_{0j}, k_{1j} are end points of the same arc, and l_{01}, l_{02},..., l_{0p}, l_{1p}, l_{1p-1},..., l_{11} are going to denote end points, on c_P, of the arcs in $F^+_{Q,1}$ where l_{0j}, l_{1j} are end points of the same arc.*

Lemma 2. *Let Q be a reducing sphere for T such that $N(Q,T^-,0)=n>N(Q,T^-,\infty)=m>N(Q,T^-,1)=0$. Then $\{f_{ij}|i=0,1\quad j=1,m\}\subseteq\{e_{ij}|i=0,1\quad j=2,...,n-1\}$.*

Proof. Suppose that $\{f_{ij}|i=0,1\quad j=1,m\}\nsubseteq\{e_{ij}|i=0,1\quad j=2,...,n-1\}$ (see figure 4). Then c_Q does not have a "wave" τ [VKF] with respect to the curve Y or the curve Z. Therefore c_Q can not bound a disc in V, a contradiction. □

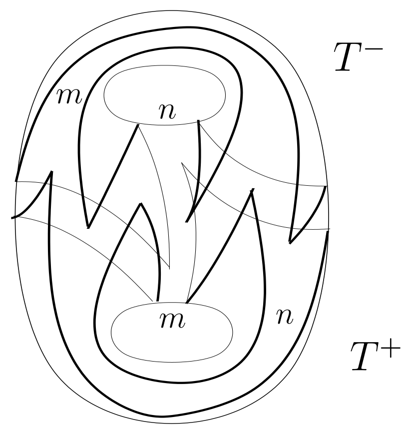


FIGURE 4.

Proposition 2. *Let v, $\tilde{v}$ be any two distinct vertices of Γ such that $v \cdot \tilde{v} \neq 4$. Then there exists unique vertex u of Γ such that*

- **i:** $u \cdot v = 4$
- **ii:** $u \cdot \tilde{v} < v \cdot \tilde{v}$
- **iii:** $u \cdot \tilde{v} < v' \cdot \tilde{v}$ *for any vertex v' of Γ such that $v' \neq u$ and $v' \cdot v = 4$*

Moreover, there is at most one vertex v'' of Γ satisfying $v \cdot v'' = 4$ and $u \cdot \tilde{v} < v \cdot v'' \leq v \cdot \tilde{v}$. In this case $v'' \cdot u = 4$.

Proof. Let v, $\tilde{v}$ be any two vertices of Γ such that $v \neq \tilde{v}$ and $v \cdot \tilde{v} \neq 4$. Since the group $\mathcal{H}_2$ is transitive on the vertices of Γ, we may assume that $v = v_P$ and $\tilde{v}$ is a vertex of Γ such that $\tilde{v} \neq v_P$ and $v_P \cdot \tilde{v} \neq 4$. Then for some word w in α, γ, β and δ, $w(P) \in \tilde{v}$. Let Q denote the reducing sphere $w(P)$. Since Q is not isotopic to P there must be some arcs in $c_Q \cap T^{\pm}$. By [Sc, Lemma 4] there is an arc of c_Q of slope 0 either on T^- or on T^+. Suppose it is on T^-. Let e_{ij}, g_{dq}, k_{rs}, f_{tu}, h_{yv}, l_{wz} denote the end points of the arcs of $c_Q \cap T^{\pm}$ as in the Notation 1. Possible cases for the arc families in $c_Q \cap T^{\pm}$ and their configurations, upto a power of β, are the following:

Case I. If $N(Q, T^-, 0) = m$, $N(Q, T^-, 1/k) = a$ and $N(Q, T^-, 1/(k+1)) = b$ where $k \geq 1$ then $N(Q, T^+, \infty) = m$, $N(Q, T^+, k) = a$ and $N(Q, T^+, k+1) = b$ by Proposition 1. Scharlemann in [Sc, Lemma 5] constructs a reducing sphere R satisfying (i) and (ii) (i.e. $v_R \cdot v_P = 4$ and $v_R \cdot v_Q < v_P \cdot v_Q$). We will show that upto isotopy the reducing sphere R also satisfies (iii). Let $n = a + b$.

I.A. If $n \neq 0$: Let us label end points of the arcs in $c_Q \cap T^+$ of slope different from ∞ as d_1, d_2,..., d_{2n}. Then it is not hard to show $\{e_{ij}\} \nsubseteq \{d_i\}$ by an argument similar to the proof of Lemma 2.

I.A.1. If $\{e_{ij}\} \cap \{h_{ij}\} \neq \emptyset$ (see figure 5): Set $p = \frac{|\{e_{ij}\} \cap \{h_{ij}\}|}{2}$ then $1 \leq p < m$. Consider the curve ξ shown in figure 5. It is easy to see that ξ bounds a disc in V and a disc in W. So ξ is the intersection of a reducing sphere S with T. Denote ξ by c_S. The reducing sphere S satisfies $v_S \cdot v_Q \leq |c_S \cap c_Q| = 2(n - m + 2p) < 2(n + m) = v_P \cdot v_Q$ and $v_S \cdot v_P = 4$.

