Resilient Reducibility in Nuclear Multifragmentation

L.G. Moretto, L. Phair, and G.J. Wozniak
Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA
(February 8, 2008)

The resilience to averaging over an initial energy distribution of reducibility and thermal scaling observed in nuclear multifragmentation is studied. Poissonian reducibility and the associated thermal scaling of the mean are shown to be robust. Binomial reducibility and thermal scaling of the elementary probability are robust under a broad range of conditions. The experimental data do not show any indication of deviation due to averaging.

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The complexity of nuclear multifragmentation underwent a remarkable simplification when it was *empirically* observed that many aspects of this process were: a) "reducible"; and b) "thermally scalable" [1–6].

"Reducibility" means that a given many-fragment probability can be expressed in terms of a corresponding one-fragment probability, i.e., the fragments are emitted essentially independent of one another.

"Thermal scaling" means that the one-fragment probability so extracted has a thermal-like dependence, i.e., it is essentially a Boltzmann factor.

Both "reducibility" and "thermal scaling" were observed in terms of a global variable, the transverse energy E_t (defined as $E_t = \sum_i E_i \sin^2\theta_i$, i.e. the sum of the kinetic energies E of all charged particles in an event weighted by the sine squared of their polar angles θ), which was assumed (see below) to be proportional to the excitation energy of the decaying source(s) [1–3].

In particular, it was found that the Z-integrated multiplicity distributions $P(n)$ were binomially distributed, and thus "reducible" to a one-fragment probability p. With higher resolution, it was noticed that for each individual fragment species of a given Z, the n_Z-fragment multiplicities $P(n_Z)$ obeyed a nearly Poisson distribution, and were thus "reducible" to a single-fragment probability proportional to the mean value $\langle n_Z \rangle$ for each Z [4].

The one-fragment probabilities p showed "thermal scaling" by giving linear Arrhenius plots of $\ln p$ vs $1/\sqrt{E_t}$ where it is assumed that $\sqrt{E_t} \propto T$. Similarly n-fragment charge distributions $P_n(Z)$ were shown to be both "reducible" to a one-fragment Z distribution as well as "thermally scalable" [5]. Even the two-fragment angular correlations $P_{1,2}(\Delta\phi)$ were shown to be expressible in terms of a one-body angular distribution with amplitudes that are "thermally scalable" [6]. Table I gives a summary of the "reducible" and "thermal scaling" observables.

reducibility	thermal scaling	reference
$P(n) \to p$	$\ln p \propto 1/\sqrt{E_t}$	[1]
$P(n_Z) \to \langle n_Z \rangle$	$\ln \langle n_Z \rangle \propto 1/\sqrt{E_t}$	[4]
$P_n(Z) \to P_1(Z) \propto e^{-\alpha Z}$	$\alpha \propto 1/\sqrt{E_t}$	[5]
$P_{1,2}(\Delta\phi) \to \int P_1(\phi) P_2(\phi + \Delta\phi)$	amplitude $\propto 1/E_t$	[6]

TABLE I. Summary of reducible and thermal scaling observables in nuclear multifragmentation.

Empirically, "reducibility" and "thermal scaling" are pervasive features of nuclear multifragmentation. "Reducibility" proves nearly stochastic emission. "Thermal scaling" gives an indication of thermalization.

Recently, there have been some questions on the significance (not the factuality) of "reducibility" and "thermal scaling" in the *binomial* decomposition of Z-integrated multiplicities [7]. For instance, had the original distribution in the true excitation-energy variable been binomially distributed and thermally scalable, wouldn't the process of transforming from excitation energy E to transverse energy E_t through an (assumedly) broad transformation function $P(E, E_t)$ destroy both features?

Specifically, under a special choice of averaging function (Gaussian), for a special choice of parameters (variance from GEMINI [8]), and for special input p (the excitation energy dependent one-fragment emission probability) and m (the number of "throws" or attempts) to the binomial function, the binomial parameters *extracted* from the averaged binomial distribution are catastrophically altered, and the initial thermal scaling is spoiled [7]. This "spoiling" in [7] is not due to detector acceptance effects (which has been commented on extensively in [3]), but rather is due to the intrinsic width of correlation between E_t and E as discussed below.

It should be pointed out that, while the decomposition of the many-fragment emission probabilities $P(n)$ into p and m may be sensitive to the averaging process, the quantity $\langle mp \rangle$ is not [7]. However, both p and $\langle mp \rangle$ are known to give linear Arrhenius plots with essentially the same slope (see below). This by itself demonstrates that no damaging average is occurring.

