

On \mathbb{Q} -conic bundles

By

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Abstract

A \mathbb{Q} -conic bundle is a proper morphism from a threefold with only terminal singularities to a normal surface such that fibers are connected and the anti-canonical divisor is relatively ample. We study the structure of \mathbb{Q} -conic bundles near their singular fibers. One corollary to our main results is that the base surface of every \mathbb{Q} -conic bundle has only Du Val singularities of type A (a positive solution of a conjecture by Iskovskikh). We obtain the complete classification of \mathbb{Q} -conic bundles under the additional assumption that the singular fiber is irreducible and the base surface is singular.

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Communicated by

2000 Mathematics Subject Classification(s): 14J30, 14E35, 14E30

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§1. Introduction

In this paper we study the local structure of extremal contractions from threefolds to surfaces. Such contractions naturally appear in the birational classification of three-dimensional algebraic varieties of negative Kodaira dimension. More precisely, according to the minimal model program every algebraic projective threefold V with $\kappa(V) = -\infty$ is birationally equivalent to a \mathbb{Q} -factorial terminal threefold X having a K_X -negative extremal contraction to a lower dimensional variety Z . There are three cases:

- a) Z is a point and then X is a \mathbb{Q} -Fano variety with $\rho(X) = 1$,
- b) Z is a smooth curve and then X/Z is a del Pezzo fibration,
- c) Z is a normal surface and then X/Z is a rational curve fibration.

We study the last case.

(1.1) Definition. By a \mathbb{Q} -conic bundle we mean a projective morphism $f: X \rightarrow Z$ from an (algebraic or analytic) threefold to a surface that satisfies the following properties:

- (i) X is normal and has only terminal singularities,
- (ii) $f_*\mathcal{O}_X = \mathcal{O}_Z$,
- (iii) all fibers are one-dimensional,
- (iv) $-K_X$ is f -ample.

For $f: X \rightarrow Z$ as above and for a point $o \in Z$, we call the *analytic germ* $(X, f^{-1}(o)_{\text{red}})$ a \mathbb{Q} -conic bundle germ.

The easiest example of \mathbb{Q} -conic bundles is a *standard Gorenstein conic bundle*: Z is smooth and X is embedded in the projectivization $\mathbb{P}_Z(\mathcal{E})$ of a rank 3 vector bundle so that the fibers $X_z, z \in Z$ are conics in $\mathbb{P}_Z(\mathcal{E})_z$. More complicated examples can be constructed as quotients:

(1.1.1) Example-Definition (toroidal example). Consider the following action of μ_m on $\mathbb{P}_x^1 \times \mathbb{C}_{u,v}^2$:

$$(x; u, v) \longmapsto (\varepsilon x; \varepsilon^a u, \varepsilon^b v),$$

where ε is a primitive m -th root of unity and $\gcd(m, a) = \gcd(m, b) = 1$. Let $X := \mathbb{P}^1 \times \mathbb{C}^2 / \mu_m$, $Z := \mathbb{C}^2 / \mu_m$ and let $f: X \rightarrow Z$ be the natural projection. Since μ_m acts freely in codimension one, $-K_X$ is f -ample. Two fixed points on $\mathbb{P}^1 \times \mathbb{C}^2$ gives two cyclic quotient singularities of types $\frac{1}{m}(1, a, b)$ and $\frac{1}{m}(-1, a, b)$ on X . These points are terminal if and only if $a + b \equiv 0 \pmod{m}$. In this case, f is a \mathbb{Q} -conic bundle and the base surface Z has a Du Val singularity of type A_{m-1} . We say that a \mathbb{Q} -conic bundle germ is *toroidal* if it is biholomorphic to $f: (X, f^{-1}(0)_{\text{red}}) \rightarrow (Z, 0)$ above (with $a + b \equiv 0 \pmod{m}$).

Our first main result is a complete classification of \mathbb{Q} -conic bundle germs with irreducible central fiber under the assumption that the base surface is singular:

(1.2) Theorem. *Let $f: (X, C) \rightarrow (Z, o)$ be a \mathbb{Q} -conic bundle germ, where C is irreducible. Assume that (Z, o) is singular. Then one of the following holds:*

Cases where X is locally primitive.

(1.2.1) (X, C) is toroidal.

(1.2.2) (X, C) is biholomorphic to the quotient of the smooth \mathbb{Q} -conic bundle

$$X' = \{y_1^2 + uy_2^2 + vy_3^2 = 0\} \subset \mathbb{P}_{y_1, y_2, y_3}^2 \times \mathbb{C}_{u, v}^2 \longrightarrow \mathbb{C}_{u, v}^2$$

by μ_m -action:

$$(y_1, y_2, y_3, u, v) \longmapsto (\varepsilon^a y_1, \varepsilon^{-1} y_2, y_3, \varepsilon u, \varepsilon^{-1} v).$$

Here $m = 2a + 1$ is odd and ε is a primitive m -th root of unity. The singular locus of X consists of two cyclic quotient singularities of types $\frac{1}{m}(a, -1, 1)$ and $\frac{1}{m}(a + 1, 1, -1)$. The base surface \mathbb{C}^2 / μ_m has a singularity of type A_{m-1} .

Cases where X is not locally primitive. Let $P \in X$ be the imprimitive point and let m , s and \bar{m} be its index, splitting degree and subindex, respectively. In this case, P is the only non-Gorenstein point and X has at most one Gorenstein singular point. We refer to (5.3.1) for the definition of types (IA^\vee) – (IE^\vee) .

(1.2.3) (X, C) is of type (IE^\vee) at P , $s = 4$, $\bar{m} = 2$, (Z, o) is Du Val of type A_3 , X has a cyclic quotient singularity P of type $\frac{1}{8}(5, 1, 3)$ and has no other singular points. Furthermore, (X, C) is the quotient of the index-two \mathbb{Q} -conic bundle germ given by the following two equations in $\mathbb{P}(1, 1, 1, 2)_{y_1, \dots, y_4} \times \mathbb{C}_{u, v}^2$

$$\begin{aligned} y_1^2 - y_2^2 &= u\psi_1(y_1, \dots, y_4; u, v) + v\psi_2(y_1, \dots, y_4; u, v), \\ y_1 y_2 - y_3^2 &= u\psi_3(y_1, \dots, y_4; u, v) + v\psi_4(y_1, \dots, y_4; u, v) \end{aligned}$$

by μ_4 -action:

$$y_1 \mapsto -iy_1, \quad y_2 \mapsto iy_2, \quad y_3 \mapsto -y_3, \quad y_4 \mapsto iy_4, \quad u \mapsto iu, \quad v \mapsto -iv,$$

(see Example (7.7.1)).

(1.2.4) (X, C) is of type (ID^\vee) at P , $s = 2$, $\bar{m} = 1$, (Z, o) is Du Val of type A_1 , (X, C) is a quotient of a Gorenstein conic bundle given by the following equation in $\mathbb{P}_{y_1, y_2, y_3}^2 \times \mathbb{C}_{u, v}^2$

$$y_1^2 + y_2^2 + \psi(u, v)y_3^2 = 0, \quad \psi(u, v) \in \mathbb{C}\{u^2, v^2, uv\},$$

by μ_2 -action:

$$u \mapsto -u, \quad v \mapsto -v, \quad y_1 \mapsto -y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto y_3.$$

Here $\psi(u, v)$ has no multiple factors. In this case, (X, P) is the only singular point and it is of type $cA/2$ or $cAx/2$.

(1.2.5) (X, C) is of type (IA^\vee) at P , $\bar{m} = 2$, $s = 2$, (Z, o) is Du Val of type A_1 , (X, P) is a cyclic quotient singularity of type $\frac{1}{4}(1, 1, 3)$, and (X, C) is the quotient of the index-two \mathbb{Q} -conic bundle germ given by the following two equations in $\mathbb{P}(1, 1, 1, 2)_{y_1, \dots, y_4} \times \mathbb{C}_{u, v}^2$

$$\begin{aligned} y_1^2 - y_2^2 &= u\psi_1(y_1, \dots, y_4; u, v) + v\psi_2(y_1, \dots, y_4; u, v), \\ y_3^2 &= u\psi_3(y_1, \dots, y_4; u, v) + v\psi_4(y_1, \dots, y_4; u, v) \end{aligned}$$

by μ_2 -action:

$$y_1 \mapsto y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto y_3, \quad y_4 \mapsto -y_4, \quad u \mapsto -u, \quad v \mapsto -v$$

(see Example (7.7.2)).

(1.2.6) (X, C) is of type (II^\vee) at P , $\bar{m} = 2$, $s = 2$, (Z, o) is Du Val of type A_1 , (X, P) is a singularity of type $cAx/4$, and (X, C) is the quotient of the same form as in (1.2.5) (see Example (7.7.3)).

All the cases (1.2.1) – (1.2.6) occur.

By running MMP over the base Z we immediately obtain the following fact which was conjectured by Iskovskikh:

(1.2.7) Theorem. *Let $f: X \rightarrow Z$ be a \mathbb{Q} -conic bundle (possibly with reducible fibers). Then Z has only Du Val singularities of type A .*

We also note that the singularity (Z, o) is unbounded only in locally primitive cases (1.2.1) and (1.2.2). In all other cases (Z, o) is either of type A_1 or A_3 . Theorem (1.2.7) has important applications to rationality problem of conic bundles [Isk96].

Note that the condition that X has only terminal singularities is essential in Theorem (1.2.7): put in Example (1.1.1) $a = 1$ and $b = -2$ (m is odd). We get an extremal contraction having two singular points which are canonical Gorenstein of type $\frac{1}{m}(1, 1, -2)$ and terminal of type $\frac{1}{m}(-1, 1, -2)$. The base surface has a singularity of type $\frac{1}{m}(1, -2)$ which is not Du Val.

(1.2.8) Corollary. *If in notation of (1.2) the base (Z, o) is not of type A_1 , then X has only cyclic quotient singularities.*

In the case of smooth base our results are not so strong:

(1.3) Theorem. *Let $(X, C \simeq \mathbb{P}^1)$ be a \mathbb{Q} -conic bundle germ over a smooth base (Z, o) . Then (X, C) is locally primitive and the configuration of singular points is one of the following (notation (IA) – (III) are explained in (5.2.1)):*

(1.3.1) $\emptyset, \text{(III)}, \text{(III)+(III)}$ (X is Gorenstein).

(1.3.2) $\text{(IA)}, \text{(IA)+(III)}, \text{(IA)+(III)+(III)}$.

(1.3.3) $\text{(IIA)}, \text{(IIA)+(III)}$.

(1.3.4) $\text{(IC)}, \text{(IIB)}$.

(1.3.5) (IA)+(IA) of indices 2 and odd $m \geq 3$.

(1.3.6) (IA)+(IA)+(III) of indices 2, odd $m \geq 3$ and 1.

In contrast with Theorem (1.2) we can say nothing about the existence of \mathbb{Q} -conic bundles as in (1.3.3) – (1.3.6). There are examples of index-two \mathbb{Q} -conic bundles as in (1.3.2) (see [Pro97a, §3] and (12.1)). One can also easily construct examples of Gorenstein standard conic bundles of type (1.3.1).

(1.3.7) Proposition (Reid’s conjecture about general elephant. *Let $(X, C \simeq \mathbb{P}^1)$ be a \mathbb{Q} -conic bundle germ. Then, except possibly for cases (1.3.4), (1.3.5), and (1.3.6), a general member of $|-K_X|$ has only Du Val singularities. In these exceptional cases a general member of $D \in |-2K_X|$ does not contain C and the log divisor $K_X + \frac{1}{2}D$ is log terminal.*

Proposition (1.3.7) follows from Remarks (7.6.1), (8.6.1) and Theorem (10.10).

(1.4) Comments on the approach. Contractions similar to \mathbb{Q} -conic bundles were considered in [Mor88]. In fact, [Mor88] deals with birational contractions of threefolds $f: X \rightarrow (Z, o)$ such that X has only terminal singularities, $-K_X$ is f -ample, and $C := f^{-1}(o)_{\text{red}}$ is a curve. In this case, we have vanishings $R^1 f_* \mathcal{O}_X = 0$ and $R^1 f_* \omega_X = 0$. (cf. (2.3)). Though the former vanishing was used all over the places in [Mor88], the latter vanishing $R^1 f_* \omega_X = 0$ was used only occasionally. It is easy to find the places where the corollaries of $R^1 f_* \omega_X = 0$ were used. In this paper we follow the arguments of [Mor88] paying special attention to those corollaries of $R^1 f_* \omega_X = 0$ and furthermore give comments to modify the arguments when the corollaries are used.

Though fewer vanishing conditions are available, we have new tools Lemma (2.8) and Theorem (4.4) for \mathbb{Q} -conic bundles. These results together with [Mor88] form the basis of our approach.

§2. Preliminaries

(2.2) Let $f: (X, C) \rightarrow (Z, o)$ be a \mathbb{Q} -conic bundle germ. The following is an immediate consequence of the Kawamata-Viehweg vanishing theorem.

(2.3) Theorem. $R^i f_* \mathcal{O}_X = 0$ for $i > 0$.

(2.3.1) Corollary (cf. [Mor88, Remark 1.2.1, Cor. 1.3]).

- (i) If J be an ideal such that $\text{Supp } \mathcal{O}_X/J \subset C$, then $H^1(\mathcal{O}_X/J) = 0$.
- (ii) $p_a(C) = 0$ and C is a union of smooth rational curves.
- (iii) $\text{Pic } X \simeq H^2(C, \mathbb{Z}) \simeq \mathbb{Z}^\rho$, where ρ is the number of irreducible components of C .

(2.3.2) Remark. If C is reducible, then $\rho(X/Z) > 1$ and for every closed curve $C' \subsetneq C$ the germ (X, C') is an extremal neighborhood (isolated or divisorial). These were classified in [Mor88] and [KM92] under the condition that C' is irreducible.

(2.3.3) Remark. In general, we do not assume that X is \mathbb{Q} -factorial (i.e., a Weil divisor on X is not necessarily \mathbb{Q} -Cartier). In fact, the following are equivalent

- (i) X is \mathbb{Q} -factorial and $\rho(X/Z) = 1$,
- (ii) the preimage of an arbitrary irreducible curve $\Gamma \subset Z$ is also irreducible.

Indeed, the implication (i) \Rightarrow (ii) is obvious. To show (ii) \Rightarrow (i), consider a \mathbb{Q} -factorialization $Y \rightarrow X$ [Kaw88] and run the MMP over (Z, o) . If (i) does not hold, $\rho(Y/Z) > 1$. On the last step of the MMP we get a divisorial contraction $Y_{n-1} \rightarrow Y_n$ over (Z, o) . Let E be the corresponding exceptional divisor and let Γ be its image on Z . Then Γ is an irreducible curve and $f^{-1}(\Gamma)$ has two components.

(2.4) We need the following easy construction which is to be used throughout the paper. First, we claim that (Z, o) is a quotient singularity. Indeed, the general hyperplane section $H \subset X$ is smooth and the restriction $f_H: H \rightarrow Z$ is a finite morphism. Thus (Z, o) is a log terminal singularity [KM98, Prop. 5.20]. Therefore, (Z, o) is a quotient of a smooth germ (Z', o') by a finite group G which acts freely outside of o' [Kaw88, Th. 9.6]. Consider the base change

$$(2.4.1) \quad \begin{array}{ccc} (X', C') & \xrightarrow{g} & (X, C) \\ \downarrow f' & & \downarrow f \\ (Z', o') & \xrightarrow{h} & (Z, o) \end{array}$$

where X' is the normalization of $X \times_Z Z'$ and $C' := f'^{-1}(C)_{\text{red}}$. The group G naturally acts on X' so that $X = X'/G$. Since X has only isolated singularities, g is étale in codimension 2. Moreover, $K_{X'} = g^*K_X$ and singularities of X' are terminal. In particular, $f': (X', C') \rightarrow (Z', o')$ is a \mathbb{Q} -conic bundle germ.

(2.4.2) Corollary ([Cut88]). *Let $f: X \rightarrow Z$ be a \mathbb{Q} -conic bundle. If X is Gorenstein (and terminal), then Z is smooth and there is a vector bundle \mathcal{E} of rank 3 on Z and an embeddings $X \hookrightarrow \mathbb{P}(\mathcal{E})$ such that every scheme fiber X_z , $z \in Z$ is a conic in $\mathbb{P}(\mathcal{E})_z$.*

Sketch of the proof. The question is local, so we assume that $f: (X, C) \rightarrow (Z, o)$ is a \mathbb{Q} -conic bundle germ. If (Z, o) is smooth, the assertion can be proved in the standard way: f is flat because X is Cohen-Macaulay and we can put $\mathcal{E} = f_*\mathcal{O}_X(-K_X)$ (see, e.g., [Cut88]). Assume that (Z, o) is singular. Consider the base change (2.4.1). Then (Z', o') is smooth and $G \neq \{1\}$. Since X is Gorenstein terminal, the action of G on X' and C' is free. On the other hand, X' is also Gorenstein. By the above arguments f' is a standard Gorenstein conic bundle. In particular, the central fiber $C' := f'^{-1}(o')_{\text{red}}$ is a conic. If C'

is reducible, then the singular point $P' \in C'$ is G -invariant, a contradiction. Hence, $C' \simeq \mathbb{P}^1$. This contradicts the fact that the action of G on C' is free. \square

(2.5) Definition ([Mor88, (0.4.16), (1.7)]). Let (X, P) be a terminal 3-dimensional singularity of index m and let $C \subset X$ be a smooth curve passing through P . We say that C is (locally) *primitive* at P if the natural map

$$\varrho: \mathbb{Z} \simeq \pi_1(C \setminus \{P\}) \rightarrow \pi_1(X \setminus \{P\}) \simeq \mathbb{Z}/m\mathbb{Z}$$

is surjective and *imprimitive* at P otherwise. The order s of $\text{Coker } \varrho$ is called the *splitting degree* and the number $\bar{m} = m/s$ is called the *subindex* of $P \in C$.

It is easy to see that the splitting degree coincides with the number of irreducible components of the preimage C^\sharp of C under the index-one cover $X^\sharp \rightarrow X$ near P . If P is primitive, we put $s = 1$ and $\bar{m} = m$.

(2.6) From now on we assume that $f: (X, C) \rightarrow (Z, o)$ is a \mathbb{Q} -conic bundle germ with $C \simeq \mathbb{P}^1$. There are two cases (cf. [Mor88, (1.12)]):

(2.6.1) Case: C' is irreducible.

(2.6.2) Case: $C' = \cup_{i=1}^s C'_i$, where $s > 1$ and $C'_i \simeq \mathbb{P}^1$. In this case, G acts on $\{C'_i\}$ transitively.

(2.6.3) Claim. *In the case (2.6.2), all the irreducible components C'_i pass through one point P' and do not intersect each other elsewhere.*

Proof. Since G acts on $\{C'_i\}$ transitively and $p_a(C') = 0$, each component C'_i meets the closure of $C' - C'_i$ at one point. \square

(2.6.4) Proposition (cf. [Mor88, 1.11-1.13]). *Notation as in (2.4) and (2.6.3).*

- (i) *In the case (2.6.2), C is imprimitive at $g(P')$. Conversely, if C is imprimitive at some point P , then f is such as in (2.6.2) and $P = g(P')$. Moreover, the splitting degree s coincides with the number of irreducible components of C' .*
- (ii) *(X, C) has at most one imprimitive point.*

(2.7) Proposition ([Pro97a, Lemma 1.10]). *(Z, o) is a cyclic quotient singularity.*

Proof. It is sufficient to show that G is a cyclic group. In the case where X is locally primitive, G effectively acts on $C' \simeq \mathbb{P}^1$ and on the tangent space $T_{Z', o'} \simeq \mathbb{C}^2$. This gives us two embeddings: $G \subset PGL(2, \mathbb{C})$ and $G \subset GL(2, \mathbb{C})$. Assume that G is not cyclic. By the classification of finite subgroups in $PGL(2, \mathbb{C})$ G is either \mathfrak{A}_5 , \mathfrak{S}_4 , \mathfrak{A}_4 , or the dihedral group \mathfrak{D}_n of order $2n$ (see, e.g., [Spr77]). In all cases there are at least two different elements of order two in G . But then at least one of them is a reflection in $G \subset GL(2, \mathbb{C})$, a contradiction.

In the case where f is not locally primitive, by Claim (2.6.3), G has a fixed point $P' \in X'$. Let $P = g(P')$ and let $U \ni P$ be a small neighborhood. There is a surjection $\pi_1(U \setminus \{P\}) \rightarrow G$. On the other hand, $\pi_1(U \setminus \{P\})$ is cyclic [Kaw88, Lemma 5.1]. \square

Thus we may assume that $G = \mu_d$ and $Z \simeq \mathbb{C}^2/\mu_d$, where the action of μ_d on $\mathbb{C}^2 \simeq Z'$ is free outside of 0. We call this d the *topological index* of $f: (X, C) \rightarrow (Z, o)$.

Let $\text{Cl}^{\text{sc}} X$ be the subgroup of the divisor class group $\text{Cl} X$ consisting of Weil divisor classes which are \mathbb{Q} -Cartier.

(2.7.1) Corollary. $\pi_1(X \setminus \text{Sing } X) \simeq \mu_d$ and $\text{Cl}^{\text{sc}} X \simeq \mathbb{Z} \oplus \mathbb{Z}_d$, where $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$ and d is the topological index of f .

(2.7.2) Corollary. In the above notation, let P_1, \dots, P_l be all the non-Gorenstein points and let m_1, \dots, m_l be their indices (the case $l = 1$ is not excluded). Assume that P_2, \dots, P_l are primitive. Let s_1 and \bar{m}_1 be the splitting degree and the subindex of P_1 .