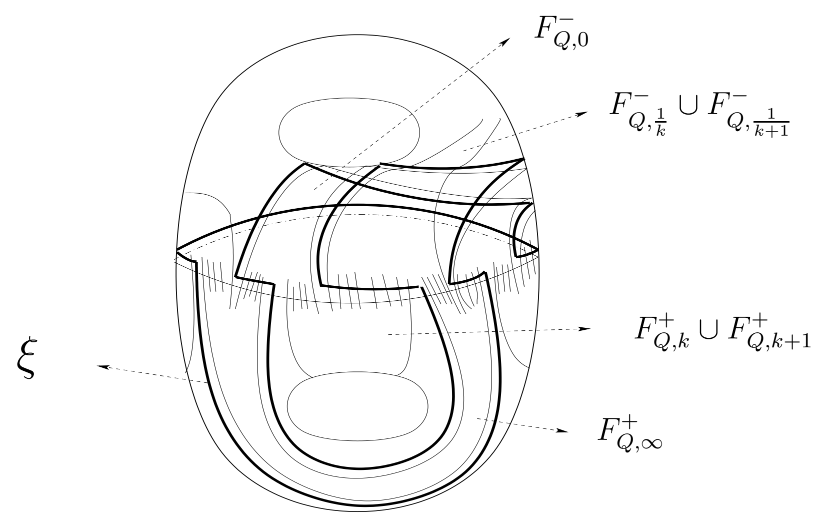


FIGURE 5.

Claim 1. $v_S \cdot v_Q = |c_S \cap c_Q|$.

Claim 2. $v_{\beta^i(S)} \cdot v_Q,\ \ v_{\beta^i\gamma(S)} \cdot v_Q > 2(n + m)$ for $i \neq 0$.

Proof of Claim 1. It suffices to show that there is no bigon on T formed by the curves c_S and c_Q. We may assume that c_S intersects c_Q in a neighborhood $N \subseteq T$ of c_P where $N \cap (B \cup Z \cup C \cup Y) = \emptyset$. The neighborhood N has two boundary components N^-, N^+. Say $N^\pm \subset T^\pm$. The set $c_S \cap N$ consists of four arcs ν_1, ν_2, ν_3, ν_4. Assume that end points of the arcs ν_1, ν_2, ν_3, ν_4 on N^- are lined up consecutively as $N^- \cap \nu_1$, $N^- \cap \nu_2$, $N^- \cap \nu_3$, $N^- \cap \nu_4$. The curve c_S has two arcs a_1, a_2 on T^- of slope 0 and two arcs b_1, b_2 on T^+ of slope ∞. Assume that $\nu_i \cap a_1 \neq \emptyset$ for $i = 1, 2$ and $\nu_1 \cap b_1 \neq \emptyset$. See figure 6. There are eight regions D_1,...,D_8 on N that can contain a vertex of a bigon. The regions D_1,...,D_8 are shown in figure 6. Any bigon should contain two of them. After an isotopy, we may assume that $\alpha(c_Q) = c_Q$ and $\alpha(c_S) = c_S$. Then $\alpha(D_i) = D_{i+2}$ for $i = 1, 2$ and $\Theta(\{D_i \mid i = 1, ..., 4\}) =$

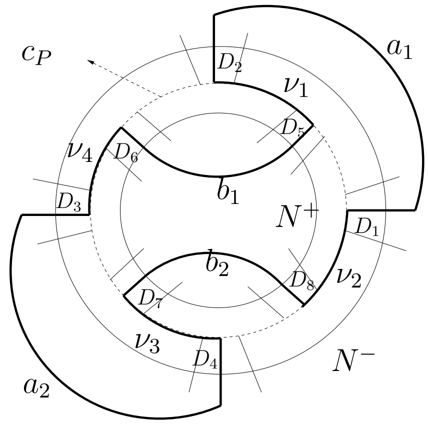


FIGURE 6.

$\{D_i \mid i = 5, ..., 8\}$ (see Definition 5 for Θ). So it is enough to check if D_i is a part of a bigon for $i = 1, 2$.

D_1**:** The region D_1 is part of a region $\widetilde{D}_1$ in T whose four consecutive sides are x_1, a_1, x_2, x_3 where $x_1 \in F^+_{Q,k} \cup F^+_{Q,k+1}$, $x_2 \in F^-_{Q,0}$ and $x_3 \in F^+_{Q,k} \cup F^+_{Q,k+1}$. See figure 7(a). If $\widetilde{D}_1$ is a bigon then $v_Q \cdot v_P < 2(n + m)$, a contradiction.

D_2**:**
- If $b = 0$ then $a \neq 0$. Then D_2 is part of a region $\widetilde{D}_2$ whose eight sides are x_1, y_1, a_1, z_1, x_2, a_2, y_2, z_2 where $x_1, x_2, z_1, z_2 \in F^+_{Q,\infty}$, $y_1, y_2 \in F^-_{Q,1/k}$. See figure 7(b). Therefore $\tilde{D}_1$ can not be a bigon.
- If $a, b \neq 0$ then D_2 is part of a region $\widetilde{D}_2$ whose four sides are x_1, a_2, y_1, z_1 where $x_1 \in F^+_{Q,\infty}$, $y_1 \in F^-_{Q,1/(k+1)}$, and z_1 is either a piece of c_S or $z_1 \in F^+_{Q,k} \cup F^+_{Q,k+1}$. See figure 7(c). If z_1 is an arc in c_S then $\widetilde{D}_2$ can not be a bigon. If $z_1 \in F^+_{Q,\infty} \cup F^+_{Q,k} \cup F^+_{Q,k+1}$ and $\widetilde{D}_2$ is a bigon then $v_Q \cdot v_P < 2(n + m)$, a contradiction.

By the above cases, $v_S \cdot v_Q = |c_S \cap c_Q|$.