Furthermore, we have observed that by restricting the definition of "fragment" to a single Z, the multiplicity distributions become nearly Poissonian and thus are characterized by the average multiplicity $\langle mp \rangle$ which gives well behaved Arrhenius plots [4]. *Thus, the linearity of the Arrhenius plots of both p and $\langle mp \rangle$ extracted from all fragments, and the linearity of the Arrhenius plots of $\langle mp \rangle$ for each individual Z value eliminate observa-*

tionally the criticisms described above. In fact, it follows that no visible damage is inflicted by the true physical transformation from E to E_t. Therefore, the experimental Poisson "reducibility" of multiplicity distributions for each individual Z and the associated "thermal scaling" of the means eliminates observationally these criticisms.

We proceed now to show in detail that: 1) binomial reducibility and thermal scaling are also quite robust under reasonable averaging conditions; 2) the data do not show any indication of pathological behavior.

We first discuss the possible origin and widths of the averaging distribution.

It is not apparent why the variance of $P(E, E_t)$ calculated from GEMINI [8] should be relevant. GEMINI is a low energy statistical code and is singularly unable to reproduce intermediate mass fragment (IMF:$3 \leq Z \leq 20$) multiplicities, the magnitudes of E_t, and other multifragmentation features. There is no reason to expect that the variance in question is realistic.

Apparently, E_t does not originate in the late thermal phase of the reaction. Rather, it seems to be dominated by the initial stages of the collision. Consequently its magnitude may reflect the geometry of the reaction and the consequent energy deposition in terms of the number of primary nucleon-nucleon collisions. This is attested to by the magnitude of E_t which is several times larger than predicted by any thermal model. Thus, the worrisome "thermal widths" are presumably irrelevant.

Since there is no reliable way to determine the actual resolution of the correlation between E_t and E, experimentally or via simulation calculations [7], instead of using large or small variances, we will show:

a) which variables control the divergence assuming a Gaussian distribution, and in what range of values the averaging is "safe", i.e. it does not produce divergent behavior;

b) that the use of Gaussian tails is dangerous and improper unless one shows that the physics itself *requires* such tails.

The input binomial distribution is characterized by m, the number of throws (assumed constant in the calculations in [7]), and p which has a characteristic energy dependence of

$$\log \frac{1}{p} \propto \frac{B}{\sqrt{E_t'}}. \qquad (1)$$

We denote the one-to-one image of E in E_t space with a prime symbol.

The averaging in [7] is performed by integrating the product of an exponential folded with a Gaussian (Eq. (12) of [7]).

$$\langle p \rangle \propto \int \exp\left(-\frac{B}{\sqrt{x}} - \frac{(x-x_0)^2}{2\sigma^2}\right) dx. \qquad (2)$$

If the slope of the exponential is large, there will be 1) a substantial shift ϵ in the peak of the integrand, and 2) a great sensitivity to the tail of the Gaussian.