- (i) For each prime p the number of the m_i 's divisible by p is ≤ 2 .
- (ii) There is a \mathbb{Q} -Cartier Weil divisor D on X such that $D \cdot C = d/m_1 \cdots m_l$. Moreover, D generates $\text{Cl}^{\text{sc}} X / \text{Torsion} = \text{Cl}^{\text{sc}} X / \cong$.
- (iii) $\prod_{i=1}^l m_i = d \cdot \text{lcm}(\bar{m}_1, m_2, \dots, m_l)$.

Proof. Let H be an ample generator of $\text{Pic } X$ so that $H \cdot C = 1$. Clearly, the following sequence

$$(2.7.3) \quad 0 \longrightarrow \text{Pic } X \longrightarrow \text{Cl}^{\text{sc}} X \xrightarrow{\mathcal{S}} \bigoplus_i \text{Cl}^{\text{sc}}(X, P_i) \longrightarrow 0$$

is exact. Here $\text{Cl}^{\text{sc}}(X, P_i) \simeq \mathbb{Z}_{m_i}$ by [Kaw88, Lemma 5.1]. Then (i) immediately follows by (2.7.1).

Let us prove (ii). We have $\text{Cl}^{\text{sc}} X/\mathbb{Z}_d \simeq \mathbb{Z}$ by (2.7.1) and the order of $(\text{Cl}^{\text{sc}} X/\mathbb{Z}_d)/\text{Pic } X$ is $\frac{1}{d} \prod m_i$ by (2.7.3). Let D be an ample Weil divisor generating $\text{Cl}^{\text{sc}} X/\mathbb{Z}_d$. Since $H \cdot C = 1$, We have $\frac{1}{d} \prod m_i D \cdot C = H \cdot C = 1$. This proves (ii).

Finally, by [Mor88, 1.9, 1.7] the \mathbb{Z} -module $\text{Cl}^{\text{sc}} X$ is generated by H and some ample Weil divisors D_1, \dots, D_l with relations $m_i D_i - n_i H \sim 0$, where $n_i = m_i D_i \cdot C$, $\gcd(m_1, n_1) = s_1$, and $\gcd(m_i, n_i) = 1$ for $i = 2, \dots, l$. Now (iii) can be proved by considering the p -prime component of $\text{Cl}^{\text{sc}} X$ for each prime p . \square

(2.7.4) Corollary. *In the locally primitive case, μ_d has exactly two fixed points on $(X', C' \simeq \mathbb{P}^1)$. Therefore, there are two points on (X, C) whose indices are divisible by d . Conversely, if there are two primitive points on (X, C) whose indices divisible by r , then r divides d .*

(2.7.5) Corollary. *In the case (X, C) is imprimitive at $P = g(P')$, the splitting degree $s (> 1)$ divides d , and let $r := d/s$. Put $X^{\flat} := X'/\mu_r$, $Z^{\flat} := Z'/\mu_r$, and $C^{\flat} := C'/\mu_r$. We have the following decomposition:*

$$(2.7.6) \quad \begin{array}{ccccc} (X', C') & \xrightarrow{g''} & (X^{\flat}, C^{\flat}) & \xrightarrow{g^{\flat}} & (X, C) \\ f' \downarrow & & f^{\flat} \downarrow & & f \downarrow \\ (Z', o') & \xrightarrow{h'} & (Z^{\flat}, o^{\flat}) & \xrightarrow{h^{\flat}} & (Z, o) \end{array}$$

and the following hold:

- (i) *The group μ_r does not permute components of C' , so C^{\flat} has exactly s irreducible components C_i^{\flat} passing through one point $P^{\flat} = g''(P')$. The group $\mu_s = \mu_d/\mu_r$ naturally acts on X^{\flat} so that $X = X^{\flat}/\mu_r$.*
- (ii) *If $d > s$, then μ_r has two fixed points on each component $C'_i \subset C'$, P' and $Q'_i \neq P'$.*
- (iii) *(X^{\flat}, C_i^{\flat}) is a locally primitive extremal neighborhood (2.3.2), $X^{\flat} \rightarrow X$ is étale outside P^{\flat} and $C_i^{\flat} \rightarrow C$ is an isomorphism.*

The base change g^{\flat} as in (2.7.6) is called the *splitting cover* [Mor88, 1.12.1].

Proof. Let $G \subset \mu_d$ be the stabilizer of some component $C'_i \subset C'$. Then $G = \mu_r$ and $X^{\flat} := X'/G$ satisfies the desired properties. \square

(2.8) Lemma. *Let (X, C) be a \mathbb{Q} -conic bundle germ with $C \simeq \mathbb{P}^1$. Let d be the topological index of (X, C) and let m_1, \dots, m_r be indices of all the non-Gorenstein points. Assume that f is not toroidal. Then*

$$-K_X \cdot C = d/m_1 \cdots m_r.$$

Proof. Take D as in (ii) of Corollary (2.7.2). Then $-K_X \equiv rD$ for some $r \in \mathbb{Z}_{>0}$. We claim that $r = 1$. Indeed, for the general fiber L we have $2 = -K_X \cdot L = rD \cdot L$. Since $D \cdot L$ is an integer, $r = 1$ or 2 . If $r = 2$, then $D \cdot L = 1$, i.e., D is f -ample with $\deg = 1$ on the general fiber. Apply construction (2.4.1). Then $D' := f'^*D$ satisfies the same property: it is f' -ample with $\deg = 1$ on the general fiber. Since $X' \setminus f'^{-1}(o') \rightarrow Z' \setminus \{o'\}$ is a standard conic bundle (see (2.4.2)), this implies that all the fibers over $Z' \setminus \{o'\}$ are smooth rational curves. In particular, the morphism f' is smooth outside of C' . We claim that f' is smooth everywhere. Denote $\mathcal{F} := \mathcal{O}_{X'}(D')$. Then locally near a singular point $P' \in X'$, \mathcal{F} is a direct summand of $\pi_*\mathcal{F}^\sharp$, where $\pi: (X^\sharp, P^\sharp) \rightarrow (X', P')$ is the index-one cover and \mathcal{F}^\sharp is the lifting of \mathcal{F} . Since \mathcal{F}^\sharp is Cohen-Macaulay and Z' is smooth, \mathcal{F} is flat over Z' . Then by the base change theorem ([Mum66, Lect. 7, (iii), p. 51]) $f'_*\mathcal{F}$ is locally free. Put $\hat{X} := \mathbb{P}(f'_*\mathcal{F})$ with natural projection $\hat{f}: \hat{X} \rightarrow Z'$. We have a bimeromorphic map $\hat{X} \dashrightarrow X'$ over Z' that induces an isomorphism $(\hat{X} \setminus \hat{C}) \simeq (X' \setminus C')$, where $\hat{C} = \hat{f}^{-1}(o)$. Since f', \hat{f} are projective and $\rho(X'/Z') = \rho(\hat{X}/Z') = 1$, we have $\hat{X} \simeq X'$. But then X' is smooth and so is the morphism f' . This proves our claim.

Thus we may assume $X' \simeq Z' \times \mathbb{P}^1$. Recall that X is the quotient of X' by μ_d . By [Pro97a, §2] the action of μ_d is as in (1.1.1) and f is toroidal, a contradiction to our assumption. Therefore, $r = 1$ and $-K_X \cdot C = D \cdot C$. This proves our equality. \square

(2.8.1) Corollary. *If X has a unique non-Gorenstein point which is imprimitive of splitting degree s and subindex \bar{m} , then $2\bar{m} \equiv 0 \pmod{s}$.*

Proof. Let $f'^{-1}(o')$ be the scheme fiber. Then $f'^{-1}(o') \equiv rC'$ for some $r \in \mathbb{Z}_{>0}$. Thus $2 = -K_{X'} \cdot f'^{-1}(o') = -rK_{X'} \cdot C' = -rsK_X \cdot C = rs/\bar{m}$. This proves our statement. \square

The following fact will be used freely.

(2.9) Proposition ([Pro97a, Th. 2.4]). *In notation of (2.4.1) assume that X' is Gorenstein (we do not assume that C is irreducible). Assume further that $d > 1$. Then (X, C) is in one of the cases (1.2.1), (1.2.2), (1.2.4).*

Sketch of the proof. By (2.4.2) there is a μ_d -equivariant embedding $X' \hookrightarrow \mathbb{P}^2 \times Z'$ over Z' . Then one can choose a suitable coordinate system in \mathbb{P}^2 and $Z' \simeq \mathbb{C}^2$. \square

§3. Numerical invariants i_P , w_P and w_P^*

For convenience of the reader we recall some basic notation of [Mor88].

(3.1) Let X be an analytic threefold with terminal singularities and let $C \subset X$ be a reduced curve. Let $I_C \subset \mathcal{O}_X$ be the ideal sheaf of C and let $I_C^{(n)}$ be the symbolic n th power of I_C , that is, the saturation of I_C^n in \mathcal{O}_X . Put $\text{gr}_C^n \mathcal{O} := I_C^{(n)} / I_C^{(n+1)}$. Further, let $F^n \omega_X$ be the saturation of $I_C^n \omega_X$ in ω_X and let $\text{gr}_C^n \omega := F^n \omega_X / F^{n+1} \omega_X$. Let m be the index of K_X . There are natural homomorphisms

$$\begin{aligned} \alpha_1 &: \bigwedge^2 \text{gr}_C^1 \mathcal{O} && \longrightarrow \text{Hom}_{\mathcal{O}_C}(\Omega_C^1, \text{gr}_C^0 \omega), \\ \alpha_n &: S^n \text{gr}_C^1 \mathcal{O} && \longrightarrow \text{gr}_C^n \mathcal{O}, \quad n \geq 2, \\ \beta_0 &: (\text{gr}_C^0 \omega)^{\otimes m} && \longrightarrow (\omega_X^{\otimes m})^{**} \otimes \mathcal{O}_C, \\ \beta_n &: \text{gr}_C^0 \omega \otimes S^n \text{gr}_C^1 \mathcal{O} && \longrightarrow \text{gr}_C^n \omega, \quad n \geq 1, \end{aligned}$$

where M^* for an \mathcal{O}_X -module M denotes its dual, $\text{Hom} \mathcal{O}_X(M, \mathcal{O}_X)$. Denote

$$\begin{aligned} i_P(n) &:= \text{len}_P \text{Coker } \alpha_n, & w_P(0) &:= \text{len}_P \text{Coker } \beta_0 / m, \\ w_P(n) &:= \text{len}_P \text{Coker } \beta_n, & w_P^*(n) &:= \binom{n+1}{2} i_P(1) - w_P(n), \quad n \geq 1. \end{aligned}$$

Assume that $C \simeq \mathbb{P}^1$. Then we have by [Mor88, 2.3.1]

$$(3.1.1) \quad -\deg \text{gr}_C^0 \omega = -K_X \cdot C + \sum_P w_P(0)$$

$$(3.1.2) \quad 2 + \deg \text{gr}_C^0 \omega - \deg \text{gr}_C^1 \mathcal{O} = \sum_P i_P(1).$$

$$(3.1.3) \quad \deg \text{gr}_C^n \mathcal{O} = \frac{1}{2} n(n+1) \deg \text{gr}_C^1 \mathcal{O} + \sum_P i_P(n), \quad n \geq 2,$$

and therefore the following corollaries to $\text{rk} \text{gr}_C^i \mathcal{O} = i+1$ and $R^1 f_* \mathcal{O}_X = 0$:

$$(3.1.4) \quad \sum_{i=1}^n (\deg \text{gr}_C^i \mathcal{O} + i + 1) \geq 0, \quad n \geq 1,$$

$$(3.1.5) \quad 4 \geq -\deg \operatorname{gr}_C^0 \omega + \sum_P i_P(1) = -K_X \cdot C + \sum_P w_P(0) + \sum_P i_P(1).$$

(3.1.6) Remark. In the case of extremal neighborhoods by the Grauert-Riemenschneider vanishing one has $\operatorname{gr}_C^0 \omega = \mathcal{O}_C(-1)$ (see [Mor88, 2.3]). This is no longer true for \mathbb{Q} -conic bundles: in Example (1.1.1) easy computations show $\deg \operatorname{gr}_C^0 \omega = -2$ (see (3.1.1)). Similarly, in (1.2.4) we also have $\deg \operatorname{gr}_C^0 \omega = -2$. We will show below that these two examples are the only exceptions (see Corollaries (4.4.3) and (7.2.2)).

(3.1.7) Lemma. *If $\operatorname{gr}_C^0 \omega = \mathcal{O}(-1)$, then*

$$(3.1.8) \quad \deg \operatorname{gr}_C^n \omega = \frac{1}{2}(n+1)(n-2) - \sum_P w_P^*(n), \quad n \geq 1.$$

If furthermore $H^1(\omega_X/F^{n+1}\omega_X) = 0$, then

$$(3.1.9) \quad \sum_{i=1}^n (\deg \operatorname{gr}_C^i \omega + i + 1) \geq 0, \quad n \geq 1.$$

Proof. Follows by the exact sequences

$$0 \longrightarrow \operatorname{gr}_C^i \omega \longrightarrow \omega_X/F^{i+1}\omega_X \longrightarrow \omega_X/F^i\omega_X \longrightarrow 0$$

(see [Mor88, 2.3]). □

(3.1.10) Lemma ([Mor88, 2.15]). *If (X, P) is singular, then $i_P(1) \geq 1$.*

Proof. The proof of [Mor88, 2.15] applies because it uses only local computations near P that are not based on $R^1 f_* \omega_X$. □

(3.1.11) Corollary. *A \mathbb{Q} -conic bundle germ $(X, C \simeq \mathbb{P}^1)$ has at most three singular points.*

§4. Sheaves $\operatorname{gr}_C^n \omega$

(4.1) Lemma. *Let $f: X \rightarrow Z$ be a \mathbb{Q} -conic bundle. Assume that the base surface Z is smooth. Then there is a canonical isomorphism $R^1 f_* \omega_X \simeq \omega_Z$.*

Proof. Let $g: W \rightarrow X$ be a resolution. By [Kol86, Prop. 7.6] we have $R^1(f \circ g)_* \omega_W = \omega_Z$. Since X has only terminal singularities, $g_* \omega_W = \omega_X$ and

by the Grauert-Riemenschneider vanishing, $R^i g_* \omega_W = 0$ for $i > 0$. Then the Leray spectral sequence gives us $R^1 f_* \omega_X = R^1 (f \circ g)_* \omega_W = \omega_Z$. \square

For convenience of the reader we recall basic definitions [Mor88, 8.8].

(4.2) Let (X, P) be three-dimensional terminal singularity of index m and let $\pi: (X^\sharp, P^\sharp) \rightarrow (X, P)$ be the index-one cover. Let \mathcal{L} be a coherent sheaf on X without submodules of finite length > 0 . An ℓ -structure of \mathcal{L} at P is a coherent sheaf \mathcal{L}^\sharp on X^\sharp without submodules of finite length > 0 with μ_m -action endowed with an isomorphism $(\mathcal{L}^\sharp)^{\mu_m} \simeq \mathcal{L}$. An ℓ -basis of \mathcal{L} at P is a collection of μ_m -semi-invariants $s_1^\sharp, \dots, s_r^\sharp \in \mathcal{L}^\sharp$ generating \mathcal{L}^\sharp as an \mathcal{O}_{X^\sharp} -module at P^\sharp . Let Y be a closed subscheme of X such that $P \notin \text{Ass } \mathcal{O}_Y$ and let $Y^\sharp \subset X^\sharp$ be the canonical lifting. Note that \mathcal{L} is an \mathcal{O}_Y -module if and only if \mathcal{L}^\sharp is an \mathcal{O}_{Y^\sharp} -module. We say that \mathcal{L} is ℓ -free \mathcal{O}_Y -module at P if \mathcal{L}^\sharp is a free \mathcal{O}_{Y^\sharp} -module at P^\sharp . If \mathcal{L} is ℓ -free \mathcal{O}_Y -module at P , then an ℓ -basis of \mathcal{L} at P is said to be ℓ -free if it is a free \mathcal{O}_{Y^\sharp} -basis.

Let \mathcal{L} and \mathcal{M} be \mathcal{O}_Y -modules at P with ℓ -structures $\mathcal{L} \subset \mathcal{L}^\sharp$ and $\mathcal{M} \subset \mathcal{M}^\sharp$. Define the following operations:

- $\mathcal{L} \tilde{\oplus} \mathcal{M} \subset (\mathcal{L} \oplus \mathcal{M})^\sharp$ is an \mathcal{O}_Y -module at P with ℓ -structure

$$(\mathcal{L} \tilde{\oplus} \mathcal{M})^\sharp = \mathcal{L}^\sharp \oplus \mathcal{M}^\sharp.$$

- $\mathcal{L} \tilde{\otimes} \mathcal{M} \subset (\mathcal{L} \otimes \mathcal{M})^\sharp$ is an \mathcal{O}_Y -module at P with ℓ -structure

$$(\mathcal{L} \tilde{\otimes} \mathcal{M})^\sharp = (\mathcal{L}^\sharp \otimes_{\mathcal{O}_{X^\sharp}} \mathcal{M}^\sharp) / \text{Sat}_{\mathcal{L}^\sharp \otimes \mathcal{M}^\sharp}(0),$$

where $\text{Sat}_{\mathcal{F}_1} \mathcal{F}_2$ is the saturation of \mathcal{F}_2 in \mathcal{F}_1 .

These operations satisfy standard properties (see [Mor88, 8.8.4]). If X is an analytic threefold with terminal singularities and Y is a closed subscheme of X , then the above local definitions of $\tilde{\oplus}$ and $\tilde{\otimes}$ patch with corresponding operations on $X \setminus \text{Sing } X$. Therefore, they give well-defined operations of global \mathcal{O}_Y -modules.

(4.3) Let $f: (X, C) \rightarrow (Z, o)$ be a \mathbb{Q} -conic bundle germ (we do not assume that C is irreducible).

(4.4) Theorem. *Assume that (Z, o) is smooth. Let $J \subset \mathcal{O}_X$ be an ideal such that $\text{Supp } \mathcal{O}_X/J \subset C$ and \mathcal{O}_X/J has no embedded components. Assume that $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J) \neq 0$. Then $\text{Spec}_X \mathcal{O}_X/J \supset f^{-1}(o)$, where $f^{-1}(o)$ is the scheme fiber (in other words, $J \subset \mathfrak{m}_{Z,o} \mathcal{O}_X$, where $\mathfrak{m}_{Z,o} \subset \mathcal{O}_Z$ is the maximal ideal of o).*

Proof. First we assume that $\text{Spec}_X \mathcal{O}_X/J \subsetneq f^{-1}(o)$. Denote $\Gamma := f^{-1}(o)$ and $V := \text{Spec} \mathcal{O}_X/J$. Then $\omega_\Gamma \simeq \omega_X \otimes \mathcal{O}_\Gamma$, and so $\omega_X \tilde{\otimes} \mathcal{O}_V \simeq \omega_\Gamma \tilde{\otimes} \mathcal{O}_V$ in this case.

Let \mathcal{I}_V be the ideal sheaf of V in Γ where we note $\mathcal{I}_V \neq 0$ by $V \subsetneq f^{-1}(o)$, and let \mathcal{I}_D be an associated prime of \mathcal{I}_V (i.e. $\mathcal{I}_D \in \text{Ass}(\mathcal{I}_V)$), and let $D \subset C$ be the corresponding irreducible component. By the Serre duality, we have

$$\omega_D = \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \omega_\Gamma) = \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma) \tilde{\otimes} \omega_\Gamma.$$

Hence $\mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma)$ is a torsion-free \mathcal{O}_D -module of rank 1. We also see $\mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{I}_V) \neq 0$ by $\mathcal{I}_D \in \text{Ass}(\mathcal{I}_V)$. Thus the cokernel of the inclusion

$$0 \neq \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{I}_V) \hookrightarrow \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma).$$

is of finite length and is a submodule of $\mathcal{O}_V = \mathcal{O}_\Gamma/\mathcal{I}_V$. Since $\mathcal{O}_V = \mathcal{O}_\Gamma/\mathcal{I}_V$ has no embedded primes, we have $\mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{I}_V) = \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma)$ and

$$\mathcal{I}_V = \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_\Gamma, \mathcal{I}_V) \supset \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma).$$

Considering the trace map one can see that $\mathbb{C} \simeq H^1(\omega_D) \rightarrow H^1(\omega_\Gamma)$ is an injection (and moreover $H^1(\omega_D) \simeq H^1(\omega_\Gamma) \simeq \mathbb{C}$). Since ω_Γ is ℓ -invertible, the composition map

$$v: H^1(\omega_D) \xrightarrow{\sim} H^1(\omega_\Gamma) \rightarrow H^1(\omega_\Gamma \tilde{\otimes} (\mathcal{O}_\Gamma/\mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma)))$$

is zero.

On the other hand, $\omega_\Gamma \rightarrow \omega_X \tilde{\otimes} \mathcal{O}_V$ has the following decomposition

$$\omega_\Gamma \rightarrow \omega_\Gamma \tilde{\otimes} (\mathcal{O}_\Gamma/\mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma)) \rightarrow \omega_\Gamma \tilde{\otimes} \mathcal{O}_\Gamma/\mathcal{I}_V \simeq \omega_X \tilde{\otimes} \mathcal{O}_V,$$

and the induced surjective map

$$H^1(\omega_\Gamma) \longrightarrow H^1(\omega_\Gamma \tilde{\otimes} \mathcal{O}_V) \neq 0$$

factors through v which is zero, a contradiction.

This proves that $\text{Spec}_X \mathcal{O}_X/J = f^{-1}(o)$ if $\text{Spec}_X \mathcal{O}_X/J \subset f^{-1}(o)$.