In figure 8, intersection of a reducing sphere R' with the surface T is shown. Notice that $R' \in v_{\gamma S}$ and $v_S \cdot v_{\gamma S} = 4$. By an argument similar to the proof of Claim 1 we can show that $v_{R'} \cdot v_Q = |c_{R'} \cap c_Q| = 4kb + 4(k-1)a + 2m + 2n = v_{\gamma S} \cdot v_Q \geq 2m + 2n$.

Proof of Claim 2. We will do the calculation for $i = \pm 1$. The general case is similar. We may assume that $\beta^i(c_S)$ and $\beta^i\gamma(c_S)$ intersect c_Q in a neighborhood N described in the proof of Claim 1. By an argument similar to the proof of Claim 1 we get

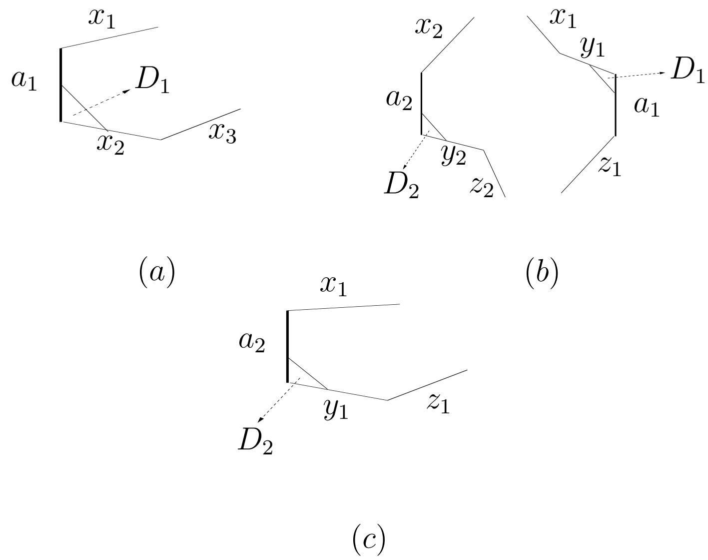


FIGURE 7.

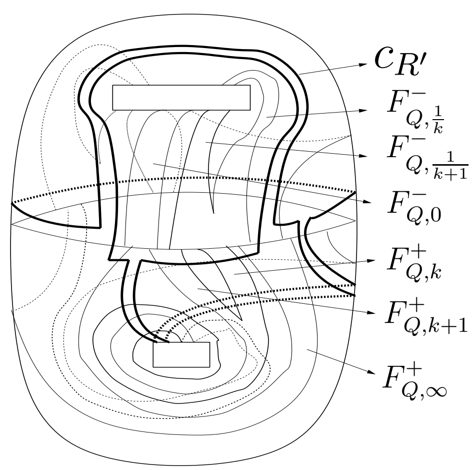
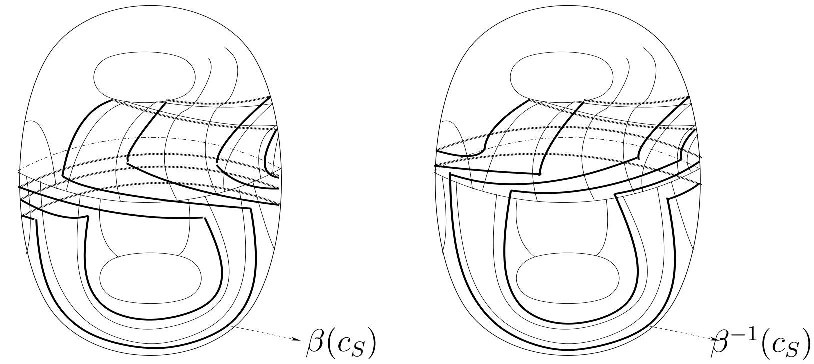


FIGURE 8.

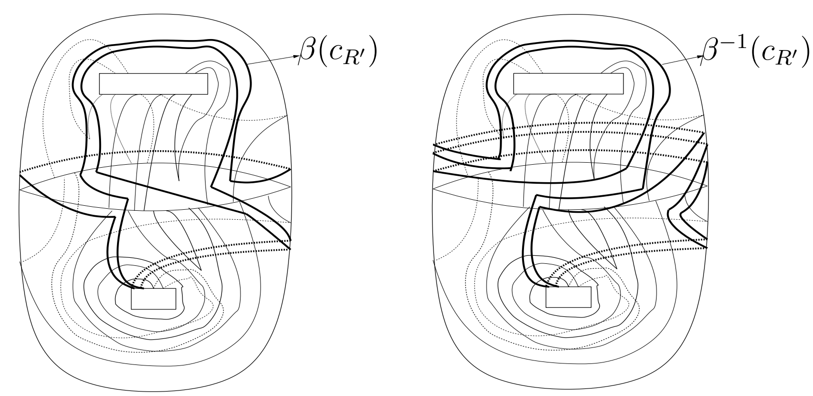
- $v_{\beta(S)} \cdot v_Q = 4p + 2m + 6n > 2(n+m)$. See figure 9 (a).
- $v_{\beta^{-1}(S)} \cdot v_Q = 6m + 2n - 4p > 2(n+m)$. See figure 9 (b).
- $v_{\beta\gamma(S)} \cdot v_Q = 4kb + 4(k-1)a + 4m + 2n + 2p > 2(n+m)$. See figure 10 (a).
- $v_{\beta^{-1}\gamma(S)} \cdot v_Q = 4kb + 4(k-1)a + 6m + 6n - 4p > 2(n+m)$. See figure 10 (b).