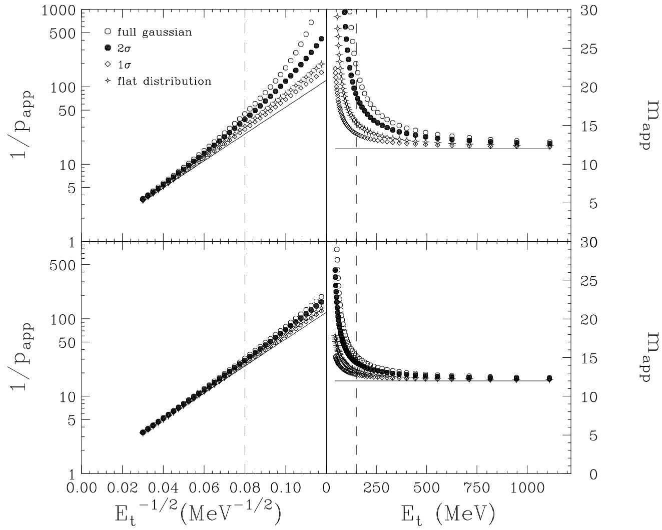

FIG. 1. Upper left panel: the distorted Arrhenius plot for a value of B=40MeV$^{1/2}$, m=12, and a fixed ratio of Γ_{E_t}/E_t=0.3. The open circles represent the apparent single fragment emission probabilities extracted after a folding of the "thermal" binomial emission probabilities with a Gaussian distribution (see Eqs. (5)-(8)). The solid circles and open diamonds show the effect of truncating the Gaussian tails at 2σ and 1σ, respectively. The star symbols demonstrate results from folding with a square distribution of the same full width at half maximum as the Gaussian. The solid line represents the undistorted Arrhenius plot. The dashed line represents an experimental "soft" limit beyond which the transverse energy may be unreliable as a measure of impact parameter [9] or deposited energy. Lower left panel: same as upper left but for Γ_{E_t}/E_t=0.2. Upper right panel: the distorted values of the number of throws m_{app} as a function of E_t for a value of B=40MeV$^{1/2}$ and a fixed ratio of Γ_{E_t}/E_t=0.3. The solid line represents the undistorted value of m=12. Lower right panel: same as upper right but for Γ_{E_t}/E_t=0.2.

The shifts $\epsilon_{\langle p\rangle}$ and $\epsilon_{\langle p^2\rangle}$ can be approximately evaluated:

$$\epsilon_{\langle p\rangle} = \frac{\sigma^2 B}{2x_0^{3/2}} \qquad (3)$$

$$\epsilon_{\langle p^2\rangle} = 2\frac{\sigma^2 B}{2x_0^{3/2}}. \qquad (4)$$

This illustrates the divergence at small values of x_0 both in the shift of the integrand in $\langle p\rangle$ and $\langle p^2\rangle$ and the corresponding divergence in $\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2$. The scale of the divergence is set by the product $\sigma^2 B$. Thus one can force a catastrophic blowup by choosing a large value of σ^2, of B, or of both. This is what has been shown to happen with large values of σ^2 and B. The counterpart to this is that there possibly exists a range of values for B and σ^2 which leads to a "safe" averaging process.

In order to illustrate this, we have calculated the "apparent" values of the single fragment emission probability p_{app} for widths characterized by the ratio of the full width at half maximum Γ_{E_t} over E_t. Specifically we have extracted p_{app}:

$$p_{app} = 1 - \frac{\sigma_n^2}{\langle n \rangle} \tag{5}$$

and m_{app}:

$$m_{app} = \frac{\langle n \rangle}{p_{app}} \tag{6}$$

by calculating the observed mean:

$$\langle n \rangle = \int \sum_{n=0}^{m} n P_n^m(E_t') g(E_t') dE_t' \tag{7}$$

and variance:

$$\sigma_n^2 = \left[\int \sum_{n=0}^{m} n^2 P_n^m(E_t') g(E_t') dE_t' \right] - \langle n \rangle^2 \tag{8}$$

for "thermal" emission probabilities P_n^m folded with a Gaussian distribution $g(E_t)$. We have assumed m is constant.

For a value of Γ_{E_t}/E_t=0.3, m=12, and B=40MeV$^{1/2}$ (consistent with the upper limits of the slopes observed in the Xe induced reactions [2,3]), the onset of divergence is observed in the Arrhenius plot at small values of E_t (top left panel of Fig. 1, open circles). For Γ_{E_t}/E_t=0.2 (open circles in bottom left panel of Fig. 1), the divergent behavior is "shifted" to even lower energies and the resulting Arrhenius plot remains approximately linear. Therefore, the thermal signature survives. For both widths, the linear (thermal) scaling survives *in the physically explored range* of $1/\sqrt{E_t} \leq 0.08$ ($E_t \geq 150$ MeV) shown by the dashed lines in Fig. 1. As we shall see below, the effect is weaker for even lower values of B which are commonly seen experimentally.

The divergent behavior manifests itself as well in the parameter m, the number of "throws" in the binomial description. Values of m_{app} are plotted (open circles) as a function of E_t in the right column of Fig. 1 for Γ_{E_t}/E_t=0.3 (top panel) and Γ_{E_t}/E_t=0.2 (bottom panel).

While the distortions depend mostly on the variance of the energy distribution, distributions with similar widths can be associated with very different variances. For instance, a Lorentzian distribution with finite Γ has infinite variance. Its use would lead to a divergence even for infinitely small values of Γ. Thus, even innocent trimmings to the (non-physical) tails of a Gaussian can produce big differences in the variance of the distribution and in the ensuing corrections. We exemplify this point in two ways.

a) We use a "square" distribution with a width equal to the full width at half maximum of the Gaussian. As can be seen by the star symbols of Fig. 1 this simple exercise dramatically extends the range over which the average can be performed safely.