Now we treat the general case. By Nakayama's lemma $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes_{\mathcal{O}_Z} \mathcal{O}_Z/\mathfrak{m}_{Z,o} \neq 0$. Since H^1 is right exact for \mathcal{O}_X -sheaves, we see that

$$H^1((\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes_{\mathcal{O}_X/\mathfrak{m}_{Z,o}} \mathcal{O}_X) \simeq H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes_{\mathcal{O}_Z} \mathcal{O}_Z/\mathfrak{m}_{Z,o} \neq 0.$$

Let us consider the homomorphism

$$(\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes_{\mathcal{O}_X/\mathfrak{m}_{Z,o}} \mathcal{O}_X \rightarrow \omega_X \tilde{\otimes} \mathcal{O}_X/J^s,$$

where J^s is the saturation of $J + \mathfrak{m}_{Z,o}\mathcal{O}_X$ in \mathcal{O}_X . It is surjective and its kernel is supported at a finite number of points. Thus

$$H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J^s) \simeq H^1((\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes \mathcal{O}_X/\mathfrak{m}_{Z,o}\mathcal{O}_X) \neq 0$$

and $J^s \supset \mathfrak{m}_{Z,o}\mathcal{O}_X$. By the special case treated above we have $J + \mathfrak{m}_{Z,o}\mathcal{O}_X \subset J^s = \mathfrak{m}_{Z,o}\mathcal{O}_X$, i.e., $J \subset \mathfrak{m}_{Z,o}\mathcal{O}_X$. □

(4.4.1) Corollary. *Assume that (Z, o) is smooth. If $H^1(\mathrm{gr}_C^0 \omega) \neq 0$, then $C = f^{-1}(o)$.*

Proof. Apply Theorem (4.4) with $J = I_C$. □

(4.4.2) Lemma ([Kol99, Prop. 4.2]). *If X is not Gorenstein, then X has index > 1 at all singular points of C .*

Proof. If C has at least three irreducible components, the assertion follows by Remark (2.3.2) and [Mor88, Cor. 1.15]. Thus we assume that $C = C_1 \cup C_2$ and X is Gorenstein at $P \in C_1 \cap C_2$. First we consider the case when (Z, o) is smooth. By our assumption $\mathrm{gr}_C^0 \omega = \omega_X \otimes \mathcal{O}_C$ is invertible at P . Consider the injection $\varphi: \mathrm{gr}_C^0 \omega \hookrightarrow \mathrm{gr}_{C_1}^0 \omega \oplus \mathrm{gr}_{C_2}^0 \omega$. Recall that (X, C_i) is an extremal neighborhood by Remark (2.3.2). Then by [Mor88, Prop. 1.14] $\mathrm{gr}_{C_i}^0 \omega = \mathcal{O}_{C_i}(-1)$, so $H^0(\mathrm{Coker} \varphi) = H^1(\mathrm{gr}_C^0 \omega)$. On the other hand, $\mathrm{Coker} \varphi$ is a sheaf of finite length supported at P . Since $\mathrm{gr}_C^0 \omega$ is invertible, $\mathrm{Coker} \varphi$ is non-trivial. So, $H^1(\mathrm{gr}_C^0 \omega) \neq 0$ and by Corollary (4.4.1) $C_1 \cup C_2 = f^{-1}(o)$. Thus X is smooth outside of $\mathrm{Sing} C$. Since P is the only singular point of C by Corollary (2.3.1), we are done.

Now we assume that (Z, o) is singular. Consider the base change (2.4.1). Since X is Gorenstein terminal at P , so is X' at all the points $P'_i \in g^{-1}(P)$. Moreover, g is étale over P . Hence, the central curve C' is singular at P'_i . By the above, X' is Gorenstein and by Corollary (2.4.2) $f': X' \rightarrow Z'$ is a standard Gorenstein conic bundle. In particular, C' is a plane conic. Since the set $g^{-1}(P)$ is contained in the singular locus of C' , it consists of one point, a contradiction. □

(4.4.3) Corollary (cf. [Mor88, Prop. 1.14]). *Assume that C is irreducible. If $\mathrm{gr}_C^0 \omega \not\cong \mathcal{O}_C(-1)$, then in notation of (2.4.1) we have $f'^{-1}(o') = C'$. If furthermore (X, C) is locally primitive, then it is toroidal.*

Proof. Let m be the index of X . Since there is an injection $(\mathrm{gr}_C^0 \omega)^{\otimes m} \hookrightarrow \mathcal{O}_C(mK_X)$, $\deg \mathrm{gr}_C^0 \omega < 0$. Since $\mathrm{gr}_C^0 \omega \not\cong \mathcal{O}_C(-1)$, $H^1(\mathrm{gr}_C^0 \omega) \neq 0$. In notation of (2.4.1) we have $H^1(\mathrm{gr}_{C'}^0 \omega) \neq 0$ (because $H^1(\mathrm{gr}_C^0 \omega) = H^1(\mathrm{gr}_{C'}^0 \omega) \mu_d$).

By Corollary (4.4.1) $C' = f'^{-1}(o')$. If f is locally primitive, C' is irreducible (see (2.6.1)). So $C' \simeq \mathbb{P}^1$ and X' is smooth. Up to analytic isomorphism we may assume that $X' \simeq Z' \times \mathbb{P}^1$. Then in some coordinate system the action of μ_d on X' is as in (1.1.1) (see [Pro97a, §2]), so f is toroidal. \square

(4.4.4) Remark. In notation of Theorem (4.4) assume that the map $H^0(I_C) \rightarrow H^0(I_C/J)$ is zero. Then $\mathrm{Spec}_X \mathcal{O}_X/J \subset f^{-1}(o)$. Therefore, the nonvanishing $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J) \neq 0$ implies $\mathrm{Spec}_X \mathcal{O}_X/J = f^{-1}(o)$.

(4.4.5) Corollary. *Notation as in (4.3). Assume that (Z, o) is smooth. If the map $H^0(I_C) \rightarrow H^0(\mathrm{gr}_C^1 \mathcal{O})$ is zero, then $H^1(\mathrm{gr}_C^1 \omega) = 0$.*

Proof. Assume that $H^1(\mathrm{gr}_C^1 \omega) \neq 0$. In notation of Theorem (4.4), put $J = I_C^{(2)}$ and $V := \mathrm{Spec}_X \mathcal{O}_X/I_C^{(2)}$. From the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{gr}_C^1 \mathcal{O} \tilde{\otimes} \omega_X & \longrightarrow & \mathcal{O}_X/I_C^{(2)} \tilde{\otimes} \omega_X & \longrightarrow & \mathcal{O}_C \tilde{\otimes} \omega_X \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & \mathrm{gr}_C^1 \omega & & \mathcal{O}_V \tilde{\otimes} \omega_X & & \mathrm{gr}_C^0 \omega \end{array}$$

and $\deg \mathrm{gr}_{C_i}^0 \omega < 0$ for each i we get $H^1(\mathcal{O}_V \tilde{\otimes} \omega_X) \neq 0$. Then by Theorem (4.4) and Remark (4.4.4) $V = f^{-1}(o)$. Let $P \in C$ be a general point. Then in a suitable coordinate system (x, y, z) near P we may assume that C is the z -axis. So, $I_C = (x, y)$ and $I_C^{(2)} = (x^2, xy, y^2)$. But then V is not a local complete intersection near P , a contradiction. \square

(4.4.6) Corollary. *Assume that (Z, o) is smooth and C is irreducible. If $\sum_P i_P(1) \geq 3$, then $\sum w_P^*(1) \leq 1$.*

Proof. By (3.1.5) $\mathrm{gr}_C^0 \omega = \mathcal{O}(-1)$. Further, by (3.1.2) $\deg \mathrm{gr}_C^1 \mathcal{O} \leq -2$. Hence, $H^0(\mathrm{gr}_C^1 \mathcal{O}) = 0$ (cf. [Mor88, Remark 2.3.4]). Now the desired inequality follows by Corollary (4.4.5) and (3.1.8). \square

§5. Preliminary classification of singular points

(5.1) Notation. Let $f: (X, C \simeq \mathbb{P}^1) \rightarrow (Z, o)$ be a \mathbb{Q} -conic bundle germ. Let $P \in C$ be a point of index $m \geq 1$. Let s and \bar{m} be the splitting degree

and subindex, respectively. Thus $m = s\bar{m}$. Consider the canonical μ_m -cover $\pi: (X^\sharp, P^\sharp) \rightarrow (X, P)$ and let $C^\sharp := \pi^{-1}(C)$. Take normalized ℓ -coordinates (x_1, \dots, x_4) and t and let ϕ be an ℓ -equation of $X \supset C \ni P$ (see [Mor88, 2.6]). Put $a_i = \text{ord } x_i$.

Note that $a_i < \infty$ and $\text{wt } x_i \equiv a_i \pmod{\bar{m}}$. If $m = 1$, then $X = X^\sharp$. In this case, P is said to be of type (III).

(5.2) Primitive point. Consider the case when P is primitive and $m > 1$. Then $s = 1$ and $\bar{m} = m$. In this case, the classification coincides with that in [Mor88] as shown next:

(5.2.1) Proposition (cf. [Mor88, Prop. 4.2]). *Let P and m be as above. Modulo permutations of x_i 's, the semigroup $\text{ord } C^\sharp$ is generated by a_1 and a_2 . Moreover, exactly one of the following holds:*

- (IA) $a_1 + a_3 \equiv 0 \pmod{m}$, $a_4 = m$, $m \in \mathbb{Z}_{>0}a_1 + \mathbb{Z}_{>0}a_2$, where we may still permute x_1 and x_3 if $a_2 = 1$,
- (IB) $a_1 + a_3 \equiv 0 \pmod{m}$, $a_2 = m$, $2 \leq a_1$,
- (IC) $a_1 + a_2 = a_3 = m$, $a_4 \not\equiv a_1, a_2 \pmod{m}$, $2 \leq a_1 < a_2$,
- (IIA) $m = 4$, P is of type $cAx/4$, and $\text{ord } x = (1, 1, 3, 2)$,
- (IIB) $m = 4$, P is of type $cAx/4$, and $\text{ord } x = (3, 2, 5, 5)$.

Proof. If X has an imprimitive point ($\neq Q$), then P is as classified in [Mor88, Prop.4.2] by (2.7.5), (iii). So we can assume that X has no imprimitive points. If (X, C) is toroidal, then at both singular points $\text{ord } x = (1, a, m - 1, m)$. So, these points are of type (IA). Taking Corollary (4.4.3) into account, we may assume that $\text{gr}_C^0 \omega = \mathcal{O}_C(-1)$. By (3.1.1) and (3.1.5) we have $w_P(0) < 1$ and $i_P(1) \leq 3$. We claim that C^\sharp is planar, i.e., $\text{ord } C^\sharp$ is generated by two elements. Indeed, in the contrary case by [Mor88, Lemma 3.4] we have $i_P(1) = 3$. Hence P is the only singular point (see (3.1.5)) and (Z, o) is smooth (Corollary (2.7.4)). By Corollary (4.4.6) $w_P^*(1) \leq 1$. In this case, arguments of [Mor88, 3.5] work. This shows that C^\sharp is planar. Now we can apply [Mor88, Proof of 4.2] and obtain the above classification. \square

(5.3) Imprimitive point. Now assume that P is imprimitive. Then in diagram (2.4.1) the central fiber C' has exactly s (> 1) irreducible components. Note that the classification is different from that in [Mor88] only in the case $C' = f'^{-1}(o')$.

(5.3.1) Proposition (cf. [Mor88, Prop. 4.2]). *Let P , C^\sharp , and s be as above. Modulo permutations of x_i 's and changes of ℓ -characters, the semi-group $\text{ow } C^\sharp$ is generated by $\text{ow } x_1$ and $\text{ow } x_2$ except for the case (IE^\vee) below. Moreover, exactly one of the following holds:*

(IA $^\vee$) $\bar{m} > 1$, $\text{wt } x_1 + \text{wt } x_3 \equiv 0 \pmod{m}$, $\text{ow } x_4 = (\bar{m}, 0)$, $\text{ow } C^\sharp$ is generated by $\text{ow } x_1$ and $\text{ow } x_2$, and $w_P(0) \geq 1/2$.

(IC $^\vee$) $s = 2$, \bar{m} is an even integer ≥ 4 , and

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \text{wt} & 1 & -1 & 0 \pmod{m} \\ \text{ord} & 1 & \bar{m} - 1 & \bar{m} \pmod{m} \end{array}$$

(II $^\vee$) $\bar{m} = s = 2$, P is of type $cAx/4$, and

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \text{wt} & 1 & 3 & 3 \pmod{4} \\ \text{ord} & 1 & 1 & 2 \end{array}$$

(ID $^\vee$) $\bar{m} = 1$, $s = 2$, P is of type $cA/2$ or $cAx/2$, and

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \text{wt} & 1 & 1 & 1 \pmod{2} \\ \text{ord} & 1 & 1 & 1 \end{array}$$

(IE $^\vee$) $\bar{m} = 2$, $s = 4$, P is of type $cA/8$, and

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \text{wt} & 5 & 1 & 3 \pmod{8} \\ \text{ord} & 1 & 1 & 2 \end{array}$$

Moreover, we are in the case (ID $^\vee$) or (IE $^\vee$) if only if $C' = f'^{-1}(o')$. In this case, P is the only non-Gorenstein point.

(5.3.2) Remark. It is easy to show that in cases (IC $^\vee$) and (IE $^\vee$) the point (X, P) is a cyclic quotient singularity (cf. [Mor88, Lemma 4.4]).

Proof. First assume that $C' \neq f'^{-1}(o')$. By Corollary (4.4.1) we have $H^1(\text{gr}_C^0 \omega) = 0$. Therefore,

$$H^1(\text{gr}_C^0 \omega) = H^1(\text{gr}_{C'}^0 \omega)^{\mu_a} = 0.$$

This implies $\mathrm{gr}_C^0 \omega = \mathcal{O}_C(-1)$ and $w_P(0) < 1$. In particular, $\bar{m} > 1$ (see [Mor88, Cor. 2.10]). Let $g^b: (X^b, C^b) \rightarrow (X, C)$ be the splitting cover. Consider the exact sequence

$$(5.3.3) \quad 0 \longrightarrow \mathrm{gr}_{C^b}^0 \omega \xrightarrow{\varphi} \bigoplus_{i=1}^s \mathrm{gr}_{C_i^b}^0 \omega \longrightarrow \mathrm{Coker} \varphi \longrightarrow 0.$$

Note that $\mathrm{gr}_{C_i^b}^0 \omega = \mathcal{O}(-1)$ (see [Mor88, 2.3.2]). Hence,

$$H^0(\mathrm{Coker} \varphi) = H^1(\mathrm{gr}_{C^b}^0 \omega) = H^1(\mathrm{gr}_{C'}^0 \omega)^{\mu_{d/s}} = 0.$$

Since the support of $\mathrm{Coker} \varphi$ is zero-dimensional, φ is an isomorphism. Therefore, the classification [Mor88, 4.2] holds for (X, P) in this case (see [Mor88, 3.6-3.8]).

Now we consider the case where $C' = f'^{-1}(o')$. If $\bar{m} = 1$, then by Lemma (4.4.2) the splitting cover X^b is Gorenstein and $-K_{X^b} \cdot C_i^b$ is an integer for any component $C_i^b \subset C^b$. Hence, $2 = -K_{X^b} \cdot C^b = -sK_{X^b} \cdot C_i^b$. This implies $s = 2$. We get the case (ID^\vee) . Furthermore by Proposition (2.9) we are in the case (1.2.4) and hence P is the only non-Gorenstein point. From now on we assume that $\bar{m} > 1$.

(5.3.4) We claim that P is the only non-Gorenstein point. Indeed, assume first that there are at least two non-Gorenstein points other than P on C . Then on the splitting cover X^b any irreducible component C_i^b of C^b contains at least three non-Gorenstein points by $\bar{m} > 1$ (2.7.5), (iii). Since the extremal neighborhood (X^b, C_i^b) (2.7.5), (iii) can have at most two non-Gorenstein points [Mor88, Thm. 6.2], this is impossible. So we assume that (X, C) contains exactly two non-Gorenstein points, P and Q . Let n be the index of Q . Clearly, $-K_{X'} \cdot C' = 2$. On the other hand, by Lemma (2.8) $-K_{X'} \cdot C' = -sK_X \cdot C = sd/mn$. Let $r = \mathrm{gcd}(\bar{m}, n)$. Then $d = rs$ and $s = 2n\bar{m}/r$. Since $\bar{m} > 1$, we have $s = 2s_1$, where $s_1 = n\bar{m}/r > 1$. Consider the quotient X'' of X^b by $\mu_{s_1} \subset \mu_s$. We get extremal neighborhoods (X'', C_i'') with two non-Gorenstein points: imprimitive of index $\bar{m}s_1$ and primitive of index n . By [Mor88, Th. 6.7, 9.4] this is impossible. Thus the claim is proved. In particular, $X^b = X'$.

As above we have $-K_{X'} \cdot C' = 2 = s/\bar{m}$. Hence, $s = 2\bar{m}$. In particular, $m = 2\bar{m}^2 \neq 4$ and P is not of type $cAx/4$. Up to permutation of x_i 's we may assume that $\mathrm{wt} x_4 \equiv \mathrm{wt} x_1 x_3 \equiv \mathrm{wt} \phi \equiv 0 \pmod{m}$. Since $-K_X \cdot C = 1/\bar{m}$ and P is the only non-Gorenstein point, $\mathrm{ord}(x_1 \cdots x_4/\phi) \equiv -\bar{m}K_X \cdot C \equiv 1 \pmod{\bar{m}}$ (see [Mor88, Corollary 2.10]). So, $a_2 \equiv 1 \pmod{\bar{m}}$.

Consider the map $\varphi: \text{gr}_{C'}^0 \omega \rightarrow \bigoplus \text{gr}_{C'_i}^0 \omega$ (see (5.3.3)) and the induced map

$$(5.3.5) \quad \Phi: \text{gr}_{C'}^0 \omega = (\mathcal{O}_{C^\sharp} \bar{\omega})^{\mu_{\bar{m}}} \rightarrow \bigoplus_{i=1}^s \text{gr}_{C'_i}^0 \omega \otimes \mathbb{C}(P'),$$

where $\bar{\omega}$ is a semi-invariant generator of ω_{X^\sharp} at P^\sharp . For example we can take

$$\bar{\omega} = \frac{dx_2 \wedge dx_3 \wedge dx_4}{\partial \phi / \partial x_1}.$$

Since $\bar{m} w_{P'_{(i)}}(0) = \bar{m} - 1$, we see that $\Phi(\nu \bar{\omega}) = 0$ for $\mu_{\bar{m}}$ -semi-invariants ν with $\text{ord } \nu \geq \bar{m}$.

(5.3.6) Lemma. *If $C' = f'^{-1}(o')$, then*

$$H^1(\text{gr}_{C'}^0 \omega) = H^0(\text{Coker } \varphi) = H^0(\text{Coker } \Phi) = \mathbb{C}.$$

Proof. Since $H^j(\text{gr}_{C'_i}^0 \omega) = 0$, we have

$$\begin{aligned} H^0(\text{Coker } \varphi) &\simeq H^1(\text{gr}_{C'}^0 \omega) \simeq H^1(\omega_{X'} \otimes \mathcal{O}_{f'^{-1}(o')}) \\ &\simeq (R^1 f'_* \omega_{X'}) \otimes \mathbb{C}(o') \simeq \omega_{Z'} \otimes \mathbb{C}(o') \simeq \mathbb{C}. \end{aligned}$$

(We used the base change theorem and Lemma (4.1).) □

There are two cases.

(5.3.7) Case $a_2 \geq \bar{m}$. Clearly, $a_1 + a_3 \geq \bar{m}$. Thus we may assume that $a_1 \leq a_3 \geq \bar{m}/2$. In this case, Φ factors through

$$\left(\mathcal{O}_{C^\sharp, P^\sharp} / (x_1^{\bar{m}}, x_1 x_3, x_3^2, x_2, x_4) \cdot \bar{\omega} \right)^{\mu_{\bar{m}}} \simeq \mathbb{C}(P') x_1^\lambda \cdot \bar{\omega} \oplus (\mathbb{C}(P') x_3 \cdot \bar{\omega})^{\mu_{\bar{m}}}$$

for a unique $0 < \lambda < \bar{m}$ such that $\lambda a_1 + \text{wt } \bar{\omega} \equiv 0 \pmod{\bar{m}}$. Since $\dim \text{Coker } \Phi \leq 1$, by (5.3.5) we have $2\bar{m} = s \leq 2 + 1 = 3$, a contradiction.