This implies that the vertex $v_R = v_S$ and satisfies the conditions of Proposition 2.



(a) (b)

FIGURE 9.



(a) (b)

FIGURE 10.

I.A.2. If $\{d_i\} \subseteq \{e_{ij}\}$ (see figure 11): Set $p = |\{e_{0j}\} \cap \{h_{0j}\}|$. Then $0 < p \leq m - n$. Either $p < m - n - p$ or $m - n - p < p$. Assume $p < m-n-p$. Consider the curve ξ shown in figure 11. The curve ξ is an intersection of a reducing sphere S with T. Denote ξ by c_S. Notice that $v_S \cdot v_P = 4$.

By an argument similar to the proof of the case I.A.1. we get

- $v_S \cdot v_Q = |c_S \cap c_Q| = 2(m - n - 2p) < v_P \cdot v_Q = 2(n + m)$
- $v_S \cdot v_{\gamma(S)} = 4$
- $v_{\gamma(S)} \cdot v_Q = 4kb + 4(k-1)a + 2(m+n) \geq 2(m+n)$ (see figure 12)
- $v_{\beta^i(S)} \cdot v_Q, \quad v_{\beta^i\gamma(S)} \cdot v_Q > 2(n+m)$ for $i \neq 0$.

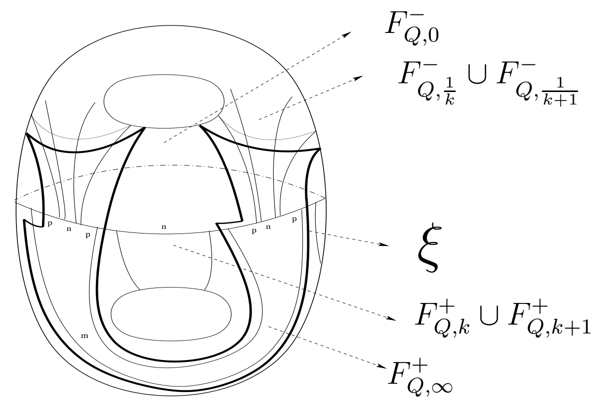


FIGURE 11.

This implies that the vertex $v_R = v_S$ and satisfies the conditions of Proposition 2.

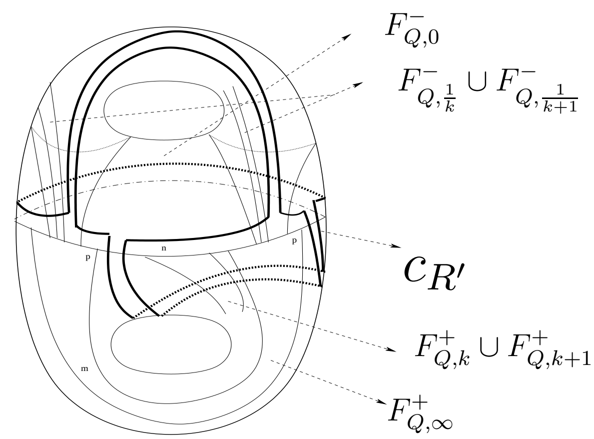


FIGURE 12. The curve $c_{R'}$ in the figure is $R' \cap T$ for some reducing sphere R' for T satisfying $R' \in v_{\gamma S}$

I.B. If $n = 0$: This is a special case of I.A.3.

Case II. If $N(Q, T^-, 0) = m$, $N(Q, T^-, \infty) = n \neq 0 = N(Q, T^-, 1)$ then $N(Q, T^+, 0) = n$, $N(Q, T^+, \infty) = m \neq 0 = N(Q, T^+, 1)$ by Proposition 1. By Lemma 1, $m \neq n$. Suppose $m < n$. By Lemma 2, $\{e_{ij} | i = 0, 1\ j = 1, .., m\} \subseteq \{f_{ij} | i = 0, 1\ j = 2, .., n-1\}$. By the argument in [Sc, Lemma 5] we get two non-isotopic reducing spheres for T that satisfy (i) and (ii). Let us call S the one having an arc on T^- of slope 0 and S' the one having an arc on T^+ of slope 0. In the figure 13 intersections of two reducing spheres R and R' with T are shown. It is easy to see that $R \in v_S$ and $R' \in v_{S'}$.

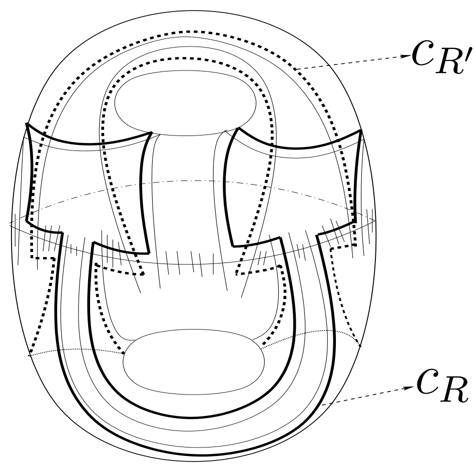


FIGURE 13.

Let $p = |\{g_{0j}\} \cap \{f_{0j}\}|$. Then $0 < p \leq m - n$. Either $p < m - n - p$ or $m - n - p < p$. Assume $p < m - n - p$. Then by an argument similar to the proof of the case I.A.1. we can show that $2n + 2m = v_P \cdot v_Q > v_R \cdot v_Q = 2n - 2m > v_{R'} \cdot v_Q = 2(n - m - 2p)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q$, $v_{\beta^i(R')} \cdot v_Q > 2n + 2m$ for $i \neq 0$.