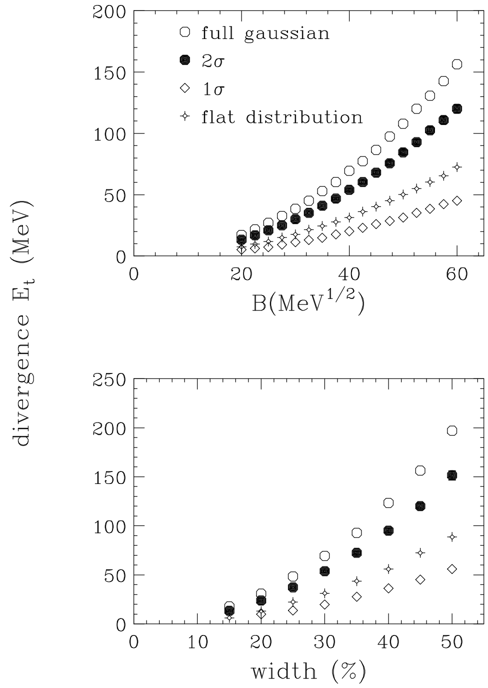

FIG. 2. Top panel: the divergence energy (the energy at which m_{app} and p_{app} change sign) as a function of the slope parameter B for a fixed ratio of Γ_{E_t}/E_t=0.3. Bottom panel: the divergence energy as a function of the width Γ_{E_t}/E_t for a fixed value of B=40MeV$^{1/2}$.

b) We truncate the tails of the Gaussian at 1σ (diamonds) and 2σ (solid circles) in Fig. 1. Already the cut at 2σ shows a dramatic improvement over a full Gaussian. The 1σ cut actually makes things even better than the square distribution (as seen in Fig. 1).

To illustrate the conditions under which the "thermal" scaling survives (i.e. linear Arrhenius plots as a function of $1/\sqrt{E_t}$), we have traced the evolution of the "divergence energy" (or the point at which m_{app} and p_{app} change sign) as a function of the two parameters which control the strength of the divergence: the slope parameter B and the variance σ^2 (hereafter characterized by its full width at half maximum value $\Gamma_{E_t} \approx 2.35\sqrt{\sigma^2}$).

A particular example for Γ_{E_t}/E_t=0.3 is shown by the open circles in the top panel of Fig. 2. In addition, values of the divergence energy for 1σ and 2σ truncations of the Gaussian as well as a square distribution are also plotted. For all intents and purposes, divergencies that occur at less than 100 MeV do not alter substantially the linear Arrhenius plots as they have been observed to date [1–3] in the E_t range of 150 to 1600 MeV.

In a similar manner, the dependence of the divergence energy can also be determined as a function of the relative width Γ_{E_t}/E_t (for a fixed value of B). This behavior is demonstrated in the bottom panel of Fig. 2.

A more global view of the parameter space is shown

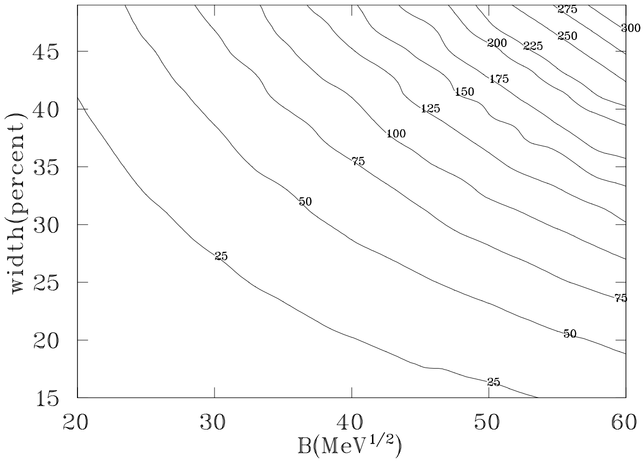

FIG. 3. The divergence energies (contour line) as a function of the width Γ_{E_t}/E_t and the slope B for m=12 and a Gaussian distribution truncated at 2σ.