(5.3.8) Case $a_2 = 1$. As above, Φ factors through

$$R := \left(\mathcal{O}_{C^\sharp, P^\sharp} / (x_1^{\bar{m}}, x_1 x_3, x_3^2, x_2^{\bar{m}}, x_4) \cdot \bar{\omega} \right)^{\mu_{\bar{m}}}.$$

This R is generated by the images of

$$x_1^i x_2^{\bar{m}-1-a_1 i}, \quad x_3^j x_2^{\bar{m}-1-a_3 j}, \quad 0 \leq i \leq (\bar{m}-1)/a_1, \quad 1 \leq j \leq (\bar{m}-1)/a_3.$$

Therefore,

$$2\bar{m} = s \leq \dim R + 1 \leq \frac{\bar{m}-1}{a_1} + 1 + \frac{\bar{m}-1}{a_3} + 1 \leq \frac{\bar{m}-1}{1} + \frac{\bar{m}-1}{1} + 2 = 2\bar{m}.$$

This immediately implies $a_1 = a_3 = 1$. Since $a_1 + a_3 \equiv 0 \pmod{\bar{m}}$, we have $\bar{m} = 2$, $s = 4$, and $m = 8$. Changing ℓ -characters [Mor88, 2.5] and permuting x_1 and x_3 , we may assume that $\text{wt } x_2 \equiv 1 \pmod{8}$ and $\text{wt } x_1 \equiv 1$ or $5 \pmod{8}$. If $\text{wt } x_1 \equiv 1 \pmod{m}$, then $\text{ow } C^\sharp$ is generated by $\text{ow } x_1$ and $\text{ow } x_3$. In particular, x_1/x_2 is constant on C^\sharp . This means that R is generated by x_1 and x_3 . Hence, by (5.3.5) $4 = s \leq \dim R + 1 \leq 3$, a contradiction. Therefore, we have the case (IE^\vee) . □

§6. Deformations of \mathbb{Q} -conic bundles

We recall the following

(6.1) Proposition ([Mor88, 1b.8.2]). *Let (X, C) a the \mathbb{Q} -conic bundle germ and let $P \in C$. Then every deformation of germs $(X, P) \supset (C, P)$ can be extended to a deformation of (X, C) so that the deformation is trivial outside some small neighborhood of P .*

Proof (cf. [KM92, 11.4.2]). Let $P_i \in X$ be singular points. Consider the natural morphism

$$\Psi: \text{Def } X \longrightarrow \prod \text{Def}(X, P_i).$$

It is sufficient to show that Ψ is smooth (in particular, surjective). The obstruction to globalize a deformation in $\prod \text{Def}(X, P_i)$ lies in $R^2 f_* T_X$. Since f has only one-dimensional fibers, $R^2 f_* T_X = 0$. □

(6.2) Proposition. *Let $f: (X, C \simeq \mathbb{P}^1) \rightarrow (Z, o)$ be a \mathbb{Q} -conic bundle germ. Let (X_t, C_t) , $t \in \mathfrak{T} \ni 0$ be an one-parameter deformation as in Proposition (6.1) and let $\mathfrak{X} \rightarrow \mathfrak{T}$ be the corresponding family so that $\mathfrak{X}_0 = X$. There exists a contraction $\mathfrak{f}: \mathfrak{X} \rightarrow \mathfrak{Z}$ over $\mathfrak{T} \ni 0$ such that $\mathfrak{Z}_0 = Z$ and for all $t \in \mathfrak{T} \ni 0$, $\mathfrak{f}_t: \mathfrak{X}_t \rightarrow \mathfrak{Z}_t$ is a \mathbb{Q} -conic bundle germ.*

Proof. Consider the base change (2.4.1). Let $\Gamma' = f'^{-1}(o')$ be the scheme fiber (so that $\Gamma'_{\text{red}} = C'$) and let $g^{-1}(P_i) = \{P'_{i,1}, \dots, P'_{i,s_i}\}$. We claim that arbitrary deformation in $\text{Def}(X, P_i)$ determines a μ_d -equivariant deformation in $\prod_j \text{Def}(X', P'_{i,j})$. Indeed, the total space \mathfrak{Q} of a deformation of a terminal singularity (X, P) is \mathbb{Q} -Gorenstein (see [Ste88, §6]) and index-one cover of \mathfrak{Q} is the total deformation space of the index-one cover (X^\sharp, P^\sharp) of (X, P) . Therefore every deformation of a terminal singularity of index m is induced by some μ_m -equivariant deformation of its index-one cover. This proves our claim. This

implies that a deformation in $\text{Def}(X, P_i)$ determines a deformation of X' which must be μ_d -equivariant. Therefore, the cover $X' \rightarrow X$ induces a cover $\mathfrak{X}' \rightarrow \mathfrak{X}$ so that $\mathfrak{X} = \mathfrak{X}'/\mu_d$.

Since Γ' is a complete intersection in \mathfrak{X}' , the conormal sheaf $\mathcal{N}_{\Gamma'/\mathfrak{X}'}^*$ is locally free. We have the exact sequence

$$(6.2.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N}_{X'/\mathfrak{X}'}^*|_{\Gamma'} & \longrightarrow & \mathcal{N}_{\Gamma'/\mathfrak{X}'}^* & \longrightarrow & \mathcal{N}_{\Gamma'/X'}^* \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \mathcal{O}_{\Gamma'} & & & & \mathcal{O}_{\Gamma'}^{\oplus 2} \end{array}$$

Since $\text{Ext}^1(\mathcal{O}_{\Gamma'}^{\oplus 2}, \mathcal{O}_{\Gamma'}) = H^1(\mathcal{O}_{\Gamma'}^{\oplus 2}) = 0$, the sequence splits.

Therefore the germ \mathfrak{D} of the Douady space of \mathfrak{X}' at $[\Gamma']$ is smooth, where $[\Gamma']$ is the point representing Γ' . Let $\mathfrak{U} \rightarrow \mathfrak{D}$ be the corresponding universal family. There is a natural embedding $\mathfrak{U} \subset \mathfrak{X}' \times \mathfrak{D}$ such that $\mathfrak{U} \rightarrow \mathfrak{D}$ is induced by the projection $\mathfrak{X}' \times \mathfrak{D} \rightarrow \mathfrak{D}$. Thus we have the following diagram:

$$\begin{array}{ccc} \mathfrak{X}' \times \mathfrak{D} & \supset & \mathfrak{U} \xrightarrow{\alpha} \mathfrak{X}' \supset \Gamma' \\ \downarrow \text{pr}_2 & \swarrow \beta & \\ \mathfrak{D} & & \end{array}$$

The natural embedding $\Gamma' = \Gamma' \times [\Gamma'] \subset \mathfrak{U}$ induces an isomorphism $\alpha|_{\Gamma'}: \Gamma' \rightarrow \Gamma'$. Further, $\Gamma' = \Gamma' \times [\Gamma'] \subset \mathfrak{U}$ is a fiber of β , so $\mathcal{N}_{\Gamma'/\mathfrak{U}}^*$ is locally free and isomorphic to $\mathcal{O}_{\Gamma'}^{\oplus 3}$. Hence,

$$d\alpha: \mathcal{N}_{\Gamma'/\mathfrak{X}'}^* \rightarrow \mathcal{N}_{\Gamma'/\mathfrak{U}}^*$$

is an isomorphism. Shrinking $\mathfrak{U} \supset \Gamma'$ and $\mathfrak{X}' \supset \Gamma'$ we may assume that α is an isomorphism. This induces a μ_d -equivariant contraction morphism $\Phi = \alpha^{-1}\beta: \mathfrak{X}' \rightarrow \mathfrak{D}$ such that $\Phi(\Gamma')$ is a point. Put $\mathfrak{Z} := \mathfrak{D}/\mu_d$. Since the morphism $\mathfrak{p}: \mathfrak{X} \rightarrow \mathfrak{Z}$ maps X to 0, by shrinking \mathfrak{X} we may assume that \mathfrak{p} is constant on fibers of \mathfrak{f} . Then \mathfrak{p} defines $\mathfrak{Z} \rightarrow \mathfrak{Z}$. We obtain the following diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{\tau} & \mathfrak{X} \\ \downarrow \mathfrak{f}' & \searrow \mathfrak{p} & \downarrow \mathfrak{f} \\ \mathfrak{D} & \xrightarrow{\mathfrak{q}} & \mathfrak{Z} \\ & \searrow & \downarrow \\ & & \mathfrak{Z} \end{array}$$

We have

$$\mathfrak{f}_* \mathcal{O}_{\mathfrak{X}} = \mathfrak{f}_*(\tau_* \mathcal{O}_{\mathfrak{X}'})^{\mu_d} = (\mathfrak{f}_* \tau_* \mathcal{O}_{\mathfrak{X}'})^{\mu_d} = (\mathfrak{q}_* \mathfrak{f}'_* \mathcal{O}_{\mathfrak{X}'})^{\mu_d} = (\mathfrak{q}_* \mathcal{O}_{\mathfrak{D}})^{\mu_d} = \mathcal{O}_{\mathfrak{Z}}.$$

Therefore \mathfrak{f} has connected fibers. Clearly, \mathfrak{X} is \mathbb{Q} -Gorenstein and $-K_{\mathfrak{X}}$ is \mathfrak{f} -ample. \square

(6.2.2) Remark. In general, it is not true that $f_t^{-1}(o_t)_{\text{red}} = C_t$. It is possible that $f_t^{-1}(o_t)$ is reducible and C_t is one of its components. In this case, (X_t, C_t) is an extremal neighborhood by Remark (2.3.2).

§7. The case where X is not locally primitive

In this section we consider the case where X is not locally primitive. We classify configurations of singular points and prove Theorem (1.2) except for one case when a \mathbb{Q} -conic bundle germ has two non-Gorenstein points. Some weaker results were obtained in [Pro97b].

(7.1) Notation. Let $f: (X, C) \rightarrow (Z, o)$ be a \mathbb{Q} -conic bundle germ, where C is irreducible. Assume that (X, C) contains an imprimitive point P . Let m, \bar{m} and s be the index, the subindex and the splitting degree of P , respectively.

(7.1.1) First we note that if $\bar{m} = 1$, then by Lemma (4.4.2) X' is Gorenstein and we have the case (1.2.4) by (2.9). From now on we assume that $\bar{m} > 1$ (in particular, P is not of type (ID^\vee)).

(7.1.2) Lemma (cf. [Mor88, Th. 6.1 (ii)]). *In notation of (7.1), assume that X has a singular point $Q \neq P$. Then $\text{Sing } X = \{P, Q\}$ and Q is of type (IA) or (III). If Q is of type (IA), then $\text{siz}_Q = 1$.*

Recall [Mor88, 4.5] that a point $P \in X$ is said to be *ordinary* iff (X, P) is either an ordinary double point or a cyclic quotient singularity.

Proof. Assume that (X, C) has two more singular points Q and R . Then by Proposition (5.3.1), P is of type (IA^\vee) , (IC^\vee) , or (II^\vee) . By Proposition (2.6.4) both Q and R are primitive. Replace (X, C) with L -deformation [Mor88, Prop.-Def. 4.7] so that P, Q, R are ordinary (cf. [Mor88, Rem. 4.5.1]). If this new (X, C) is an extremal neighborhood, the assertion follows by [Mor88, Th. 6.1 (ii)]. Thus we may assume that (X, C) is a \mathbb{Q} -conic bundle germ. Consider the cover $g: (X', C') \rightarrow (X, C)$ from (2.4.1). By Lemma (4.4.2) we may assume that $P' \in X'$ is not Gorenstein (otherwise (X', C') is a standard Gorenstein conic bundle germ and then (X, C) is as in (1.2.4), see Proposition (2.9)). Let $C'_i \subset C'$ be any irreducible component. By (2.7.5) (iii), (X', C'_i) is an extremal neighborhood with at least three singular points. Then by [Mor88, (2.3.2)] $\deg \text{gr}_{C'_i}^1 \mathcal{O} \leq -2$. Hence, $H^0(\text{gr}_{C'_i}^1 \mathcal{O}) = 0$ [Mor88, Remark 2.3.4].

This implies that $H^0(\mathrm{gr}_{C'}^1 \mathcal{O}) \subset \bigoplus_i H^0(\mathrm{gr}_{C'_i}^1 \mathcal{O}) = 0$. Therefore, $H^0(I_{C'}^{(2)}) = H^0(I_{C'})$ and $H^0(\mathcal{O}_{X'}/I_{C'}^{(2)}) = \mathbb{C}$. By Corollary (4.4.5) we have $H^1(\mathrm{gr}_{C'}^1 \omega) = 0$. Therefore, $H^1(\mathrm{gr}_C^1 \omega) = H^1(\mathrm{gr}_{C'}^1 \omega) \mu_a = 0$. With this extra condition, the proof of [Mor88, Th. 6.1 (i)] (resp. [Mor88, Th. 6.1 (ii)]) works if P is of type (IC^\vee) (resp. (IA^\vee)). Thus $\mathrm{Sing} X = \{P, Q\}$.

Now assume that Q is not of type (III). Consider the splitting cover g^b from (2.7.6). Each (X^b, C_i^b) is an extremal neighborhood having two non-Gorenstein points: P^b and Q_i^b . By [Mor88, Th. 6.7, Th. 9.4] Q_i^b is of type (IA) with $\mathrm{siz} = 1$ and so is Q . \square

(7.2) Proposition. *If $C' = f'^{-1}(o')$ (and $\bar{m} > 1$) or, equivalently if we have a point of type (IE^\vee) , then f is as in (1.2.3).*

Proof. We note that $C' = f'^{-1}(o')$ (and $\bar{m} > 1$) if and only if we are in the case (IE^\vee) by Proposition (5.3.1). In some (non-normalized) coordinate system, $C^\sharp \subset \mathbb{C}_{y_1, y_2, y_3, y_4}^4$ is a complete intersection given by

$$(7.2.1) \quad y_1^2 - y_2^2 = y_1 y_2 - y_3^2 = y_4 = 0,$$

where $\mathrm{wt}(y) \equiv (5, 1, 3, 0) \pmod{8}$. Thus we may fix an embedding $C^\sharp \subset \mathbb{C}_{y_1, y_2, y_3}^3$ and $X^\sharp \subset \mathbb{C}_{y_1, \dots, y_4}^4$. Let (u, v) be μ_8 -semi-invariant coordinates in $Z' = \mathbb{C}^2$. Since $C' = f'^{-1}(o')$, we may regard u, v as μ_8 -semi-invariant generators of the ideal of C^\sharp in X^\sharp . Therefore the ideal of C^\sharp in $\mathbb{C}_{y_1, \dots, y_4}^4$ has two systems of semi-invariant generators:

$$y_1^2 - y_2^2, y_1 y_2 - y_3^2, y_4 \quad \text{and} \quad u, v, \phi.$$

Up to permutation of u and v we may assume that

$$\begin{aligned} \mathrm{wt} u &\equiv \mathrm{wt} y_1^2 \equiv 2, & \mathrm{wt} v &\equiv \mathrm{wt} y_1 y_2 \equiv -2 \pmod{8}, \\ \phi &= (\mathrm{unit})y_4 + (y_1^2 - y_2^2)\phi_1 + (y_1 y_2 - y_3^2)\phi_2 \end{aligned}$$

because $\mathrm{wt} y_1^2, \mathrm{wt} y_3^2 \not\equiv \mathrm{wt} y_4 \pmod{8}$. In particular, X^\sharp is smooth and (X, P) is a cyclic quotient of type $\frac{1}{8}(5, 1, 3)$. Hence (X', P') is a singularity of type $\frac{1}{2}(1, 1, 1)$ and coordinates y_1, y_2, y_3 can be regarded as sections of $|-K_{X'}|$ on X' . Note that the linear system $|-K_{X'}|$ has a unique base point P' and $|-2K_{X'}|$ is base point free. Let z be a section of $|-2K_{X'}|$. Then y_1, y_2, y_3, z define a map $\vartheta: X' \dashrightarrow \mathbb{P} \times \mathbb{C}^2$, where $\mathbb{P} := \mathrm{Proj} \mathbb{C}[y_1, y_2, y_3, y_4] = \mathbb{P}(1, 1, 1, 2)$. Since this ϑ is regular on each component of C' and on the tangent space to

C' at P' , it is an embedding. Therefore, X' can be naturally embedded into $\mathbb{P} \times \mathbb{C}^2$ and by (7.2.1) the defining equations are of the form

$$\begin{aligned} y_1^2 - y_2^2 &= u\psi_1 + v\psi_2, \\ y_1y_2 - y_3^2 &= u\psi_3 + v\psi_4, \end{aligned}$$

where $\psi_i = \psi_i(y_1, y_2, y_3, z, u, v)$. This proves our proposition. \square

(7.2.2) Corollary. *In the notation of (7.1) the following are equivalent: (i) $\text{gr}_C^0 \omega \not\cong \mathcal{O}(-1)$, (ii) we are in the case (1.2.4), and (iii) P is a point of type (ID^\vee) .*

Proof. Assume that $\text{gr}_C^0 \omega \not\cong \mathcal{O}(-1)$. Then by (4.4.3) we see that $f'^{-1}(o') = C'$. If $\bar{m} > 1$, then we are in the case (1.2.3) by Proposition (7.2), in which case we have $w_P(0) = 1/2$ and hence $\text{gr}_C^0 \omega = \mathcal{O}(-1)$ by (3.1.1) and Lemma (2.8). Hence $\bar{m} = 1$, then we are in the case (1.2.4) as explained in (7.1.1). Thus (i) implies (ii) and (iii).

If P is of type (ID^\vee) , then $\bar{m} = 1$ and again by (7.1.1) we are in the case (1.2.4). Then by [Mor88, (2.10)] we have $w_P(0) \geq 1$ and so $\deg \text{gr}_C^0 \omega < -1$ (see (3.1.1)). \square

(7.3) Proposition. *In notation of (7.1) (X, C) has no type (IC^\vee) points.*

Proof. Assume that $P \in (X, C)$ is a type (IC^\vee) point. Recall that $s = 2$ and \bar{m} is even ≥ 4 in this case. Then (X, C) has at most one more (primitive) singular point. Applying L -deformation we may assume one of the following:

(7.3.1) P is the only singular point of X , or

(7.3.2) (X, C) has one more ordinary singular point Q of index $n > 1$.

By [Mor88, Th. 6.1 (i)] this new (X, C) is a conic bundle germ. Following the proof of [Mor88, (i) Th. 6.1] we get $H^1(\mathcal{O}_X/I_C^{(2)} \tilde{\otimes} \omega_X) \neq 0$. Hence, $H^1(\mathcal{O}_{X'}/I_{C'}^{(2)} \tilde{\otimes} \omega_{X'}) \neq 0$.

(7.3.3) Let $V := \text{Spec } \mathcal{O}_{X'}/I_{C'}^{(2)}$. By Theorem (4.4) we have $f'^{-1}(o') \subset V$. Moreover, as in the proof of (4.4.5) one can see that V is not a local complete intersection at the general point. Therefore, $f'^{-1}(o') \neq V$. Since $V \equiv 3C'$ (as a cycle), we have

$$2 = -K_{X'} \cdot f'^{-1}(o') < -K_{X'} \cdot V = -3K_{X'} \cdot C'.$$

Taking account of $-K_{X'} \cdot C' = d^2/mn$ (see (2.8)) we get $2nm < 3d^2$, where we put $n = 1$ in the case (7.3.1). Recall that $s = 2$. Write $m = s\bar{m} = 2\bar{m}$,

$n = rn'$ and $\bar{m} = r\bar{m}'$, where $r = \gcd(\bar{m}, n)$. Then $d = 2r$ (see Corollary (2.7.2) (iii)) and $n'\bar{m}' < 3$. Note that \bar{m}' is the index of P' . If $\bar{m}' = 1$, then X' is Gorenstein by (4.4.2) and (X, P) cannot be of type (IC^\vee) by Proposition (2.9). So, $\bar{m}' = 2$, $n' = 1$, and $\bar{m} = 2n$. In particular, $n > 1$ and the case (7.3.1) is impossible.

Note that (X, P) is a cyclic quotient singularity by [Mor88, Lemma 4.4]. Thus we may assume that $X^\sharp = \mathbb{C}_{x_1, x_2, x_4}^3$ and C^\sharp is given by the equations $x_4 = x_2^2 - x_1^{m-2} = 0$. Thus C' near P' is isomorphic to $\{x_4 = x_2^2 - x_1^{m-2} = 0\}/\mu_2(1, 1, 1)$. Putting $w_1 = x_1^2$, $w_2 = x_2^2$, $w_3 = x_1x_2$ we get that near P' the curve $C' \subset \mathbb{C}_{w_1, w_2, w_3}^3$ can be given by two equations $w_2 = w_1^{\bar{m}-1}$ and $w_1w_2 = w_3^2$. Eliminating w_2 we obtain $C' := \{w_3^2 = w_1^{2\bar{m}}\}$. It is easy to see that C' has an ordinary double point at the origin only if $n = 1$. This contradicts $p_a(C') = 0$. \square

The following lemma was proved in [Pro97b, §3]. However it was implicitly assumed in the proof that X is \mathbb{Q} -factorial. Below is a corrected version.

(7.3.4) Lemma. *In notation of (7.1) assume that X is \mathbb{Q} -factorial. Then $s = 2^k$. If furthermore X has two non-Gorenstein points, then $s = 2$.*

Proof. Write $d = sr$ and $s = 2^kq$, where q is odd. We will derive a contradiction assuming $q > 1$. Consider the quotient X^\dagger/Z^\dagger of X'/Z' from (2.4.1) by $\mu_{2^k r} \subset \mu_d$:

$$\begin{array}{ccc} X^\dagger & \xrightarrow{g^\dagger} & X \\ f^\dagger \downarrow & & f \downarrow \\ Z^\dagger & \xrightarrow{h^\dagger} & Z \end{array}$$

where $h^\dagger: Z^\dagger \rightarrow Z$ is a μ_q -cover. Then $C^\dagger := g^{\dagger-1}(C)$ has q irreducible components because X^\dagger is a μ_{2^k} -quotient of the splitting cover (2.7.5) and therefore, $\rho(X^\dagger/Z^\dagger) = q$ by Corollary (2.3.1). There is a curve $V \subset Z^\dagger$ such that $f^{\dagger-1}(V)$ has exactly two components, say E_1 and $E_{1'}$. For a general point $z \in V$ the preimage $f^{\dagger-1}(z)$ is a reducible conic, so $f^{\dagger-1}(z) = \ell_1 + \ell_2$. Consider the orbit $\{E_1, E_2, \dots, E_t\}$ of E_1 under the action of μ_q . Obviously, every $f^\dagger(E_i)$ is a curve on Z^\dagger . Further, $\sum_{i=1}^t E_i \sim_{\mathbb{Q}} g^{\dagger*}M$, where M is a Weil \mathbb{Q} -Cartier divisor on X . On the other hand, $\rho(X/Z) = 1$ and M is f -vertical. Hence, $M \sim_{\mathbb{Q}} 0$ and $\sum E_i \sim_{\mathbb{Q}} 0$. We can choose components $\ell_1, \ell_2 \subset f^{\dagger-1}(z)$, $z \in V$ so that $\ell_1 \cdot E_1 < 0$ and $\ell_1 \cdot E_{1'} > 0$. This gives us $\ell_1 \cdot E_i > 0$ for some $E_i \in \{E_1, E_2, \dots, E_t\}$. Then $E_{1'} = E_i$, i.e., there exists $\sigma \in \mu_q$ such that $\sigma(E_1) = E_{1'}$. From the symmetry we get that the orbit $\{E_1, E_2, \dots, E_t\}$ may be divided into pairs of divisors E_j ,

$E_{j'}$, such that $f^\dagger(E_j) = f^\dagger(E_{j'})$ is a curve. Thus both t and q are even. This proves the first statement.