Case III. If $N(Q, T^-, 0) = m$, $N(Q, T^-, \infty) = n$, $N(Q, T^-, 1) = p$ where $m, n, p \neq 0$, then $N(Q, T^+, 0) = n$, $N(Q, T^+, \infty) = m$, $N(Q, T^+, 1) = p$ by Proposition 1. By Lemma 1, $m \neq n$. Say $m > n$.

The curves A, B, C and c_P divide T into four punctured discs T_f^-, T_b^-, T_f^+, T_b^+ where $T_f^- \cup T_b^- = T^-$ and $T_f^+ \cup T_b^+ = T^+$. This division also gives two pairs of pants $T_f^- \cup T_f^+ = P_f$ and $T_b^- \cup T_b^+ = P_b$. Let $c_f = P_f \cap c_P$ and $c_b = P_b \cap c_P$.

Let K be a reducing sphere intersecting the interior of T^- in a simple closed curve parallel to c_P. The reducing sphere K divides T into two parts. Denote the one containing the curve B by t^- and the one containing the curve C by t^+. Let $c_K^f = T_f^- \cap K$ and $c_K^b = T_b^- \cap K$.

Suppose that $F_{Q,0}^- \cap t^- \cap A = F_{Q,1}^- \cap t^- \cap A = \emptyset$, $|F_{Q,\infty}^- \cap (c_K^f \setminus A)|$ $= |F_{Q,\infty}^- \cap (c_K^b \setminus A)| = |F_{Q,\infty}^- \cap t^- \cap A| = n$ and that k'_{01}, k'_{02},..., k'_{0p}, e'_{01}, e'_{02},..., e'_{0m} and g'_{01}, g'_{02},..., g'_{0n} are consecutive intersection points of the arcs in $F_{Q,1}^-$, $F_{Q,0}^-$ and $F_{Q,\infty}^-$ with c_K^f respectively. Locate arcs of c_Q on T^+ in such a way that $|F_{Q,\infty}^+ \cap (c_P^f \setminus A)| = |F_{Q,\infty}^+ \cap (c_P^b \setminus A)| = |F_{Q,\infty}^+ \cap A| = m$ and $|F_{Q,0}^+ \cap A| = |F_{Q,1}^+ \cap A| = 0$. Suppose that l_{01},...,

l_{0p}, f_{01},..., f_{0n} and h_{01},..., h_{0m} are consecutive intersection points of the arcs in $F^+_{Q,1}$, $F^+_{Q,0}$ and $F^+_{Q,\infty}$ with c_f respectively. Let τ be an arc in $F^-_{Q,1}$ whose intersection with c^f_K is k'_{01}. Suppose that $\tau \cap (t^+ \setminus T^+) \cap A \neq \emptyset$. See figure 14. By applying a power of β we can assume that $2 \leq |c_Q \cap A \cap (t^+ \setminus T^+)| < 2(p+n+m)$. By the argument in [Sc, Lemma 5] we get two non-isotopic reducing spheres for T that satisfy (i) and (ii). Let us call S the one having an arc on T^- of slope 0 and S' the one having an arc on T^+ of slope 0.

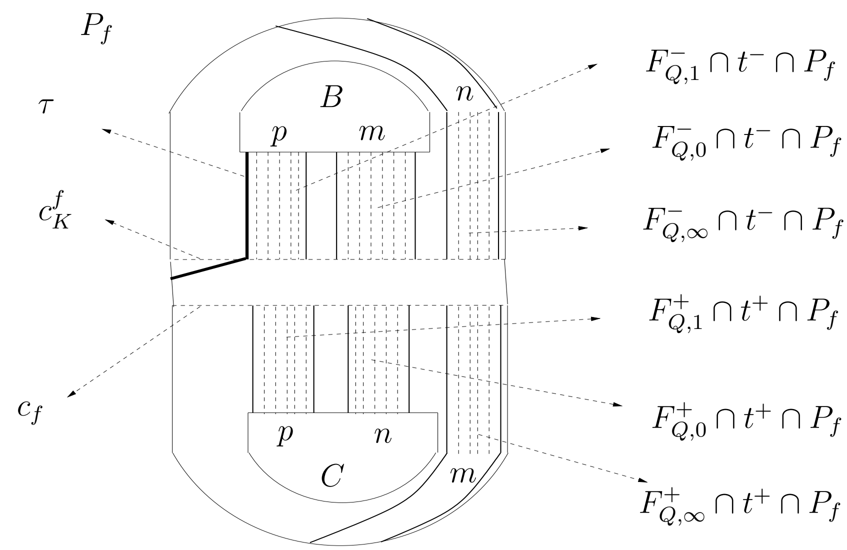


FIGURE 14.

In the below figures, intersections of two reducing spheres R, R' with T are shown. It is easy to see that $R \in v_S$ and $R' \in v_{S'}$.

III.A: If $\{g_{ij}\} \subseteq \{h_{ij}\}$ (see figure 15): Let $x = |\{h_{ij}\} \cap \{k_{ij}\}|/2$. Then by an argument similar to the proof of the case I.A.1. we get $2(n+m+p) = v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(m+p-n) > v_R \cdot v_Q = 2(m+p-n-2x)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q$, $v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

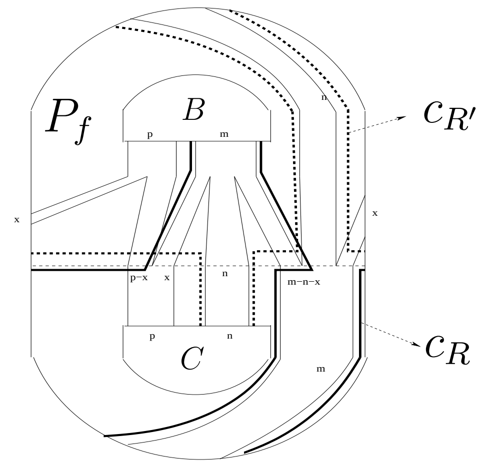


FIGURE 15.