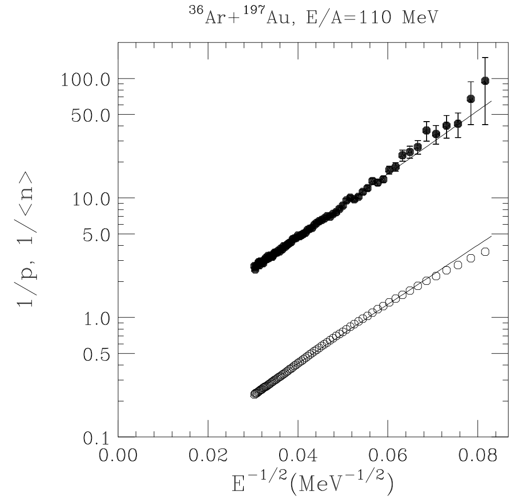

FIG. 4. The inverse of the single fragment emission probability (solid circles) and the inverse of the average fragment multiplicity (open circles) as a function of $1/\sqrt{E_t}$ for the reaction Ar+Au at E/A=110 MeV. The solid lines are linear fits to the data.

in Fig. 3 where the divergence energy is plotted (contour lines) as a function of the width Γ_{E_t}/E_t and slope B. The shape of the contour lines reflects the $\sigma^2 B$ scale deduced in Eqs. (3) and (4). The calculation in ref. [7] sits nearly in the upper right hand corner of the graph. But, as is clearly demonstrated, large regions exist where binomial reducibility and thermal scaling survive (roughly given by the region with divergence energies less than 100 MeV).

From the above exercises it is concluded that *there is abundant room for the survival of binomiality and thermal scaling.*

In this second part, we show that *none* of the symptoms of divergence are present in the available experimental data [1–3]. Furthermore, the average fragment multiplicity $\langle n \rangle$ is expected to be "distortion free" [7]. As such, it provides a baseline reference with which to compare the "distorted" variable, p_{app} (to verify whether the label "distorted" is appropriate). In addition, we can force the divergence to appear in the data, by artificially broadening the E_t bins, thus establishing that it is not present with ordinary (small) E_t bins. Finally, we show that *thermal scaling* is *present* and *persists* in the data even when the divergence is forced.

First, we draw attention again to the two pathologic features arising from excessive averaging. 1) The quantity m diverges near E_t=0. 2) The quantity $1/p$ suffers a corresponding discontinuity at the same low energy.

Inspection of the published data shows that:

1) *m never diverges near E_t = 0.* To the contrary m remains relatively constant or *actually decreases with decreasing E_t*. This is particularly true for all of the Xe induced reactions [2,3] (see Fig. 6);

2) $\log 1/p$ is nearly linear vs. $1/\sqrt{E_t}$ over the experimental E_t range without the indications of trouble suggested by the calculations in the previous section.

Thus the experimental data do not show any signs of pathological features.

The quantity $\langle n \rangle = m_{app} p_{app}$ does not suffer from the distortions due to averaging. In fact, $\langle n \rangle$ is a suitable alternative for constructing an Arrhenius plot in those cases where m depends only weakly on E_t (as observed in many of the data sets we have studied). A comparison of the Arrhenius plots constructed from $1/p_{app}$ and $1/\langle n \rangle$ is shown in Fig. 4. The striking feature of this comparison is that the $1/p_{app}$ values *have the same slope* as the "distortion-free" case of $1/\langle n \rangle$. Similar observations can be made for all the other reactions studied so far. *As a consequence both the "fragile" p and the "robust" $\langle mp \rangle$ survive the physical transformation $P(E, E_t)$ unscathed.*

When the probability becomes small, the binomial distribution reduces to a Poisson distribution. This can be achieved experimentally by limiting the selection to a single Z [4]. The observed average multiplicity is now experimentally equal to the variance. Thus we are in the Poisson reducibility regime and can check the thermal scaling directly on $\langle n_Z \rangle$. For a Poisson distribution, $\log \langle n_Z \rangle$ should scale linearly with $1/\sqrt{E_t}$. This can be seen experimentally for the average yield of individual elements of a given charge (see Fig. 5) for the reaction Ar+Au at E/A=110 MeV. For the case of a single species, the reducibility is Poissonian, and the thermal (linear) scaling with $1/\sqrt{E_t}$ is readily apparent. As pointed out at the outset of the paper, this evidence, together with that of Fig. 4 *indicates that no significant averaging is occurring even in the case of binomial decomposition.*