Now assume that X has two non-Gorenstein points and let $s = 2^k$, $k \geq 2$. Consider the quotient (X'', C'') of (X', C') by $\mu_{2^{k-1}}$. Then the central fiber C'' is reducible and every germ (X'', C''_i) is an extremal neighborhood having two non-Gorenstein points: imprimitive and primitive. By the classification [Mor88, Th. 6.7, 9.3] this is impossible. \square

(7.4) Proposition (cf. [Mor88, Th. 6.1 (iii)]). *Notation as in (7.1). Assume that (X, C) has one more non-Gorenstein point Q . Then P is of type (IA^\vee) , $\text{siz}_P = 1$, and $w_P(0) \geq 2/3$.*

(7.4.1) Corollary. *In the above notation we have $w_Q(0) < 1/3$. In particular, the index of Q is ≥ 4 .*

Proof. (7.4.1) immediately follows from (7.4) and (7.2.2). We assume that $\text{siz}_P \geq 2$ or $w_P(0) < 2/3$ and we will derive a contradiction. Let n be the index of Q . By Lemma (7.1.2) $\text{Sing } X = \{P, Q\}$ and Q is of type (IA) and by Propositions (7.2), (7.3) and Corollary (7.2.2), P is of type (IA^\vee) or (II^\vee) . Replacing (X, C) with L -deformation, we may assume that X has only ordinary points (in particular, P is of type (IA^\vee)). If this new (X, C) is an extremal neighborhood, the assertion follows by [Mor88, Th. 6.1 (iii)]. Thus we may assume that (X, C) is again a \mathbb{Q} -conic bundle germ.

If $H^1(\mathcal{O}_X/I_C^{(2)} \tilde{\otimes} \omega_X) = 0$, then following the proof of [Mor88, (iii) Th. 6.1] we derive a contradiction. Hence, in notation of (2.4.1) we have $H^1(\mathcal{O}_{X'}/I_{C'}^{(2)} \tilde{\otimes} \omega_{X'}) \neq 0$. Let $V := \text{Spec } \mathcal{O}_{X'}/I_{C'}^{(2)}$. As in (7.3.3) by Theorem (4.4) $f'^{-1}(o') \subset V$, and so

$$2 = -K_{X'} \cdot f'^{-1}(o') < -K_{X'} \cdot V = -3K_{X'} \cdot C'.$$

Taking account of $-K_{X'} \cdot C' = d^2/mn$ (see (2.8)) we obtain

$$2mn < 3d^2.$$

Since X has only ordinary points of index > 1 , X is \mathbb{Q} -factorial. By Lemma (7.3.4) $s = 2$. Write $m = s\bar{m} = 2\bar{m}$, $n = rn'$ and $\bar{m} = r\bar{m}'$, where $r = \text{gcd}(\bar{m}, n)$. By Corollary (2.7.2) (iii) we have $d = sr = 2r$. Then $n'\bar{m}' < 3$. Since \bar{m}' is the index of P' , we may assume that $\bar{m}' > 1$ (otherwise by Lemma (4.4.2) and Proposition (2.9) we have the case (1.2.4)). Therefore, $\bar{m}' = 2$, $n' = 1$, and $\bar{m} = 2n$.

We may assume that $X^\sharp = \mathbb{C}_{x_1, x_2, x_3}^3$ at P^\sharp , the curve C^\sharp is given by the equations $x_3 = x_2^{a_1 s} - x_1^{a_2 s} = 0$, and $(X', P') = \mathbb{C}_{x_1, x_2, x_3}^3 / \mu_2(1, 1, 1)$. Putting

$w_1 = x_1^2$, $w_2 = x_2^2$, $w_3 = x_1x_2$ we get that near P' the curve $C' \subset \{x_3 = 0\}/\mu_2 = \mathbb{C}_{x_1, x_2}^2(1, 1)/\mu_2$ can be given by two equations: $w_2^{a_1} = w_1^{a_2}$ and $w_1w_2 = w_3^2$. We claim that $a_1 = a_2 = 1$. Indeed, assume for example that $a_1 > 1$. Since $p_a(C') = 0$, C' has an ordinary double point at the origin. Hence, $a_2 = 1$. Eliminating w_1 we get the following equation for C' : $w_2^{a_1+1} = w_3^2$. Again the origin is an ordinary double point only if $a_1 = 1$, a contradiction. Thus, $a_1 = a_2 = 1$ and $w_P(0) = 1 - 1/\bar{m}$ by [Mor88, Th. 4.9]. This gives

$$w_Q(0) = 1 - w_P(0) + K_X \cdot C = 1/\bar{m} - 1/\bar{m} = 0,$$

a contradiction. Hence we have $\text{siz}_P = 1$ and $w_P(0) \geq 2/3$. If P is of type (II^\vee) , then $w_P(0) = 1/2$ (see [Mor88, Th. 4.9]). So, P is of type (IA^\vee) . \square

(7.5) Proposition. *In notation of (7.1) assume that P is of type (IA^\vee) . Then P is of index 4, splitting degree 2 and subindex 2. Moreover, (X, P) is a cyclic quotient and*

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \text{wt} & 1 & -1 & -1 & 0 \pmod{4} \\ \text{ord} & 1 & 1 & 1 & 2 \end{array}$$

Proof. By (10.7.2) below P is the only non-Gorenstein point on X . * Since $-K_X \cdot C = 1/\bar{m}$, $w_P(0) = 1 - 1/\bar{m}$. Hence we have $a_2 = 1$ by [Mor88, Th. 4.9.(i)]. The general member $F \in |-K_{(X,P)}|$ has only Du Val singularity of type A_{mk-1} , $k \in \mathbb{Z}_{>0}$. It is easy to see that F^\sharp is given by $x_2 = 0$, so $F \cdot C = 1/\bar{m}$. Hence, $K_X + F$ is a numerically trivial Cartier divisor. Since $\text{Pic } X \simeq H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$, $K_X + F \sim 0$. Thus, the general member $F \in |-K_X|$ does not contain C and has only Du Val singularity of type A_{mk-1} . Consider the double cover $f|_F: (F, P) \rightarrow (Z, o)$. Diagram (2.4.1) induces the following

$$\begin{array}{ccc} (F', P') & \xrightarrow{g_{F'}} & (F, P) \\ f'_{F'} \downarrow & & f_F \downarrow \\ (Z', o') & \xrightarrow{h} & (Z, o) \end{array}$$

where $F' \in |-K_{X'}|$, $P' = g^{-1}(P)$, $F' \cap C' = \{P'\}$, and $g_{F'}$ is étale outside of P' . Since $Z' \rightarrow Z$ is of degree s , s divides mk and (F', P') is of type A_{n-1} , where $n = mk/s = \bar{m}k$. We see $n = 2$ because otherwise we have a contradiction by Lemma (7.5.1) below. Thus $n = 2$ and $\bar{m} = 2$ (recall that $\bar{m} > 1$ by the

*In (10.1) – (10.7.2), no results in (7.5) – (9.4.2) are used when P is imprimitive. Thus the back reference (10.7.2) here does not cause any trouble.

assumption of (IA^\vee) (5.3.1)). In this case, by Corollary (2.8.1) $s = 4$ or 2 . If $s = 4$, then $-K_{X'} \cdot C' = s/\bar{m} = 2$ (see Lemma (2.8)). Hence, $C' = f'^{-1}(o')$ and we have the case (1.2.3) by Proposition (7.2). But then P is not of type (IA^\vee) , a contradiction. Hence $s = 2$ and the rest is easy. \square

(7.5.1) Lemma. *Let (S, Q) be a Du Val singularity of type A_{n-1} , $n \geq 3$ and let $\pi: (S, Q) \rightarrow (\mathbb{C}^2, 0)$ be a double cover. Assume that μ_d acts on (S, Q) and $(\mathbb{C}^2, 0)$ freely in codimension one and so that π is μ_d -equivariant. Then the quotient $(S, Q)/\mu_d$ cannot be Du Val of type A .*

Proof. Let $R \subset \mathbb{C}^2$ be the branch divisor of π . Since (S, Q) is of type A , the equation of R must contain a quadratic term. Hence, in some μ_d -semi-invariant coordinates u, v in \mathbb{C}^2 , the curve R can be given by $u^2 + v^n = 0$. In this case, there is a μ_d -equivariant embedding $(S, Q) \hookrightarrow (\mathbb{C}_{u,v,w}^3, 0)$ such that S is given by $w^2 = u^2 + v^n$ and w is a semi-invariant. Assume that S/μ_d is Du Val. Since K_{S/μ_d} is Cartier, we have $\text{wt}(uvw) = \text{wt } w^2 = \text{wt } u^2 = \text{wt } v^n$. This implies $\text{wt } w = \text{wt}(uv)$ and $\text{wt } v^2 = 0$. Since the action of μ_d on \mathbb{C}^2 is free in codimension one, $d = 2$ and n is even. So, $n = 2l$ for some $l \geq 2$. Hence, S/μ_d is a quotient of $\{u^2 + v^{2l} = w^2\}$ by $\mu_2(1, 1, 0)$. But if $l \geq 2$, this quotient is not of type A , a contradiction. \square

(7.6) Proposition. *In notation of (7.1) assume that P is of type (IA^\vee) (resp. (II^\vee)). Then $f: (X, C) \rightarrow (Z, o)$ is as in (1.2.5) (resp. (1.2.6)).*

Proof. In the case (II^\vee) X has no other non-Gorenstein points by (7.4). Then applying (2.4.1) we will see that X/Z is the quotient of an index-two \mathbb{Q} -conic bundle $f': (X', C') \rightarrow (Z', o')$ by μ_2 . The components of the central curve C' are permuted, so C' has two components of the same multiplicity. Hence X'/Z' is in the case (12.1.6). The action on X' is described in (12.1.12). \square

(7.6.1) Remark. We have treated all the types (IA^\vee) – (II^\vee) of imprimitive points. Finally we note that the existence of a good anicanonical divisor (Proposition (1.3.7)) in the imprimitive cases (1.2.3) – (1.2.6) can be shown exactly as in [Mor88, 7.3] (see also [Pro97b]).

(7.7) Examples. Below we propose explicit examples of \mathbb{Q} -conic bundles as in (1.2.3), (1.2.5) and (1.2.6).

(7.7.1) Example. Under the notation of (1.2.3) consider the subvariety X' defined by

$$\begin{cases} y_1^2 - y_2^2 & = uy_4 \\ y_1y_2 - y_3^2 & = vy_4. \end{cases}$$

The projection $f': X' \rightarrow \mathbb{C}^2$ is a \mathbb{Q} -conic bundle of index 2 (see (12.1.3)). Then $X'/\mu_4 \rightarrow \mathbb{C}^2/\mu_4$ is a \mathbb{Q} -conic bundle with an imprimitive point as in (1.2.3). The singular point is unique and is of type (IE^\vee) .

(7.7.2) Example. Let $X' \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2$ be the subvariety given by the equations

$$\begin{cases} y_1^2 - y_2^2 = uy_4 \\ y_3^2 = vy_4 + u^2y_2^2 + \lambda uy_1y_2, \quad \lambda \in \mathbb{C}. \end{cases}$$

Consider the action of μ_2 on X' :

$$y_1 \mapsto y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto y_3, \quad y_4 \mapsto -y_4, \quad u \mapsto -u, \quad v \mapsto -v.$$

Then $X := X'/\mu_2 \rightarrow \mathbb{C}^2/\mu_2$ is a \mathbb{Q} -conic bundle with an imprimitive point as in (1.2.5). It has a singularity of type (IA^\vee) which is the cyclic quotient $\frac{1}{4}(1, -1, 1)$. If $\lambda = 0$, then X also has a (Gorenstein) ordinary double point.

(7.7.3) Example. Let $X' \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2$ be the subvariety given by the equations

$$\begin{cases} y_1^2 - y_2^2 = u^3y_4 + vy_4 \\ y_3^2 = vy_4 + u^2y_2^2 + \lambda uy_1y_2, \quad \lambda \in \mathbb{C}. \end{cases}$$

Define the action of μ_2 on X' as in (7.7.2). Then $X := X'/\mu_2 \rightarrow \mathbb{C}^2/\mu_2$ is a \mathbb{Q} -conic bundle with an imprimitive point as in (1.2.6). The non-Gorenstein point is of type (II^\vee) . It is the only singular point if $\lambda \neq 0$. If $\lambda = 0$, then X has one more singular point which is of type (III) .

In case of a \mathbb{Q} -conic bundle with an imprimitive point, the proofs of Theorems (1.2) and (1.2.7) are completed here modulo the arguments in (10.1) – (10.7.2).

§8. The case where X is locally primitive. Possible singularities.

In this section we consider locally primitive \mathbb{Q} -conic bundles. The main result is summarized in Theorem (8.6).

(8.1) Notation. Let $f: (X, C \simeq \mathbb{P}^1) \rightarrow (Z, o)$ be a locally primitive \mathbb{Q} -conic bundle germ. Let $P \in (X, C)$ be a (primitive) non-Gorenstein singular point and let $m \geq 2$ be its index. We may assume that $\text{gr}_C^0 \omega \simeq \mathcal{O}_C(-1)$ (see (4.4.3)).

(8.1.1) Lemma. *There are at most 3 singular points of X on C .*

Proof. By $\deg \operatorname{gr}_C^0 \omega < 0$ and (3.1.5), we have $\sum_Q i_Q(1) \leq 3$, and the proof of [Mor88, 6.2(i)] works. \square

(8.1.2) Lemma. *If P is a point of type (IB) or (IC), then the base (Z, o) is smooth. In the case (IIB), (Z, o) is either smooth or Du Val of type A_1 .*

Proof. Assume that (Z, o) is singular and consider the base change as in (2.4.1). Let $P' \in g^{-1}(P)$ and let m' be the index of (X', P') . We note that (X^\sharp, P^\sharp) is also the index-one cover of (X', P') . Clearly, m' divides m .

We claim that $m' < m$. Suppose $m' = m$. Then the Galois cover $X' \rightarrow X$ is étale at P' and X' has at least two points of the same index m on C' . This means that Z' is singular by (2.7.2), a contradiction. Thus $m' < m$ as claimed.

Since C' is smooth, $m' \in \operatorname{ord} C^\sharp$. This is not possible in cases (IB) and (IC) (because modulo renumbering of a_i 's $a_4 = m$, $a_1 + a_2 \geq m$, $\gcd(a_i, m') = 1$, and $a_i > 1$ for $i = 1, 2, 3$). In the case (IIB) the only possibility is $m' = 2$ and then the topological index of f is 2. \square

(8.2) Proposition. *Assume that $P \in (X, C)$ is a type (IC) point. Let m be its index. Then (X, C) has no other singular points. Moreover, $i_P(1) = 2$, $w_P(0) = 1 - 1/m$, $a_1 = 2$, and $a_4 = m + 1$.*

Proof. Assume that (X, C) has one more singular point Q of index $n \geq 1$. By Lemma (8.1.2) the base surface Z is smooth. Since $i_P(1) \geq 2$ [Mor88, Prop. 5.5], we have $i_P(1) = 2$ and $i_Q(1) = 1$. We may assume that Q is ordinary of type (IA) or (III) by L -deformation. Further, by Corollary (4.4.6) $w_P^*(1) + w_Q^*(1) \leq 1$.

If $w_P(0) \neq 1 - 1/m$, all the arguments of [Mor88, 6.5.2] can be applied and we derive a contradiction. Assume that $w_P(0) = 1 - 1/m$. We follow the arguments of [Mor88, 6.5.3]. Since P is of type (IC), $m \geq 5$, so $w_Q(0) = 1 - w_P(0) - 1/nm = 1/m - 1/nm < 1/5$. Let Q be of type (IA) (resp. (III)). Then, for $1 \leq d \leq 4$, by [Mor88, 5.1] (resp. [Mor88, 4.9]), one has $w_Q^*(d) = d(d+1)/2$ (resp. $w_Q^*(d) = \lfloor (d+1)^2/4 \rfloor$) for $d \leq 4$. On the other hand, by [Mor88, 5.5] $w_P^*(1) = 1 - \delta_{m,5}$. Therefore, $m = 5$. Further, by [Mor88, 5.5 (v)] $w_P^*(2) = 0$ and $w_P^*(3) = 4$. Thus,

$$\begin{aligned} \sum_{d=1}^3 (1 + d + \deg \operatorname{gr}_C^d \omega) &= \sum_{d=1}^3 \frac{d(d+1)}{2} - 4 - \sum_{d=1}^3 \frac{d(d+1)}{2} = -4 < 0 \\ \left(\text{resp. } \sum_{d=1}^3 \frac{d(d+1)}{2} - 4 - \sum_{d=1}^3 \left\lfloor \frac{(d+1)^2}{4} \right\rfloor &= -1 < 0 \right). \end{aligned}$$

Therefore, $H^1(\mathcal{O}_X/I_C^{(4)} \tilde{\otimes} \omega_X) \neq 0$. Let $V := \operatorname{Spec} \mathcal{O}_X/I_C^{(4)}$. Then by Theorem (4.4) $V \supset f^{-1}(o)$. Moreover, $V \neq f^{-1}(o)$ because V is not a local complete

intersection inside X . Hence,

$$2 = -K_X \cdot f^{-1}(o) < -K_X \cdot V \leq -10K_X \cdot C = 10/nm.$$

This gives $m \leq nm < 5$, a contradiction. Hence P is the only singular point.

By (2.8) and (3.1.1) $w_P(0) = 1 - 1/m$. Hence, $a_4 \equiv 1 \pmod{m}$ ([Mor88, Th. 4.9]). Now assume that $i_P(1) = 3$. By [Mor88, Prop. 5.5] $w_P^*(1) \geq 2$. This contradicts (4.4.6). Thus, $a_1 = i_P(1) = 2$ ([Mor88, Prop. 5.5]) and $m + 1 \in \text{ord } C^\sharp$. By normalizedness $a_4 = m + 1$ (see (5.1)). \square

(8.3) Proposition. *(X, C) has no type (IB) points.*

Proof. By [Mor88, Prop. 4.7] we can deform (X, C) to $(X_\lambda, C_\lambda \simeq \mathbb{P}^1)$, where X_λ has at least two non-Gorenstein points of the same index m . If (X_λ, C_λ) is an extremal neighborhood, the assertion follows as in the proof of [Mor88, Th. 6.3]. Otherwise (X_λ, C_λ) is a \mathbb{Q} -conic bundle germ over a singular base (Z_λ, o_λ) by (2.7.2) (iii). But Z is smooth by Lemma (8.1.2) and Z_λ is a deformation of Z , a contradiction. \square

(8.4) Proposition. *If (X, C) has a point P of type (IIB), then P is the only singular point and the base surface is smooth.*

Proof. Assume that (X, C) has a singular point $Q \neq P$ of index $m \geq 1$. By [Mor88, Prop. 4.7] we can deform (X, C) to $(X_\lambda, C_\lambda \simeq \mathbb{P}^1)$, where X_λ has three singular points P_λ, P'_λ and Q_λ of indices 2, 4 and m . If (X_λ, C_λ) is an extremal neighborhood, the assertion follows by [Mor88, Th. 6.2]. Assume that (X_λ, C_λ) is a \mathbb{Q} -conic bundle germ over (Z_λ, o_λ) . Since the indices of P_λ and P'_λ are not coprime, the base surface (Z_λ, o_λ) is singular by (2.7.2) (iii). So is the base surface (Z, o) of (X, C) . By Lemma (8.1.2) (Z, o) is of type A_1 . This implies that m is even and $\text{Cl}^{\text{sc}} X \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ (see Corollary (2.7.1)). Then (X_λ, C_λ) contains three non-Gorenstein points of even indices. But in this case, the map ς in (2.7.3) cannot be surjective, a contradiction. \square

(8.5) Proposition (cf. [Mor88, Th. 6.6]). *Let $P \in X$ be a type (IA) point of index m . Then $\text{siz}_P = 1$. If moreover P is the only non-Gorenstein point on X , then (Z, o) is smooth, $w_P(0) = 1 - 1/m$ and $a_2 = 1$.*

Proof. Assume that $\text{siz}_P \geq 2$ and let m be the index of P .

(8.5.1) First we consider the case when (Z, o) is smooth. We claim that P is the only singular point of X . Let $Q \in X$ be a singular point of index $n \geq 1$.

To derive a contradiction we note that the proof of [Mor88, Th. 6.6] works whenever

$$H^1(\omega_X/F^2\omega_X) = H^1(\omega_X/F^3\omega_X) = 0.$$

Assume that one of the above vanishings does not hold.

Let $V := \text{Spec } \mathcal{O}_X/I_C^{(j)}$, $j = 2$ or 3 . By Theorem (4.4) (cf. (7.3.3)) we have $f^{-1}(o) \subset V$. Hence,

$$2 = -K_X \cdot f^{-1}(o) < -K_X \cdot V = -6K_X \cdot C.$$

Taking account of $-K_X \cdot C = 1/mn$ (see (2.8)) we obtain $mn < 3$. So, $m = 2$ and $n = 1$. On the other hand, a point of type (IA) and index two has $\text{ord } x = (1, 1, 1, 2)$. Such a point is of size 1, a contradiction. Thus P is the only singular point of X . By (2.8) and (3.1.1) $w_P(0) = 1 - 1/m$. Therefore, $a_2 \equiv 1 \pmod{m}$. On the other hand, $a_2 < m$ by definition of (IA) point. Hence, $a_2 = 1$, $m - a_1a_2 = a_3 \in \text{ord } C^\sharp$, and $\text{siz}_P = 1$, a contradiction.