III.B: If $\{g_{ij}\} \cap \{h_{ij}\} \neq \emptyset$, $\{g_{ij}\} \cap \{f_{ij}\} \neq \emptyset$, $\{e_{ij}\} \cap \{h_{ij}\} = \emptyset$ (see figure 16): Let $x = |\{k_{ij}\} \cap \{h_{ij}\}|/2$. Then by an argument similar to the proof of the case I.A.1. we get $2(n+m+p) = v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(p+n-m+2x) > v_R \cdot v_Q = 2(p+n-m)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q$, $v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

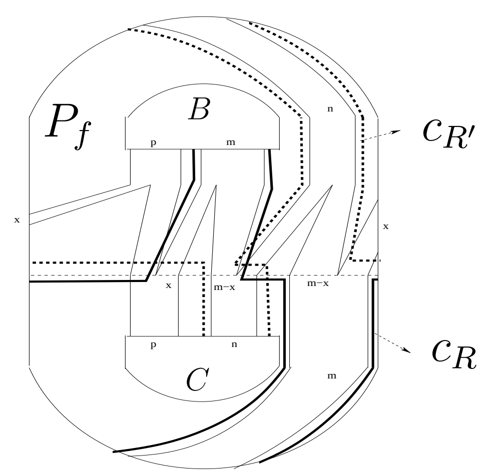


FIGURE 16.

III.C: If $\{g_{ij}\} \cap \{h_{ij}\} \neq \emptyset$, $\{g_{ij}\} \cap \{f_{ij}\} \neq \emptyset$, $\{e_{ij}\} \cap \{h_{ij}\} \neq \emptyset$ (see figure 17): Let $x = |\{f_{ij}\} \cap \{g_{ij}\}|/2$. Then by an argument similar to the proof of the case I.A.1. we get $2(n+m+p) = v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(m-n+2x+p) > v_R \cdot v_Q = 2(m-n-p+2x)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q,\ v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

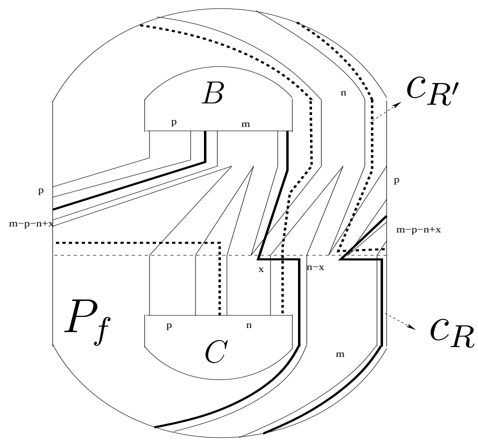


FIGURE 17.

III.D: If $\{g_{ij}\} \cap \{f_{ij}\} \neq \emptyset$, $\{g_{ij}\} \cap \{l_{ij}\} \neq \emptyset$, $\{e_{ij}\} \cap \{l_{ij}\} \neq \emptyset$, $\{e_{ij}\} \cap \{h_{ij}\} = \emptyset$ (see figure 18): Let $x = |\{g_{ij}\} \cap \{l_{ij}\}|/2$. Then by an argument similar to the proof of the case I.A.1. we get $2(n+m+p) = v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(p+n+m-2x) > v_R \cdot v_Q = 2(p+n-m)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q,\ v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

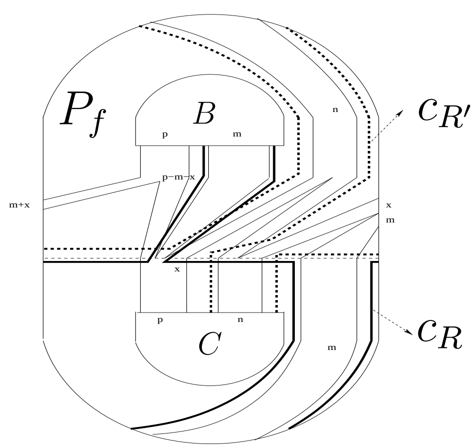


FIGURE 18.

III.E: If $\{g_{ij}\} \cap \{f_{ij}\} \neq \emptyset$, $\{g_{ij}\} \cap \{l_{ij}\} \neq \emptyset$, $\{e_{ij}\} \cap \{l_{ij}\} \neq \emptyset$, $\{e_{ij}\} \cap \{h_{ij}\} \neq \emptyset$ (see figure 19): Let $x = |\{g_{ij}\} \cap \{l_{ij}\}|/2$. Then by an argument similar to the proof of the case I.A.1. we get $v_{R'} \cdot v_Q = 2(m+n+p-2x)$, $v_R \cdot v_Q = 2(m+n-p+2x)$ and $v_{\beta^i(R)} \cdot v_Q,\ v_{\beta^i(R')} \cdot v_Q > 2(n+m+p)$ for $i \neq 0$. So $v_{R'} \cdot v_Q = v_R \cdot v_Q$ if and only if $p = 2x$. If p is equal to $2x$ then by an argument given in the proof of Lemma 2 we can show that c_Q does not bound a disc in V. Therefore either $v_{R'} \cdot v_Q > v_R \cdot v_Q$ or $v_{R'} \cdot v_Q < v_R \cdot v_Q$. Notice that $v_R \cdot v_{R'} = 4$.