The data can be "encouraged" to demonstrate the sort of catastrophic failures described here. By widening the bins in transverse energy (ΔE_t), we can induce an artificial broadening to mimic a broad correlation between E and E_t. For example, the behavior of p_{app} and m_{app} is shown in Fig. 6 for three different widths and two different reactions. The divergencies of p_{app} and m_{app} are

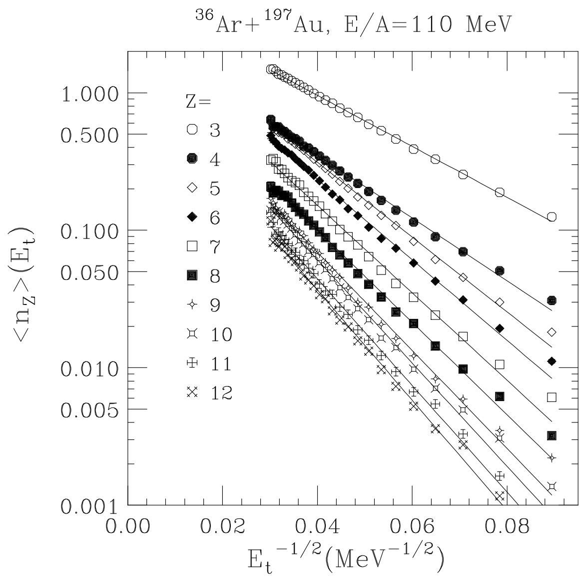

FIG. 5. The average yield per event of elements with Z between 3 and 12 as a function of $1/\sqrt{E_t}$. The lines are linear fits to the data.

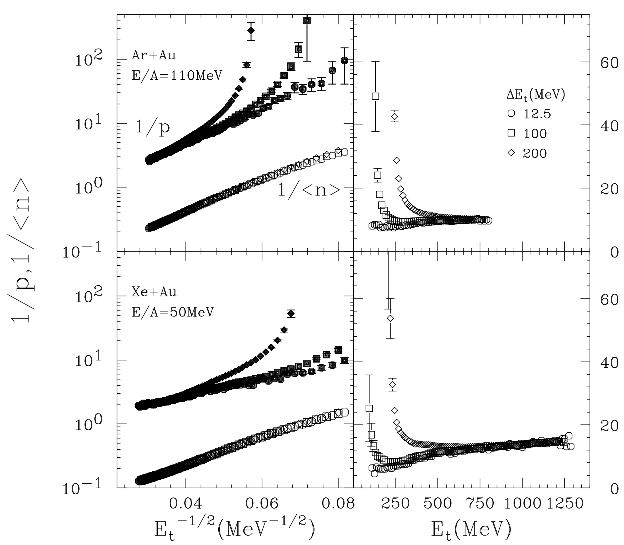

FIG. 6. Left panels: the inverse of the single fragment emission probability (solid symbols) and the inverse average fragment multiplicity (open symbols) as a function of $1/\sqrt{E_t}$ for the indicated bin widths ΔE_t and systems. Right panels: the observed values of m as a function of E_t for the indicated bin widths ΔE_t and systems

readily visible for large ΔE_t values, but are noticeably absent for small values. The spectacularly large binning in E_t (100 MeV!) necessary to force the anticipated pathologies to appear is reassuring indeed. *Notice that here the absolute width, not the relative width, was kept fixed even at the lowest energies!* Furthermore, the stability of $\langle n \rangle$ is readily apparent from the complete overlap of the values of $\langle n \rangle$ extracted for different windows of E_t (open symbols of Fig. 6).

In summary:

a) Binomial reducibility and the associated thermal scaling survive in a broad range of parameter space. The single case shown in [7] is an extreme one based on unsupported assumptions about the averaging function.

b) The experimentally observed simultaneous survival of the linear Arrhenius plot for parameter p and the robust average $\langle mp \rangle$ suggests that no serious damage is generated by the physical transformation $P(E, E_t)$.

c) The multiplicity distributions for any given Z value are Poissonian and the resulting average multiplicity $\langle n \rangle = \langle mp \rangle$ gives linear Arrhenius plots confirming the conclusion in b).

d) Finally, the data themselves do not show any indication of pathological behavior. This can be seen, for instance, by comparing the behavior of p with $\langle n \rangle$. The pathology can be forced upon the data by excessively widening the E_t bins. Even then, the thermal scaling survives in the average multiplicity.

Acknowledgments

This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Nuclear Physics Division of the US Department of Energy, under contract DE-AC03-76SF00098.

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