(8.5.2) Now we consider the case when (Z, o) is singular. So the topological index of f is $d > 1$. Put $m' := m/d$. By definition of size we have

$$(8.5.3) \quad 2 \leq U_P(a_1a_2) := \min\{k \mid km'd - a_1a_2 \in \text{ord } C^\sharp\}.$$

Consider the base change (2.4.1). Note that $P' = g^{-1}(P)$ is also a point of type (IA) of index m' having the same index-one cover as that of P . At P' we have

$$U_{P'}(a_1a_2) = \min\{l \mid lm' - a_1a_2 \in \text{ord } C^\sharp\}.$$

Write $l := U_{P'}(a_1a_2)$ as $l = qd - r$, where $0 \leq r < d$. Then

$$qm'd - a_1a_2 = lm' - a_1a_2 + rm' \in \text{ord } C^\sharp.$$

(We used the fact that $m' \in \mathbb{Z}_{>0}a_1 + \mathbb{Z}_{>0}a_2$, see (5.2.1).) It is easy to see now from (8.5.3) that $U_{P'}(a_1a_2) = l > qd - d \geq d(U_P(a_1a_2) - 1) \geq 2$. This contradicts the case (8.5.1) above.

(8.5.4) Finally assume that P is the only non-Gorenstein point on X . Then (Z, o) is smooth by Corollary (2.7.4). Hence by (2.8) and (3.1.1) we have $-K_X \cdot C = 1/m$ and $w_P(0) = 1 - 1/m$. Thus $a_2 = 1$, see [Mor88, Th. 4.9]. □

Summarizing the results of this section we obtain

(8.6) Theorem. *Let $f: (X, C \simeq \mathbb{P}^1) \rightarrow (Z, o)$ be a locally primitive \mathbb{Q} -conic bundle germ. Assume that X is not Gorenstein. Then the configuration of singular points is one of the following:*

- (i) type (IC) point P of size 1 and index m with $i_P(1) = 2$, $w_P(0) = 1 - 1/m$, $a_1 = 2$, and $a_4 = m + 1$;
- (ii) type (IIB) point;
- (iii) type (IA) point P of size 1 and index m with $w_P(0) = 1 - 1/m$ and $a_2 = 1$, and possibly at most two more type (III) points;
- (iv) type (IIA) point P , and possibly at most two more type (III) points;
- (v) two non-Gorenstein points which are of types (IA) or (IIA), and possibly at most one more type (III) point;
- (vi) three non-Gorenstein singular points and no other singularities (cf. (8.1.1)).

(8.6.1) Remark. The existence of a good member of $|-K_X|$ or $|-2K_X|$ in the cases (i)-(iv) can be shown as in [Mor88, 7.3]. The cases (v) and (vi) will be studied in the following sections.

§9. The case of three singular points

In this section we consider \mathbb{Q} -conic bundles with exactly three singular points (cf. (8.1.1)). The main result is the following

(9.1) Theorem (cf. [Mor88, Th. 6.2]). *Let $(X, C \simeq \mathbb{P}^1)$ be a \mathbb{Q} -conic bundle germ with three singular points. Up to permutations the configuration of singular points is one of the following:*

- (i) (IA), (III), (III) (cf. (8.6) (iii));
- (ii) (IA), (IA), (III). *In this case, the indices are 2, odd ≥ 3 , and 1.*

In both cases (Z, o) is smooth.

(9.2) Notation. To the end of this section we assume that $f: (X, C \simeq \mathbb{P}^1) \rightarrow (Z, o)$ is a \mathbb{Q} -conic bundle germ with three singular points P , Q , and R . Let k , m , n be the indices of P , Q , and R , respectively.

(9.2.1) By (3.1.5) $i_P(1) = i_Q(1) = i_R(1) = 1$ and by Lemma (7.1.2) all these points are primitive. By Propositions (8.2), (8.3), and (8.4) P , Q , R are of types (IA), (IIA), or (III). We may assume that $\mathrm{gr}_C^0 \omega \simeq \mathcal{O}_C(-1)$ (see (4.4.3)).

(9.2.2) Lemma. *A \mathbb{Q} -conic bundle germ $(X, C \simeq \mathbb{P}^1)$ cannot have three Gorenstein singular points.*

Proof. Indeed, assume that (X, C) has three Gorenstein singular points P_1, P_2, P_3 . In this case, (Z, o) is smooth and (X, C) has no other singular points. Applying L -deformation we may assume that P_i are ordinary. Then by [Mor88, Th. 4.9] $w_{P_i}^*(1) = 1$. This contradicts Corollary (4.4.6). \square

(9.2.3) Lemma (cf. [Mor88, 0.4.13.3, Th. 6.2]). *A \mathbb{Q} -conic bundle germ $(X, C \simeq \mathbb{P}^1)$ has at most two non-Gorenstein points.*

Proof (following [Mor88, 0.4.13.3]). Assume that $P, Q, R \in X$ are singular points of indices k, m and $n > 1$. By (7.1.1) we may assume that the subindex > 1 for imprimitive points and hence that (X, C) is locally primitive (cf. (5.3.4)). By L -deformation at P, Q and R , and by [Mor88, Th. (6.2)] we may assume that P, Q and R are cyclic quotient singularities. Using Van Kampen's theorem it is easy to compute the fundamental group of $X \setminus \{P, Q, R\}$:

$$\pi_1(X \setminus \{P, Q, R\}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle / \{ \sigma_1^k = \sigma_2^m = \sigma_3^n = \sigma_1 \sigma_2 \sigma_3 = 1 \}.$$

The target group has a finite quotient group G in which the images of $\sigma_1, \sigma_2, \sigma_3$ are exactly of order k, m and n , respectively (see, e.g., [Feu71]). The above quotient defines a finite Galois cover $g: (X', C') \rightarrow (X, C)$. By taking Stein factorization we obtain a \mathbb{Q} -conic bundle $f': (X', C') \rightarrow (Z', o')$ with irreducible central fiber C' . By Corollary (2.7.1) G is cyclic. This contradicts Corollary (2.7.4). \square

(9.2.4) Remark. One can check also that the arguments of [Mor88, (6.2.2)] work in this case without any changes.

(9.3) Proposition. *In notation (9.2) (X, C) cannot have three singular points of types (IIA), (III), and (III).*

Proof. Assume that X contains a type (IIA) point P and two type (III) points Q and R . The base (Z, o) is smooth by Corollary (2.7.4). Applying L -deformations at Q and R and L' deformation at P (see [Mor88, 4.12.2]) we may assume that Q, R are ordinary and $(X, P) \simeq \{y_1 y_2 + y_3^2 + y_4^3 = 0\} / \mu_4(1, 1, 3, 2)$, where C^\sharp is the y_1 -axis. This new (X, C) is again a \mathbb{Q} -conic bundle germ by [Mor88, 6.2].

We claim that $H^1(\omega_X \otimes \mathcal{O}_X / I_C^{(3)}) = 0$. Indeed, otherwise we can apply Theorem (4.4) to $V := \text{Spec}_X \mathcal{O}_X / I_C^{(3)}$ (cf. (7.3.3)):

$$2 = -K_X \cdot f^{-1}(o) < -K_X \cdot V = -6K_X \cdot C.$$

Taking account of $-K_X \cdot C = 1/4$ (see (2.8)) we obtain a contradiction. Therefore, $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/I_C^{(3)}) = 0$. This implies

$$(9.3.1) \quad \deg \operatorname{gr}_C^1 \omega + 2 + \deg \operatorname{gr}_C^2 \omega + 3 \geq 0$$

(see (3.1.9)).

By [Mor88, 4.9] $w_Q^*(1) = w_R^*(1) = 1$ and $w_Q^*(2) = w_R^*(2) = 2$. By Lemma (9.3.2) below and using (3.1.8) we obtain

$$\deg \operatorname{gr}_C^1 \omega = -2, \quad \deg \operatorname{gr}_C^2 \omega = -4.$$

This contradicts (9.3.1). \square

(9.3.2) Lemma. *Let (X, P) be a $cAx/4$ -singularity of the form $\{y_1y_2 + y_3^2 + y_4^3 = 0\}/\mu_4(1, 1, 3, 2)$ and let $C = (y_1\text{-axis})/\mu_4$. Then*

$$i_P(1) = 1, \quad w_P(1) = 2, \quad w_P^*(1) = -1, \quad i_P(2) = 1, \quad w_P(2) = 3, \quad w_P^*(2) = 0.$$

Proof. Let I_{C^\sharp} be the ideal of C^\sharp in X^\sharp . Since $I_{C^\sharp} = (y_2, y_3, y_4)$, we have $I_{C^\sharp}^{(2)} = (y_2) + I_{C^\sharp}^2$ and $I_{C^\sharp}^{(3)} = y_2I_{C^\sharp} + I_{C^\sharp}^3$. Let $\bar{\omega}$ be a semi-invariant generator of ω_{X^\sharp} . For example we can take

$$\bar{\omega} = \frac{dy_1 \wedge dy_3 \wedge dy_4}{y_1}.$$

Obviously, $\operatorname{wt} \bar{\omega} \equiv 1 \pmod{4}$. By definitions of $\operatorname{gr}_C^i \mathcal{O}$ and $\operatorname{gr}_C^i \omega$ we get

$$\operatorname{gr}_C^1 \mathcal{O} = y_3y_1 \cdot \mathcal{O}_C \oplus y_4y_1^2 \cdot \mathcal{O}_C,$$

$$S^2 \operatorname{gr}_C^1 \mathcal{O} = y_3^2y_1^2 \cdot \mathcal{O}_C \oplus y_3y_4y_1^3 \cdot \mathcal{O}_C \oplus y_4^2y_1^4 \cdot \mathcal{O}_C,$$

$$\operatorname{gr}_C^2 \mathcal{O} = y_2y_1^3 \cdot \mathcal{O}_C \oplus y_3y_4y_1^3 \cdot \mathcal{O}_C \oplus y_4^2 \cdot \mathcal{O}_C,$$

$$\operatorname{gr}_C^0 \omega = y_1^3 \bar{\omega} \cdot \mathcal{O}_C,$$

$$\operatorname{gr}_C^2 \omega = y_2y_1^2 \bar{\omega} \cdot \mathcal{O}_C \oplus y_3y_4y_1^2 \bar{\omega} \cdot \mathcal{O}_C \oplus y_4^2y_1^3 \bar{\omega} \cdot \mathcal{O}_C,$$

where we note that $y_3^2 = -y_1y_2$ in $\operatorname{gr}_C^2 \mathcal{O}$. By definitions of i_P and w_P (see (3.1)), $i_P(2) = \operatorname{len} \operatorname{Coker} \alpha_2 = 1$ and $w_P(2) = \operatorname{len} \operatorname{Coker} \beta_2 = 3$. Hence, $w_P^*(2) = 0$. Computations for $i_P(1)$, $w_P(1)$ and $w_P^*(1)$ are similar (for $i_P(1)$, see [Mor88, 2.16]). \square

(9.4) Proposition. *Let $(X, C \simeq \mathbb{P}^1)$ be a \mathbb{Q} -conic bundle germ having two non-Gorenstein points P and Q and one Gorenstein singular point R . Then*

the indices of non-Gorenstein points are 2 and odd ≥ 3 . In particular, both P and Q are of type (IA) (cf. (9.2.1)).

Proof. We use notation of (9.2). Assume $k \geq 2$, $m \geq 2$, $n = 1$ by hypothesis. Up to permutation we also may assume that $k \leq m$. Apply L -deformation so that P , Q , R become ordinary. In particular, P and Q are of type (IA) points. If this new (X, C) is an extremal neighborhood, the fact follows by [Mor88, Th. 6.2]. Thus we may assume that (X, C) is a \mathbb{Q} -conic bundle germ. By (2.8) and (2.7.2) (iii) we have $-K_X \cdot C = d/km$ and $-K_{X'} \cdot C' = d^2/km$, where $d = \gcd(k, m)$ and (X', C') is as in (2.4.1). If $k = m$, then X' is Gorenstein and X cannot have three singular points by (2.9). Thus, $d \leq k < m$. If $H^1(\omega_{X'} \otimes \tilde{\otimes} \mathcal{O}_{X'}/I_{C'}^{(2)}) \neq 0$, then as in (7.3.3) by Theorem (4.4) we have $f'^{-1}(o') \subset \text{Spec } \mathcal{O}_{X'}/I_{C'}^{(2)}$, and so

$$2 = -K_{X'} \cdot f'^{-1}(o') < -3K_{X'} \cdot C'.$$

We get $3d^2 > 2km$ and $d = k = m$, a contradiction. Therefore,

$$(9.4.1) \quad H^1(\omega_{X'} \otimes \tilde{\otimes} \mathcal{O}_{X'}/I_{C'}^{(2)}) = 0, \quad H^1(\omega_X \otimes \tilde{\otimes} \mathcal{O}_X/I_C^{(2)}) = 0.$$

If $H^1(\text{gr}_C^2 \omega) = 0$, the arguments of [Mor88, 6.2.3] apply and we are done. So we assume $H^1(\omega_X \otimes \tilde{\otimes} \mathcal{O}_X/I_C^{(3)}) \neq 0$ and

$$H^1(\omega_{X'} \otimes \tilde{\otimes} \mathcal{O}_{X'}/I_{C'}^{(3)}) \neq 0.$$

Again apply Theorem (4.4) to (X', C') with $I_{C'}^{(3)}$. We obtain $4 \leq km < 3d^2$. Thus $d > 1$ and $X \neq X'$. Note that in diagram (2.4.1) the preimage $g^{-1}(R)$ consists of d Gorenstein points. By Lemma (9.2.2) $d = 2$. Hence, $k = 2$ and $m = 4$. Clearly, $w_P(0) = 1/2$. By Lemma (2.8) and (3.1.1) $w_Q(0) = 1 - 1/2 - 1/4 = 1/4$. Therefore, near Q we have $\text{ord } x_2 = 3$, so $\text{ord } x = (1, 3, 3)$. Further, by [Mor88, 5.1 (ii), 4.9 (ii), 5.3], $w_Q^*(2) = 3$, $w_R^*(2) = 2$, and $w_P^*(2) = 0$. By (3.1.8) $\deg \text{gr}_C^2 \omega = -5$. In particular, $h^1(\text{gr}_C^2 \omega) \geq 2$ and $h^1(\text{gr}_{C'}^2 \omega') \geq 2$.

Now we claim $H^0(X', \text{gr}_{C'}^1 \omega') = 0$. Note that (X', C') has three singular points: Q' of index 2 and two (III) points R' , R'' . By [Mor88, 4.9, 5.3] we have $w_{P'}^*(1) = -1$ and $w_{R'}^*(1) = w_{R''}^*(1) = 1$. Thus, $\deg \text{gr}_{C'}^1 \omega' = -2$ (see (3.1.8)). Since by (9.4.1) $H^1(\text{gr}_{C'}^1 \omega') = 0$, we have $\text{gr}_{C'}^1 \omega' \simeq \mathcal{O}_{C'}(-1) \oplus \mathcal{O}_{C'}(-1)$ and $H^0(\text{gr}_{C'}^1 \omega') = 0$.

Finally we apply Proposition (9.4.2) below for (X', C') with $a = 2$ and derive a contradiction. \square

(9.4.2) Proposition. *Let (X, C) be a \mathbb{Q} -conic bundle germ with smooth (Z, o) (C may be reducible). Assume that there exists a positive integer a such that*

$H^j(X, \text{gr}_C^i \omega) = 0$ for all j and all $i < a$ and such that $H^1(X, \text{gr}_C^a \omega) \neq 0$. Then

$$(9.4.3) \quad H^j(X, \omega_X \tilde{\otimes} I_C^{(i)}) = \begin{cases} 0 & (j = 0) \\ \omega_Z & (j = 1, i \leq a), \end{cases}$$

and $H^1(X, \text{gr}_C^a \omega) \simeq \mathbb{C}$.

Proof. We note that the first assertion follows from $H^0(X, \omega_X) = 0$ when $j = 0$ and from Lemma (4.1) when $j = 1$ and $i = 0$. Consider the natural exact sequence

$$0 \rightarrow \omega_X \tilde{\otimes} I_C^{(i+1)} \rightarrow \omega_X \tilde{\otimes} I_C^{(i)} \rightarrow \text{gr}_C^i \omega \rightarrow 0.$$

If (9.4.3) is proved for an $i (< a)$, we have $H^1(X, \omega_X \tilde{\otimes} I_C^{(i+1)}) \simeq H^1(X, \omega_X \tilde{\otimes} I_C^{(i)})$, which proves the assertion for $i + 1$. If we set $i = a$, we have a surjection $\mathcal{O}_Z \simeq \omega_Z \rightarrow H^1(X, \text{gr}_C^a \omega)$ which kills $\mathfrak{m}_{Z,o}$. Thus $H^1(X, \text{gr}_C^a \omega) \simeq \mathbb{C}$. \square

Proof of Theorem (9.1). By Lemmas (9.2.2) and (9.2.3) X has one or two non-Gorenstein points and by (9.2.1) these points are of types (IA), (IIA), or (III). The case (IIA)+(III)+(III) is disproved in Proposition (9.3) and the cases (IA)+(IIA)+(III) and (IIA)+(IIA)+(III) are disproved in Proposition (9.4). \square

Finally we note that the existence of a good member of $|-K_X|$ or $|-2K_X|$ in the cases (i) and (ii) of Theorem (9.1) can be shown as in [Mor88, 7.3].

§10. Two non-Gorenstein points case: general (bi)elephants

In this section we consider \mathbb{Q} -conic bundles with two non-Gorenstein points and no other singularities. The main result of this section is Theorem (10.10).

(10.1) Notation. Let $(X, C \simeq \mathbb{P}^1)$ be a \mathbb{Q} -conic bundle germ having two singular points P, P' of indices $m, m' \geq 2$. We assume that (X, C) is not toroidal because in the toroidal case the existence of a good divisor in $|-K_X|$ is an easy exercise (see (i) of (10.10)). Since (X, C) has at most one imprimitive point, we will assume that P' is primitive. Let s and \bar{m} be the splitting degree and the subindex of P . Recall that $m = s\bar{m}$, that $m \geq 4$ by (7.1.1) if P is imprimitive, and that $s = 1$ if P is primitive. Let I_C be the sheaf of ideals defining C in \mathcal{O}_X . Let $\pi^\sharp: (X^\sharp, P^\sharp) \rightarrow (X, P)$ (resp. $\pi^b: (X^b, P^b) \rightarrow (X, P')$) be the index-one cover and $C^\sharp = \pi^{\sharp-1}(C)_{\text{red}}$ (resp. $C^b = \pi^{b-1}(C)_{\text{red}}$). Let I_C^\sharp (resp. I_C^b) be the canonical lifting of I_C at P (resp. P'). Take normalized ℓ -coordinates (x_1, \dots, x_4) (resp. (x'_1, \dots, x'_4)) at P (resp. P') such that $a_i = \text{ord } x_i$ (resp. $a'_i = \text{ord } x'_i$).

By (4.4.3) and (7.2.2) we have $\mathrm{gr}_C^0 \omega \simeq \mathcal{O}_C(-1)$. We note that if $m = 2$ then P is primitive as seen above and we can reduce $m = 2$ to the case $m' = 2$ by switching P and P' .

Thus we distinguish the following three cases:

(10.1.1) $m' = 2$ and m is odd,

(10.1.2) $m, m' \geq 3$, and

(10.1.3) $m' = 2, m = 2n, n \in \mathbb{Z}_{>0}$.

(10.2) The case (10.1.1) is easy. Indeed, by (7.1.1) and Corollary (7.4.1) (X, C) is primitive. Hence the base (Z, o) is smooth by Corollary (2.7.4) and both non-Gorenstein points are of type (IA) by Theorem (8.6) (v). We get the case (1.3.5). The existence of a good member in $|-2K_X|$ (Proposition (1.3.7)) can be shown exactly as in [Mor88, 7.3]. From now on we consider cases (10.1.2) and (10.1.3).

(10.3) First, we treat the case (10.1.3) till the end of (10.4).

(10.3.1) **Lemma.** *In the case (10.1.3), (X, C) is locally primitive, n is even, and we have*

$$a_1 = 1, \quad a_2 = n + 1, \quad a_3 = 2n - 1, \quad a_4 = 2n.$$

In particular, (X, P) is of type (IA).

Proof. By (7.4.1) P is primitive. Hence by (2.8) and (2.7.2), (iii) we have $-K_X \cdot C = \frac{1}{2n}$. Further, (4.4.3) implies $\mathrm{gr}_C^0 \omega \simeq \mathcal{O}_C(-1)$. Thus by (3.1.1) we have

$$w_P(0) = 1 - w_{P'}(0) + K_X \cdot C = \frac{1}{2} - \frac{1}{2n}.$$

Hence, by Proposition (8.6) and [Mor88, 4.9 (i)] the point P is of type (IA). In this case, $w_P(0) = 1 - a_2/2n$ (see [Mor88, 4.9 (i)]). This gives us $a_2 = n + 1$. Since $\mathrm{gcd}(2n, a_2) = 1$, n is even. Finally, by Proposition (8.5) $\mathrm{siz}_P = 1$. Therefore, $a_1 a_2 < 2n$ and $a_1 = 1$. The rest is obvious. \square

(10.3.2) **Lemma (cf. [KM92, 2.13.1]).** *In the case (10.1.3), we can write*

$$\begin{aligned} (X, P) &= (y_1, y_2, y_3, y_4; \phi) / \mu_m(1, a_2, -1, 0; 0) \supset (C, P) = y_1\text{-axis} / \mu_m, \\ (X, P') &= (y'_1, y'_2, y'_3, y'_4; \phi') / \mu_2(1, 1, 1, 0; 0) \supset (C, P') = y'_1\text{-axis} / \mu_2, \end{aligned}$$

where $\phi \equiv y_1 y_3 \bmod (y_2, y_3)^2 + (y_4)$.