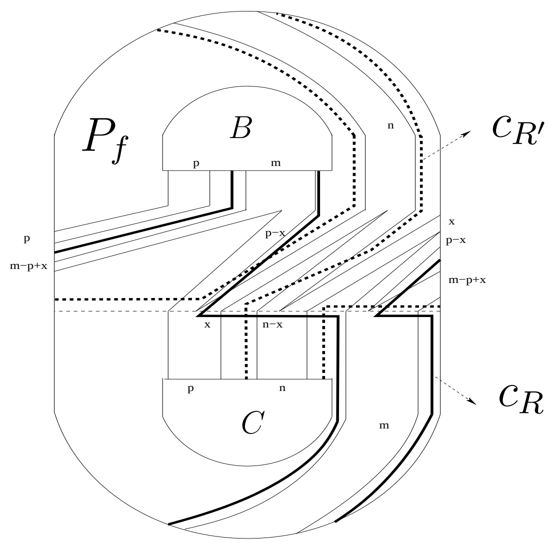


FIGURE 19.

III.F: If $\{g_{ij}\} \cap \{f_{ij}\} \neq \emptyset$, $\{g_{ij}\} \cap \{l_{ij}\} \neq \emptyset$, $\{g_{ij}\} \cap \{h_{ij}\} \neq \emptyset$ (see figure 20): Let $x = |\{g_{ij}\} \cap \{f_{ij}\}|/2$ then by an argument similar to the proof of the case I.A.1. we get $2(n+m+p) = v_P \cdot v_Q > v_R \cdot v_Q = 2(m+x+3p-n+x) > v_{R'} \cdot v_Q = 2(m+x+p-n+x)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q,\ v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

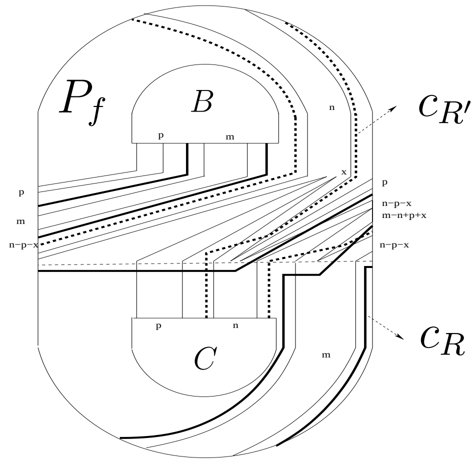


FIGURE 20.

III.G: If $\{e_{ij}, g_{ij}\} \subseteq \{l_{ij}\}$ (see figure 21): Let $x = |\{k_{ij}\} \cap \{l_{ij}\}|/2$ then by an argument similar to the proof of the case I.A.1. we get $v_P \cdot v_Q > v_{R'} \cdot v_Q = 2(p+m-n) > v_R \cdot v_Q = 2(p-m+n)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q,\ v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

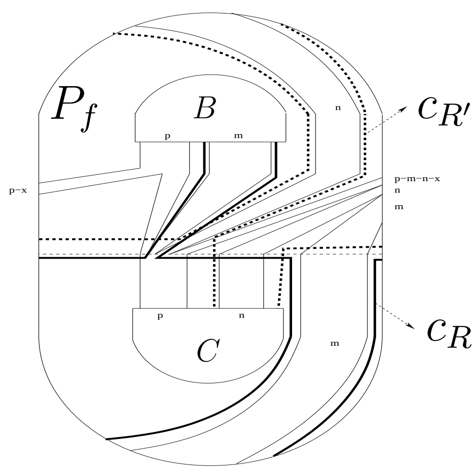


FIGURE 21.

III.H: If $\{g_{ij}\} \subseteq \{l_{ij}\}$, $\{e_{ij}\} \cap \{h_{ij}\} \neq \emptyset$: This case is eliminated by an argument given in proof of Lemma 2 (the curve c_Q does not bound a disc in V).

III.I: If $\{g_{ij}\} \cap \{l_{ij}\} \neq \emptyset$, $\{g_{ij}\} \cap \{h_{ij}\} \neq \emptyset$: After applying β^{-1} to c_Q we can assume that c_Q is as in figure 22. Let $x = |\{k_{ij}\} \cap \{l_{ij}\}|/2$ then by an argument similar to the proof of the case I.A.1. we have $2(n+m+p) = v_P \cdot v_Q > v_R \cdot v_Q = 2(m -$

$n + 3p - 2x) > v_{R'} \cdot v_Q = 2(m - n + p)$, $v_R \cdot v_{R'} = 4$ and $v_{\beta^i(R)} \cdot v_Q$, $v_{\beta^i(R')} \cdot v_Q > v_P \cdot v_Q$ for $i \neq 0$.

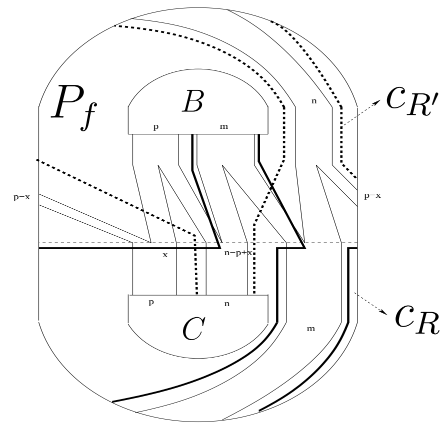


FIGURE 22.