Proof. We only need to prove the last equality, which follows from the fact that (X, P) is a point of type cA/m . \square

Denote $\ell(P) := \text{len}_P I^{\sharp(2)}/I^{\sharp 2}$, where I^{\sharp} is the ideal defining C^{\sharp} in X^{\sharp} near P^{\sharp} .

(10.3.3) Lemma (cf. [KM92, 2.13.2]). *In the case (10.1.3) we have $\ell(P) \leq 1$ and $i_P(1) = 1$.*

Proof. Follows by (10.3.2) and [Mor88, 2.16]. \square

(10.3.4) Lemma (cf. [KM92, 2.13.3]). *In the case (10.1.3) we have $\ell(P') \leq 1$ and $i_{P'}(1) = 1$.*

Proof. Assume that $r := \ell(P') \geq 2$. Then by [Mor88, 2.16] the equation of X^b near P^b has the following form: $\phi' \equiv y_1'^r y_i' \bmod (y_2', y_3', y_4')^2$, where $i = 3$ (resp. 4) if r is odd (resp. even). Consider the following L -deformation $\phi'_\lambda = \phi' + \lambda y_1'^{r-2} y_i'$. Then (X_λ, C_λ) has three singular points of indices $m = 2n, 2$, and 1. This is impossible by [Mor88, 6.2] and (9.1). Therefore, $r \leq 1$. The last statement follows by [Mor88, 2.16 (ii)]. \square

(10.3.5) Corollary. $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ in the case (10.1.3).

Proof. Follows by (3.1.2) because $H^1(\text{gr}_C^1 \mathcal{O}) = 0$. \square

(10.3.6) Let $\mathcal{L} \subset \text{gr}_C^1 \mathcal{O}$ be a (unique) subsheaf such that $\mathcal{L} \simeq \mathcal{O}$. Note that \mathcal{L} is an ℓ -invertible \mathcal{O}_C -module. Let u_1 (resp. u_1') be an ℓ -free ℓ -basis at P (resp. P'). By [Mor88, Cor. 9.1.7] there is a subbundle $\mathcal{M} \simeq \mathcal{O}(-1)$ of $\text{gr}_C^1 \mathcal{O}$ such that $\text{gr}_C^1 \mathcal{O} = \mathcal{L} \oplus \mathcal{M}$ is an ℓ -splitting. Let u_2 (resp. u_2') be an ℓ -free ℓ -basis of \mathcal{M} at P (resp. P').

(10.3.7) Lemma (cf. [KM92, 2.13.8]). $ql \deg(\mathcal{M}, P') = 1$.

Proof. Since $ql \deg(\mathcal{M}, P') < m' = 2$, it is sufficient only to disprove the case $ql \deg(\mathcal{M}, P') = 0$. Assume that $ql \deg(\mathcal{M}, P') = 0$. Since y_2, y_3, y_4 form an ℓ -basis of $\text{gr}_C^1 \mathcal{O}$ at P , we have $\mathcal{M} \simeq (-1 + iP^{\sharp})$, where $i = 0, 1$, or $m - a_2 (= n - 1)$. Recall that $\text{gr}_C^0 \omega \simeq (-1 + (m - a_2)P^{\sharp} + P^b)$. Therefore,

$$\text{gr}_C^1 \omega \simeq \text{gr}_C^1 \mathcal{O} \otimes \tilde{\text{gr}}_C^0 \omega \simeq \mathcal{L} \otimes \tilde{\text{gr}}_C^0 \omega \oplus (-2 + (m - a_2 + i)P^{\sharp} + P^b).$$

The last expression is normalized (because $m - a_2 + i = n - 1 + i < m = 2n$). Hence, $H^1(\text{gr}_C^1 \omega) \neq 0$. Put $V := \text{Spec } \mathcal{O}_X/I_C^{(2)}$ and $V' := \text{Spec } \mathcal{O}_{X'}/I_{C'}^{(2)}$.

(notation of (2.4.1)). As in the proof of Corollary (4.4.5) we get $H^1(\mathcal{O}_V \tilde{\otimes} \omega_X) \neq 0$ and therefore $H^1(\mathcal{O}_{V'} \tilde{\otimes} \omega_{X'}) \neq 0$ (notation of (2.4.1)). By Theorem (4.4) $V' \supset f'^{-1}(o')$. In particular,

$$2 = -K_{X'} \cdot f'^{-1}(o') \leq -K_{X'} \cdot V' = -3K_{X'} \cdot C' = 3/n.$$

This implies $n = 1$, a contradiction. \square

(10.3.8) Lemma (cf. [KM92, 2.13.10]). $ql \deg(\mathcal{M}, P) = m - a_2$.

Proof. First we note that the arguments of [KM92, 2.13.10.1-2] apply to our case and show in particular that if $ql \deg(\mathcal{M}, P) \neq m - a_2$ and if $H^1(\omega_X/F^4(\omega, J)) = 0$, then m is odd while m is even in our case. Hence it is enough to derive a contradiction assuming that $ql \deg(\mathcal{M}, P) \neq m - a_2$ and $H^1(\omega_X/F^4(\omega, J)) \neq 0$.

Let J be the C -laminal ideal of width 2 such that $J/I_C^{(2)} = \mathcal{L}$. Then

$$0 \neq H^1(\omega_X/F^4(\omega, J)) = H^1(\omega_X/J^2\omega_X) = H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J^{(2)}).$$

As in the proof of Lemma (10.3.7) put $V := \text{Spec}_{X'} \mathcal{O}_{X'}/J'^{(2)}$, where J' is the pull-back of J on X' (we use notation of (2.4.1)). Recall that $I_C \supset J \supset I_C^2$. Thus $I'_C \supset J' \supset I_C'^2$, where I'_C is the ideal sheaf of C' . Since $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J^{(2)}) \neq 0$, we have $H^1(\omega_{X'} \tilde{\otimes} \mathcal{O}_{X'}/J'^{(2)}) \neq 0$. By Theorem (4.4) $V \supset f'^{-1}(o')$. Let $Q' \in C'$ be a general point. Then in a suitable coordinate system (x, y, z) near Q' we may assume that C' is the z -axis. So, $I'_C = (x, y)$ and $I_C'^{(2)} = (x^2, xy, y^2)$. Since $J'/I_C'^{(2)}$ is of rank 1, by changing coordinates x, y we may assume that $J' = (x, y^2)$ near Q' . Then $J'^{(2)} = (x^2, xy^2, y^4)$ and $V \equiv lC'$, where $l = \text{len}_0 \mathbb{C}[x, y]/(x^2, xy^2, y^4) = 6$. Similarly to the proof of Lemma (10.3.7) we have

$$2 = -K_{X'} \cdot f'^{-1}(o') \leq -K_{X'} \cdot V = -6K_{X'} \cdot C' = 6/n.$$

Since n is even ≥ 2 , we get only one possibility $n = 2$.

As in the proof of Lemma (10.3.7) we see that $ql \deg(M, P) = 0$ (because $m - a_2 = 1$). Then by (10.3.7) $\mathcal{M} \simeq (-1 + P^{\sharp})$. Now we consider the base change (2.4.1). Here g is a double cover and X' is of index two. Set $P^{\sharp} := g^{-1}(P)$, the unique non-Gorenstein point of X' . (Note that in our notation $P' \in X$, which is different from the notation of (2.4)–(2.7)). Note that the index-one covers of (X, P) and (X', P^{\sharp}) coincide. Let \mathcal{M}' be the ℓ -invertible sheaf on X' , the pull-back of \mathcal{M} to X' . We have $\mathcal{M}' \simeq (-1 + 0P^{\sharp})$ and $\text{gr}_{C'}^0 \omega_{X'} \simeq (-1 + P^{\sharp})$,

where $C' := g^{-1}(C)_{\text{red}}$. Hence, $H^1(X', \omega_{X'} \tilde{\otimes} \mathcal{M}') \neq 0$. Let J' be the ideal on X' lifting J . Then taking account of the exact sequence

$$0 \longrightarrow I_{C'}/J' (= \mathcal{M}') \longrightarrow \mathcal{O}_{X'}/J' \longrightarrow \mathcal{O}_{C'} \longrightarrow 0$$

and isomorphisms $\omega_{X'} \tilde{\otimes} \mathcal{O}_{C'} \simeq \text{gr}_{C'}^0 \omega \simeq \mathcal{O}_{C'}(-1)$, we get $H^1(X', \omega_{X'} \tilde{\otimes} \mathcal{O}_{X'}/J') \simeq H^1(X', \omega_{X'} \tilde{\otimes} \mathcal{M}') \neq 0$. Hence by Theorem (4.4) we have $f'^{-1}(o') \subset \text{Spec } \mathcal{O}_{X'}/J'$ which means $2 \leq 2/2 = 1$, a contradiction. \square

(10.3.9) Remark. In the above notation the case $n = 2$ can be disproved also by considering possible actions of involutions on index two \mathbb{Q} -conic bundles $f': X' \rightarrow Z'$ (see (12.1.7) and (12.1.8)).

Thus we have proved the following

(10.4) Proposition. *In the case (10.1.3) there is an ℓ -isomorphism $\mathcal{M} \simeq \text{gr}_C^0 \omega$.*

(10.5) Lemma. *Up to permutations we may assume that P' is of type (IA) and P is of type (IA $^\vee$), (IA), or (IIA). Moreover, $\text{siz}_P = \text{siz}_{P'} = 1$.*

Proof. If (X, C) is not locally primitive, the assertion follows by (7.1.2) and (7.4). We assume that (X, C) is locally primitive. By Propositions (8.2), (8.3), and (8.4) points P and P' are of types (IA) or (IIA). If both P and P' are of type (IIA), then $w_P(0) + w_{P'}(0) = 3/2 > 1$ (see [Mor88, 4.9 (i)]). This contradicts (3.1.1). Thus we may assume that P' is of type (IA) modulo permutation of P and P' . To prove the last statement consider L -deformation $(X_\lambda, C_\lambda \ni P_\lambda, P'_\lambda)$ of $(X, C \ni P, P')$ so that P_λ, P'_λ are ordinary points. In particular, they are of type (IA). By [Mor88, 4.7] $\text{siz}_{P_\lambda} = \text{siz}_P$ and $\text{siz}_{P'_\lambda} = \text{siz}_{P'}$. If (X_λ, C_λ) is a \mathbb{Q} -conic bundle germ, the assertion follows by Proposition (8.5). Otherwise we can apply [Mor88, Th. 6.6]. \square

Temporarily we consider the following situation.

(10.6) Notation. Assume that P and P' are ordinary. Then $i_P(1) = \text{siz}_P = 1$ and $i_{P'}(1) = \text{siz}_{P'} = 1$. Hence, $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ (because $H^1(\text{gr}_C^1 \mathcal{O}) = 0$). Let $\mathcal{L} \subset \text{gr}_C^1 \mathcal{O}$ be a (unique) subsheaf such that $\mathcal{L} \simeq \mathcal{O}$. Note that \mathcal{L} is an ℓ -invertible \mathcal{O}_C -module. Let u_1 (resp. u'_1) be an ℓ -free ℓ -basis at P (resp. P').

(10.6.1) Theorem (cf. [Mor88, 9.3]). *Notation as in (10.6). Then both (C^\sharp, P^\sharp) and (C^\flat, P^\flat) are smooth, $\text{wt } u_1 \equiv -1 \pmod{m}$, $\text{wt } u'_1 \equiv -1 \pmod{m'}$ and furthermore we may assume that $a_1 = a'_1 = 1$.*

The proof follows [Mor88, 9.3]. We will treat (10.6.1) in four cases (10.6.2)–(10.6.5) below.

(10.6.2) Case: P is of type (IA^\vee) .

If both P and P' are primitive, then in a suitable coordinate system near P^\sharp the ideal I_C^\sharp is generated by $x_1^{a_2} - x_2^{a_1}$ and x_3 (because P is ordinary). Hence, either $\text{wt } u_1 \equiv a_1 a_2$ or $\text{wt } u_1 \equiv -a_1 \pmod{m}$ holds and the corresponding assertion also holds for P' . Modulo permutation of P and P' and if $a_2 = 1$ (resp. $a'_2 = 1$) modulo further permutation of a_1 and a_3 (resp. a'_1 and a'_3) there are three cases:

(10.6.3) Case: $\text{wt } u_1 \equiv a_1 a_2$, $a_2 \not\equiv \pm 1 \pmod{m}$, $\text{wt } u'_1 \equiv a'_1 a'_2$, $a'_2 \not\equiv \pm 1 \pmod{m'}$.

(10.6.4) Case: $\text{wt } u_1 \equiv a_1 a_2$, $a_2 \not\equiv \pm 1 \pmod{m}$, $\text{wt } u'_1 \equiv a'_3 \pmod{m'}$.

(10.6.5) Case: $\text{wt } u_1 \equiv a_3 \pmod{m}$, $\text{wt } u'_1 \equiv a'_3 \pmod{m'}$.

We will show that only the case (10.6.5) is possible and $a_1 = a'_1 = 1$. By [Mor88, Cor. 9.1.7] there is a subbundle $\mathcal{M} \simeq \mathcal{O}(-1)$ of $\text{gr}_C^1 \mathcal{O}$ such that $\text{gr}_C^1 \mathcal{O} = \mathcal{L} \oplus \mathcal{M}$ is an ℓ -splitting. Let u_2 (resp. u'_2) be an ℓ -free ℓ -basis of \mathcal{M} at P (resp. P').

(10.6.6) Let J be the C -laminal ideal of width 2 such that $J/I_C^{(2)} = \mathcal{L}$, and our symbols are compatible with those in [Mor88, 9.3.2].

Note that $w_P(0) = 1 - a_2/\bar{m}$ and $w_{P'}(0) = 1 - a'_2/m'$. Then by (3.1.1) and $(K_X \cdot C) < 0$ we have

$$(10.6.7) \quad 1 < \frac{a_2}{\bar{m}} + \frac{a'_2}{m'} \quad ([\text{Mor88}, 9.3.4]).$$

Using Lemma (2.8) that holds only for \mathbb{Q} -conic bundles, we have

$$(10.6.8) \quad 1 + \frac{d}{mm'} = \frac{a_2}{\bar{m}} + \frac{a'_2}{m'},$$

where $d = mm'/\text{lcm}(\bar{m}, m') = s \text{gcd}(\bar{m}, m')$ by (2.7.2), (iii).

(10.6.9) Disproof of the case (10.6.2). This case corresponds to [Mor88, 9.3.ipr], and is disproved by the same argument as [Mor88, 9.3.5].

Hence we note that $\bar{m} = m$ below till the end of the proof of (10.6.1).

(10.6.10) Disproof of the case (10.6.3). This case corresponds to [Mor88, 9.3.a], and is disproved by the same argument as [Mor88, 9.3.6].

(10.6.11) Disproof of the case (10.6.4). This case corresponds to [Mor88, 9.3.b], and is disproved by the same argument as [Mor88, 9.3.7].

(10.6.12) Treatment of the case (10.6.5). This case corresponds to [Mor88, 9.3c], and the arguments of [Mor88, 9.3.8] work except for [Mor88, 9.3.8.6], which we prove below using (10.6.8).

We have $\text{wt } u_1 \equiv a_3 \pmod{m}$, $\text{wt } u'_1 \equiv a'_3 \pmod{m'}$. We will prove that $a_1 = a'_1 = 1$. By symmetry we may assume that $a'_2/m' > 1/2$ (see (10.6.7)). Since $\text{siz}_{P'} = 1$, $m' \geq a'_1 a'_2$. This gives us $a'_1 = 1$. We will prove $a_1 = 1$. Assume that $a_1 \geq 2$. Then computations [Mor88, 9.3.8.3–9.3.8.4] apply and give us $a_1 = 2$, $a_2 = 1$. In particular, C^\sharp is smooth over P and P' . Further by [Mor88, 9.3.8.5] we have

$$(10.6.13) \quad \begin{aligned} ql_C(\mathcal{L}) &= 2P^\sharp + P'^\sharp, \\ ql_C(\mathcal{M}) &= -1 + (m-2)P^\sharp + (m' - a'_2)P'^\sharp, \end{aligned}$$

and $ql_C(\text{gr}_C^0 \omega) = -1 + (m-1)P^\sharp + (m' - a'_2)P'^\sharp$.

(10.6.14) Claim (cf. [Mor88, 9.3.8.6]). $m \geq 5$.

Proof. Assume that $m < 5$. Since $a_1 = 2$ and $\text{gcd}(m, a_1) = 1$, we have $m = 3$. Then by (10.6.8)

$$2m' + d = 3a'_2, \quad d = \text{gcd}(3, m').$$

Now one can see that computations of [Mor88, 9.3.8.6] apply and give us $\chi(F^1(\omega, J)/F^4(\omega, J)) < 0$. From the exact sequence

$$0 \longrightarrow F^1(\omega, J)/F^4(\omega, J) \longrightarrow \omega_X/F^4(\omega, J) \longrightarrow \text{gr}_C^0 \omega \longrightarrow 0$$

we get

$$\chi(\omega_X/F^4(\omega, J)) = \chi(F^1(\omega, J)/F^4(\omega, J)) + \chi(\omega_X/F^1(\omega, J)) < 0.$$

(we note that $F^1(\omega, J) = \text{Sat}_{\omega_X}(I_C \omega_X)$ and $\omega_X/F^1(\omega, J) = \text{gr}_C^0 \omega$). In particular, we have $H^1(\omega_X/F^4(\omega, J)) \neq 0$. Recall that $I_C \supset J \supset I_C^{(2)}$ and $F^4(\omega, J) = \text{Sat}_{\omega_X}(J^2 \omega_X)$. Assume that (Z, o) is smooth, i.e., $3 \nmid m'$. Then by Theorem (4.4) $f^{-1}(o) \subset \text{Spec } \mathcal{O}_X/J^{(2)} \subset \text{Spec } \mathcal{O}_X/I_C^{(4)}$. We get a contradiction (cf. (7.3.3)):

$$2 = -K_X \cdot f^{-1}(o) < -10K_X \cdot C = 10/(3m'), \quad m' = 1.$$

Now assume that $m' = 3m''$, $m'' \geq 2$. Take a Weil divisor ξ such that $ql_C \xi = P^\sharp - m''P'^\sharp$. Then ξ is a 3-torsion in $\text{Cl}^{\text{sc}} X$. Taking (10.6.13) into account we

obtain

$$\begin{aligned} ql_C(\mathcal{M} \tilde{\otimes} \xi) &= -1 + (m-1)P^\sharp + (2m'' - a'_2)P'^\sharp = \\ &= -2 + (m-1)P^\sharp + (5m'' - a'_2)P'^\sharp. \end{aligned}$$

Since $a'_2 = 2m'' + 1$, $5m'' - a'_2 = 3m'' - 1$. So, the last expression is normalized. By [Mor88, 8.9.1 (iii)] $\deg_C \mathcal{M} \tilde{\otimes} \xi = -2$. Note that $\text{gr}_C^1 \mathcal{O} \tilde{\otimes} \xi = (\mathcal{L} \tilde{\otimes} \xi) \hat{\oplus} (\mathcal{M} \tilde{\otimes} \xi)$. Hence, $H^1(\text{gr}_C^1 \mathcal{O} \tilde{\otimes} \xi) \neq 0$. This is a contradiction, and $m \geq 5$ as proved. \square

The remainder of the proof is the same as [Mor88, 9.3.8.7]. Thus Theorem (10.6.1) is proved.

Now we treat the case (10.1.2) from here till the end of (10.9).

(10.7) Proposition. *In notation and assumptions of (10.1.2) (X, C) has no (IIA) type points.*

Proof. Assume that P is of type (IIA) (and P' is of type (IA) by Lemma (10.5)). Let $d := \gcd(4, m')$. By [Mor88, Th. 4.9, (i)] $w_P(0) = 3/4$. Then by Lemma (2.8) and (3.1.1) we have

$$w_{P'}(0) = 1 - w_P(0) + K_X \cdot C = 1/4 - d/(4m') < 1/2.$$

Hence, $a'_1 = 1$ (see [Mor88, Prop. 5.1]). Moreover,

$$(10.7.1) \quad d + 3m' = 4a'_2$$

(because $w_P(0) = 1 - a'_2/m'$ by [Mor88, Th. 4.9, (i)]). Applying L -deformations at P' and L' deformation at P (see [Mor88, 4.12.2]) we may assume that P' is ordinary and $(X, P) \simeq \{y_1 y_2 + y_3^2 + y_4^3 = 0\}/\mu_4(1, 1, 3, 2)$, where C^\sharp is the y_1 -axis. This new (X, C) is again a \mathbb{Q} -conic bundle germ by [Mor88, 9.4]. Applying [Mor88, 9.4.3-9.4.5] we get an ℓ -splitting $\text{gr}_C^1 \mathcal{O} = \mathcal{L} \hat{\oplus} \mathcal{M}$, where $\mathcal{L} \simeq \mathcal{O}$ and $\mathcal{M} \simeq \mathcal{O}(-1)$. Moreover, $ql_C(\mathcal{L}) = P^\sharp + P'^\sharp$ and $ql_C(\mathcal{M}) = -1 + 2P^\sharp + (m' - a'_2)P'^\sharp$ and we may assume that y_3 is an ℓ -free ℓ -basis of \mathcal{L} at P .

Now one can see that computations of [Mor88, 9.4.6] apply and give us $\chi(F^1(\omega, J)/F^4(\omega, J)) < 0$. If $2 \nmid m'$, then we can apply the second half of the proof of (10.6.14) to get a contradiction:

$$2 = -K_X \cdot f^{-1}(o) < -10K_X \cdot C = 10/(4m'), \quad m' = 1.$$

Thus $2 \mid m'$. If $m' = 2m''$ and $2 \nmid m''$, then considering diagram (2.4.1) one has $H^1(\omega_{X'}/\text{Sat}_{\omega_{X'}}(J^2\omega_{X'})) \neq 0$ and similarly gets:

$$2 = -K_{X'} \cdot f'^{-1}(o') < -10K_{X'} \cdot C' = -20K_X \cdot C = 20/2m' = 5/m''.$$

Thus $m'' = 1$, and one sees $d = 2$ and $a'_2 = 2$ by (10.7.1), which contradicts $\gcd(m', a'_2) = 1$, a condition on (IA) points. Hence $4|m'$ and we set $m' = 4m''$.