□

4. A presentation for $\mathcal{H}_2$

We will first prove Theorem 1. Then by using Bass-Serre theory we will prove Theorem 2.

Proof of Theorem 1.

Proof. Suppose that $\tilde{\Gamma}$ is not a tree. Then there is a nontrivial loop in $\tilde{\Gamma}$. For any loop ξ in $\tilde{\Gamma}$ let $NV(\xi)$ denote the number of vertices of ξ. Then $\alpha_0 = \min\{NV(\xi) \mid \xi \text{ is a nontrivial loop in } \tilde{\Gamma}\ \} > 0$. Since each edge of Γ lies on a single 2-simplex $\alpha_0 \geq 8$. Let ξ_0 be a nontrivial loop in $\tilde{\Gamma}$ such that $NV(\xi_0) = \alpha_0$. Since ξ_0 is of minimal length all its vertices are distinct. Let v_0 be any vertex of ξ_0, and let v_0, v_1, v_2, v_3,..., $v_{\alpha_0 - 1}$ be the consecutive vertices of ξ_0. We may suppose that $v_0 \in \Gamma$. Then v_0, v_2, v_4,..., $v_{\alpha_0 - 2}$ are vertices of Γ, and $v_k \cdot v_{k+2} = v_{\alpha_0 - 2} \cdot v_0 = 4$ for $k \in \{0, 2, 4, ..., \alpha_0 - 4\}$.

Claim. $v_k \cdot v_0 < v_{k+2} \cdot v_0$ for $k \in \{0, 2, 4, ..., \alpha_0 - 4\}$.

Proof of claim. The proof will be by induction on the index k. If $k = 0$ then $v_0 \cdot v_0 = 0 < v_2 \cdot v_0 = 4$. Assume $v_k \cdot v_0 < v_{k+2} \cdot v_0$ for $k \in \{0, 2, ..., \alpha_0 - 6\}$. If $v_{k+4} \cdot v_0 \leq v_{k+2} \cdot v_0$ then $v_k \cdot v_{k+4} = 4$ by Proposition 2. Since $v_k \cdot v_{k+2} = v_{k+2} \cdot v_{k+4} = 4$, the vertices v_k, v_{k+2}, v_{k+4} form a 2-simplex $\triangle$ in Γ. Then we get a loop ξ in $\tilde{\Gamma}$ with vertices v_0, v_1,..., v_k, u, v_{k+4}, v_{k+5}, ... , $v_{\alpha_0 - 2}$, $v_{\alpha_0 - 1}$ where u is the barycenter of $\triangle$. This contradicts the minimality of α_0.

By the above claim, we get $v_0 \cdot v_{\alpha_0 - 4} < v_0 \cdot v_{\alpha_0 - 2}$. But $4 < v_0 \cdot v_{\alpha_0 - 4}$ and $v_0 \cdot v_{\alpha_0 - 2} = 4$, a contradiction. $\square$

Proof of Theorem 2.

Proof. Let v_M be a vertex of $\tilde{\Gamma}$ corresponding to the barycenter of the 2-simplex whose vertices are v_P, $v_{\delta(P)}$ and $v_{\delta^2(P)}$. Let E be the edge of $\tilde{\Gamma}$ whose vertices are v_P and v_M. Let H_P be the subgroup of $\mathcal{H}_2$ generated by the elements that stabilize v_P. Let H_M be the subgroup of $\mathcal{H}_2$ generated by the elements that preserve v_M. Let H_E be the group of elements of $\mathcal{H}_2$ that stabilize the edge E.

- Scharlemann in [Sc, Lemma 2] gives the following presentation for H_P:
  $H_P = < [\alpha],[\beta],[\gamma] \mid [\alpha]^2 = [\gamma]^2 = [\alpha\gamma]^2 = [\alpha\beta\alpha\beta^{-1}] = 1, [\gamma\beta\gamma] = [\alpha\beta] > \cong (\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.
- The subgroup H_M fixes the set $\{v_P, v_{\delta(P)}, v_{\delta^2(P)}\}$. Therefore
  $H_M =< [\delta],[\alpha],[\gamma] \mid [\delta]^3 = [\alpha]^2 = [\gamma]^2 = [\alpha\delta\alpha^{-1}\delta^{-1}] = [\alpha\gamma]^2 = 1, \ \ [\delta] = [\gamma\delta^2\gamma] > \ \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2$.
- An element h of $\mathcal{H}_2$ fixes the sets $\{v_P\}$ and $\{v_{\delta(P)}, v_{\delta^2(P)}\}$ if and only if $h \in H_E$. Hence
  $H_E =< [\alpha],[\gamma] | \ \ [\alpha]^2 = [\gamma]^2 = [\alpha\gamma]^2 = 1 > \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The action of $\mathcal{H}_2$ on the 2-complex Γ induces an action of $\mathcal{H}_2$ on the tree $\tilde{\Gamma}$. The subgroups H_P, H_M are the isotropy subgroups of $\mathcal{H}_2$ fixing the vertices v_P, v_M respectively. By the standard Bass-Serre theory [S] the group $\mathcal{H}_2$ is thus a free product of the subgroups H_P and H_M amalgamated over the subgroup H_E.

$$\mathcal{H}_2 \cong H_P \underset{H_E}{*} H_M \cong (\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \underset{\mathbb{Z}_2 \oplus \mathbb{Z}_2}{*} (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2$$

$\square$

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