Take a Weil divisor ξ such that $ql_C(\xi) = P^\sharp - m''P'^\sharp$. Then ξ is a 4-torsion in $\text{Cl}^{\text{sc}} X$. By [Mor88, 9.4.5] we have

$$ql_C(\mathcal{M}) = -1 + 2P^\sharp + (4m'' - a'_2)P'^\sharp$$

Recall that $a'_2 = 3m'' + 1$. Taking this into account we obtain

$$ql_C(\mathcal{M} \tilde{\otimes} \xi) = -1 + 3P^\sharp - P'^\sharp = -2 + 3P^\sharp + (4m'' - 1)P'^\sharp.$$

The last expression is normalized. By [Mor88, 8.9.1 (iii)] $\deg_C \mathcal{M} \tilde{\otimes} \xi = -2$. Note that $\text{gr}_C^1 \mathcal{O} \tilde{\otimes} \xi = (\mathcal{L} \tilde{\otimes} \xi) \tilde{\oplus} (\mathcal{M} \tilde{\otimes} \xi)$. Hence, $H^1(\text{gr}_C^1 \mathcal{O} \tilde{\otimes} \xi) \neq 0$. This is a contradiction. \square

(10.7.2) Corollary ([Mor88, 9.4.7]). *In notation and assumptions of (10.1.2) points P and P' are type (IA) points such that $a_1 = a'_1 = 1$, and moreover $\ell(P) < m$ and $\ell(P') < m'$.*

Proof. By (10.5) and (10.7) P is of type (IA) and P' is of type (IA) or (IA^\vee) . Replacing (X, C) with L -deformation we may assume that both P and P' are ordinary. Then by (10.6.9), (10.6.1), and [Mor88, 9.3, 9.4] P' is of type (IA) and $a_1 = a'_1 = 1$ (L -deformation does not change a_i 's because P and P' are of type (IA) or (IA^\vee)). If $\ell(P) \geq m$, then an L' -deformation (X_λ, C_λ) (see [Mor88, 4.12.2]) has at least one Gorenstein singular point besides P_λ and P'_λ . This contradicts (9.1) and [Mor88, 6.2]. Thus $\ell(P) < m$. By symmetry we also have $\ell(P') < m'$. \square

(10.7.3) Corollary ([Mor88, 9.4.8]). *In notation and assumptions of (10.1.2) we have $i_P(1) = i_{P'}(1) = 1$ and an isomorphism $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$.*

Proof. Since $\ell(P) < m$ and $\ell(P') < m'$, by [Mor88, 2.16 (ii)] we have $i_P(1) = i_{P'}(1) = 1$. Hence $\deg \text{gr}_C^1 \mathcal{O} = -1$, see (3.1.2). Taking account of $H^1(\text{gr}_C^1 \mathcal{O}) = 0$ we obtain the last statement. \square

(10.8) Proposition ([Mor88, 9.8]). *We have*

$$\ell(P) + ql \deg(\mathcal{L}, P) = \ell(P') + ql \deg(\mathcal{L}, P') = 1.$$

Proof. Similar to the proof of [Mor88, Theorem 9.8]. \square

(10.9) Proposition ([Mor88, 9.9.1]). *In the case (10.1.2), there is an ℓ -isomorphism $\mathcal{M} \simeq \text{gr}_C^0 \omega$.*

Proof. Since $\text{gr}_C^0 \omega \simeq \mathcal{O}(-1)$, it is sufficient to show that $ql \deg(\mathcal{M}, P) = ql \deg(\text{gr}_C^0 \omega, P) = R(a_2)$ and $ql \deg(\mathcal{M}, P') = ql \deg(\text{gr}_C^0 \omega, P') = R'(a'_2)$. By symmetry it is sufficient to prove for example the first equality. According to (10.8) there are two cases.

(10.9.1) Case: $\ell(P) = 0$, $ql \deg(\mathcal{L}, P) = 1$. Then (X, P) is a cyclic quotient singularity of type $\frac{1}{m}(1, a_2, -1)$. If u_1 is an ℓ -free ℓ -basis of \mathcal{L} at P , then $R(\text{wt } u_1) = 1$, so $\text{wt } u_1 \equiv -1 \pmod{m}$. An ℓ -free ℓ -basis of $\text{gr}_C^1 \mathcal{O}$ at P is x_2, x_3 . Hence we can put $u_1 = x_3$ and x_2 is an ℓ -free ℓ -basis of \mathcal{M} . Therefore, $ql \deg(\mathcal{M}, P) = R(\text{wt } x_2) = R(a_2)$.

(10.9.2) Case: $\ell(P) = 1$, $ql \deg(\mathcal{L}, P) = 0$. Then we can choose a some coordinate system so that $(X^\#, P^\#)$ is given by $\phi = 0$ with $\phi \equiv x_1 x_3 \pmod{(x_2, x_3, x_4)^2}$ and $C^\#$ is the x_1 -axis (see [Mor88, 2.16]). If u_1 is an ℓ -free ℓ -basis of \mathcal{L} at P , then $R(\text{wt } u_1) = 0$, so $\text{wt } u_1 \equiv 0 \pmod{m}$. Again an ℓ -free ℓ -basis of $\text{gr}_C^1 \mathcal{O}$ at P is x_2, x_4 . Hence we can put $u_1 = x_4$ and x_2 is an ℓ -free ℓ -basis of \mathcal{M} . Therefore, $ql \deg(\mathcal{M}, P) = R(\text{wt } x_2) = R(a_2)$. □

Taking Propositions (10.4) and (10.9) into account one can see that all the arguments and computations from [Mor88, 9.9.2-9.9.10] apply in our case. This proves (iii) of the following theorem (cf. [KM92, 2.2.4]).

(10.10) Theorem (cf. [Mor88, Th. 9.10], [KM92, 2.2.4]). *Let $(X, C \simeq \mathbb{P}^1)$ be a \mathbb{Q} -conic bundle germ having two non-Gorenstein points P, P' of indices $m, m' \geq 2$ and no other singularities. Then P, P' are of type (IA) by (10.5) and (10.7), and the following assertions hold.*

- (i) *If $m' = 2$ and m is odd, then the general member of $|-2K_X|$ does not contain C and has only log terminal singularities.*
- (ii) *If (X, C) is toroidal, then a general member $F \in |-K_X|$ does not contain C . It has two connected components. Each of them is a Du Val singularity of A-type.*
- (iii) *If (X, C) is not toroidal, we further assume either $m, m' \geq 3$ or $m' = 2$ and m is even. Then a general member $F \in |-K_X|$ is a normal surface containing C , smooth outside of $\{P, P'\}$, with Du Val points of A-type at P, P' . Furthermore C on F is contractible to a Du Val point of A-type.*

Note that (i) and (ii) of (10.10) are easy (cf. [Mor88, Th. 7.3]). In (ii) one can also take as F the sum of two horizontal toric divisors.

§11. Two non-Gorenstein points case: the classification

The following is the main result of this section.

(11.1) Theorem. *Let $(X, C \simeq \mathbb{P}^1)$ be a \mathbb{Q} -conic bundle germ having two points of indices $m, m' \geq 2$ and no other singularities. Assume either $m, m' \geq 3$ or $m' = 2$ and m is even. Then (X, C) is either toroidal or as in (1.2.2).*

The above theorem is an easy consequence of Theorem (10.10) and Proposition (11.2) below.

(11.2) Proposition (cf. [Pro97a, §4]). *Let $f: (X, C) \rightarrow (Z, o)$ be a non-Gorenstein \mathbb{Q} -conic bundle germ with $C \simeq \mathbb{P}^1$. Assume that the general element $F \in |-K_X|$ containing C has only Du Val singularities. Let $F \xrightarrow{f_1} \bar{F} \rightarrow Z$ be the Stein factorization and let $\bar{P} = f_1(C)$. Assume that (\bar{F}, \bar{P}) is a singularity of type A. Then one of the following holds:*

- (i) f is as in (1.2.2), or
- (ii) X is of index 2 and (Z, o) is smooth.

Proof of Proposition (11.2). By the inversion of adjunction [Kol92, 17.6] the log divisor $K_X + F$ is plt. Consider diagram (2.4.1) and put $F' := g^*F$. We may assume that $Z' \simeq \mathbb{C}^2$ and $Z \simeq \mathbb{C}^2/\mu_d(1, q)$, where $\gcd(d, q) = 1$. By [Kol92, 20.3] $K_{X'} + F' = g^*(K_X + F) \sim 0$ is plt. In particular, F' is normal and irreducible. Further, diagram (2.4.1) induces the following diagram

$$(11.2.1) \quad \begin{array}{ccc} (F', C') & \xrightarrow{g_{F'}} & (F, C) \\ f'_1 \downarrow & & f_1 \downarrow \\ (\bar{F}', \bar{P}') & \xrightarrow{\bar{g}} & (\bar{F}, \bar{P}) \\ f'_2 \downarrow & & f_2 \downarrow \\ (Z', o') & \xrightarrow{h} & (Z, o) \end{array}$$

where the vertical arrows are Stein factorizations of restrictions $f'|_{F'}$ and $f|_F$. It is clear that f'_2 and f_2 are double covers. By adjunction $K_{F'} \sim 0$ and f'_1 is a crepant morphism contracting C' . Since \bar{g} is étale in codimension one and (F, P) is a singularity of type A, (\bar{F}', \bar{P}') is also of type A. Note that (\bar{F}', \bar{P}') cannot be smooth (because f'_1 is non-trivial).

Consider the case $d \geq 2$. Then by Lemma (7.5.1) (\bar{F}', \bar{P}') is of type A_1 . In this case, F' is smooth and so is X' (see, e.g., [Pro97a, Lemma 1.4]). Therefore, f is the quotient of a smooth conic bundle by μ_d . By Proposition (2.9) we get the case (1.2.2).

Thus we may assume that $d = 1$ (and $X' = X$). Let $R \subset Z$ be the ramification divisor of f_2 . Since (\bar{F}, \bar{P}) is of type A , in some coordinate system on $Z = \mathbb{C}^2$, R is given by the equation $x^k + y^2 = 0$. Let $\Gamma \subset Z$ is given by $x = 0$ and let $S := f^*\Gamma$. By the inversion of adjunction the log divisor $K_Z + \Gamma + \frac{1}{2}R$ is log canonical (lc). So are $K_{\bar{F}} + f_2^*\Gamma = f_2^*(K_Z + \Gamma + \frac{1}{2}R)$ and $K_F + f^*\Gamma = K_F + S|_F$. Again by the inversion of adjunction $K_X + F + S$ is lc near F . Shrinking X we may assume that $K_X + F + S$ is lc everywhere. Replacing Γ with a general hyperplane section through o , we may assume that S is smooth outside of C . Then $K_X + S$ is plt. In particular, S is normal and has only log terminal singularities of type T [KSB88]. Let $D := F|_S$. Then $K_S + D \sim 0$ is lc and $D \supset C$. By the classification of two-dimensional log canonical singularities [Kaw88], [Kol92, Ch. 3] $K_S + C$ is plt.

The restriction $f_S: S \rightarrow \Gamma$ is a rational curve fibration such that $-K_S$ is f_S -ample. If C is a Cartier divisor on S , then S is smooth and so is X . Take the minimal positive n such that nC is Cartier. Then $nC \sim 0$. This induces an étale in codimension one μ_n -cover $\pi: S^\natural \rightarrow S$ such that $C^\natural := \pi^*C \sim 0$. The divisor $K_{S^\natural} + C^\natural = \pi^*(K_S + C)$ is plt (see, e.g., [Kol92, 20.3]). Hence, C^\natural is smooth and so is S^\natural . Thus S is a quotient of $S^\natural \simeq \mathbb{C} \times \mathbb{P}^1$ by μ_n . It is easy to see that S has singular points of types $\frac{1}{n}(1, q)$ and $\frac{1}{n}(-1, q)$, where $\gcd(n, q) = 1$. These points are of type T if and only if

$$(q+1)^2 \equiv (q-1)^2 \equiv 0 \pmod{n}$$

(see [KSB88]). This implies $n = 2$ or 4 . If $n = 2$, then S is Gorenstein and so is X , a contradiction. Hence $n = 4$, so the singularities of S are of types $\frac{1}{4}(1, 1)$ and A_3 . By [KSB88] (X, C) has exactly one non-Gorenstein point which is of index 2. \square

§12. Index two \mathbb{Q} -conic bundles

Index two \mathbb{Q} -conic bundles were classified in [Pro97a, §3]. Under the condition that the base (Z, o) is smooth, these are quotients of some elliptic fibrations by an involution. Here we propose an alternative description and sketch a different proof. (Note that a \mathbb{Q} -conic bundle of index two over a singular base is either of type (1.2.4) or toroidal [Pro97a, §3]).

(12.1) Theorem. *Let $f: (X, C) \rightarrow (Z, o)$ be a \mathbb{Q} -conic bundle germ of index*

two. Assume that (Z, o) is smooth. Fix an isomorphism $(Z, o) \simeq (\mathbb{C}^2, 0)$. Then there is an embedding

$$(12.1.1) \quad \begin{array}{ccc} X & \hookrightarrow & \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2 \\ & \searrow f & \downarrow p \\ & & \mathbb{C}^2 \end{array}$$

such that X is given by two equations

$$(12.1.2) \quad \begin{aligned} q_1(y_1, y_2, y_3) - \psi_1(y_1, \dots, y_4; u, v) &= 0, \\ q_2(y_1, y_2, y_3) - \psi_2(y_1, \dots, y_4; u, v) &= 0, \end{aligned}$$

where ψ_i and q_i are weighted quadratic in y_1, \dots, y_4 with respect to $\text{wt}(y_1, \dots, y_4) = (1, 1, 1, 2)$ and $\psi_i(y_1, \dots, y_4; 0, 0) = 0$. The only non-Gorenstein point of X is $(0, 0, 0, 1; 0, 0)$. Up to projective transformations, the following are the possibilities for q_1 and q_2 :

(12.1.3) $q_1 = y_1^2 - y_2^2$ and $q_2 = y_1 y_2 - y_3^2$; then $f^{-1}(o)$ is reduced and has exactly four irreducible components;

(12.1.4) $q_1 = y_1 y_2$ and $q_2 = (y_1 + y_2) y_3$; then $f^{-1}(o)$ has three irreducible components, one of them has multiplicity 2;

(12.1.5) $q_1 = y_1 y_2 - y_3^2$ and $q_2 = y_1 y_3$; then $f^{-1}(o)$ has two irreducible components, one of them has multiplicity 3;

(12.1.6) $q_1 = y_1^2 - y_2^2$ and $q_2 = y_3^2$; then $f^{-1}(o)$ has two irreducible components, both of multiplicity 2;

(12.1.7) $q_1 = y_1 y_2 - y_3^2$ and $q_2 = y_1^2$; then $f^{-1}(o)$ is irreducible of multiplicity 4;

(12.1.8) $q_1 = y_1^2$ and $q_2 = y_2^2$; then $f^{-1}(o)$ is also irreducible of multiplicity 4.

Conversely, if $X \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2$ is given by equations of the form (12.1.2) and singularities of X are terminal, then the projection $f: (X, f^{-1}(0)_{\text{red}}) \rightarrow (\mathbb{C}^2, 0)$ is a \mathbb{Q} -conic bundle of index 2.

Sketch of the proof. First we prove the last statement. Assume that X has only terminal singularities. Then X does not contain the surface $\{y_1 = y_2 = y_3 = 0\} = \text{Sing } \mathbb{P} \times \mathbb{C}^2$ (otherwise both ψ_1 and ψ_2 do not depend on y_4).

By the adjunction formula, $K_X = -L|_X$, where L is a Weil divisor on $\mathbb{P} \times \mathbb{C}^2$ such that the restriction $L|_{\mathbb{P}}$ is $\mathcal{O}_{\mathbb{P}}(1)$. Therefore, $X \rightarrow \mathbb{C}^2$ is a \mathbb{Q} -conic bundle. It is easy to see that the only non-Gorenstein point of X is $(0, 0, 0, 1; 0, 0)$ and it is of index 2.

Now let $f: (X, C) \rightarrow (Z, o) \simeq (\mathbb{C}^2, 0)$ be a \mathbb{Q} -conic bundle germ of index two. Let $P \in X$ be a point of index 2. We claim that P is the only non-Gorenstein point. Indeed, if C is irreducible, the assertion follows by Corollary (2.7.2), (iii). If $C = \cup C_i$ is reducible, the same holds by Lemma (4.4.2) and [KM92, Th. 4.2, Prop. 4.6]. Thus P is the only non-Gorenstein point on X . By (5.2.1) each (X, C_i) is of type (IA) at P . (The case (IB) is excluded by Proposition (8.3) and [Mor88, Th. 6.3]). Hence the general member $F \in |-K_X|$ satisfies $F \cap C = \{P\}$ and has only Du Val singularity at P (see [Mor88, Th. 7.3]).

Let $\pi: (X^\sharp, P^\sharp) \rightarrow (X, P)$ be the index-one cover and let $F^\sharp = \pi^{-1}(F)_{\text{red}}$ be the pull-back of F . Let $\Gamma := f^{-1}(o)$ be the scheme fiber and let $\Gamma^\sharp = \pi^{-1}(\Gamma)$.

(12.1.9) Lemma.

$$\mathcal{O}_{F^\sharp \cap \Gamma^\sharp} \simeq \mathbb{C}[x, y]/(xy, x^2 + y^2).$$

Furthermore μ_2 -action is given by $\text{wt}(x, y) \equiv (1, 1) \pmod{2}$.

Proof. Since (F^\sharp, P^\sharp) is a Du Val singularity, we may assume that $(F^\sharp, P^\sharp) \subset (\mathbb{C}_{x, y, z}^3, 0)$. The scheme $F^\sharp \cap \Gamma^\sharp$ is defined in $\mathbb{C}_{x, y, z}^3$ by three equations $\alpha = \beta = \gamma = 0$, where two of them are coordinates on $Z = \mathbb{C}^2$, and the rest is the defining equation of $F^\sharp \subset \mathbb{C}^3$. Since the morphism $F^\sharp \rightarrow Z$ is flat and of degree 4, we have

$$\mathcal{O}_{F^\sharp \cap \Gamma^\sharp} \simeq \mathbb{C}\{x, y, z\}/(\alpha, \beta, \gamma)$$

is of length 4. Furthermore μ_2 acts on the ring so that $\text{wt}(x, y, z, \alpha, \beta, \gamma) \equiv (1, 1, 0, 0, 0, 0) \pmod{2}$ because the quotient $(F^\sharp, P^\sharp)/\mu_2$ is Du Val, and, in particular, Gorenstein. If $\alpha, \beta, \gamma \in (x, y, z)^2$, then $\text{len } \mathbb{C}\{x, y, z\}/(\alpha, \beta, \gamma) \geq 8$, which is a contradiction. Hence, in view of the weights, we may assume that $\alpha = (\text{unit}) \cdot z + \alpha_1 \cdot \beta + \alpha_2 \cdot \gamma$ modulo permutation of α, β, γ . Thus we have

$$\mathcal{O}_{F^\sharp \cap \Gamma^\sharp} \simeq \mathbb{C}\{x, y\}/(\beta, \gamma).$$

Since $\text{wt}(x, y, \beta, \gamma) \equiv (1, 1, 0, 0) \pmod{2}$, we see $\beta, \gamma \in (x, y)^2$. Hence we may assume that $\beta \equiv xy \pmod{(x, y)^3}$ modulo coordinate change of x, y and change of β, γ . Modulo analytic change of coordinates x, y , we may assume $(\beta, \gamma) =$

$(xy, x^a + y^b)$ for some $a, b \geq 2$. Since the ring $\mathcal{O}_{F^\sharp \cap \Gamma^\sharp}$ is of length 4, we have $4 = a + b$ and hence $a = b = 2$. \square

Using this lemma one can apply arguments of [Mor75, pp. 631–633] to get the desired embedding $X \subset \mathbb{P}(1, 1, 1, 2) \times Z$ considering the graded anti-canonical \mathcal{O}_Z -algebra

$$\mathcal{R} := \bigoplus_{i \geq 0} \mathcal{R}_i, \quad \text{where } \mathcal{R}_i := H^0(\mathcal{O}_X(-iK_X)).$$

We sketch the main idea.

Let w be a local generator of $\mathcal{O}_{X^\sharp}(-K_X)$ at P^\sharp , let u, v be coordinates on $Z = \mathbb{C}^2$, and let $z = 0$ be the local equation of F^\sharp in (X^\sharp, P^\sharp) . Using the vanishing of $H^1(\mathcal{O}_X(-K_X))$ for $i > 0$ and the exact sequence

$$0 \rightarrow \mathcal{O}_X(-(i-1)K_X) \rightarrow \mathcal{O}_X(-iK_X) \rightarrow \mathcal{O}_F(-iK_X) \rightarrow 0$$

one can see

$$\mathcal{R}_i/(zw)\mathcal{R}_{i-1} \simeq H^0(\mathcal{O}_F(-iK_X)), \quad i > 0.$$

Therefore,

$$\mathcal{R}_i/(zw)\mathcal{R}_{i-1} + (u, v)\mathcal{R}_i = (\mathcal{O}_{F^\sharp \cap \Gamma^\sharp}(-iK_X))^{\mu_2}.$$

By Lemma (12.1.9) we have an embedding

$$\mathcal{R}/(zw, u, v)\mathcal{R} \hookrightarrow (\mathbb{C}[x, y, w]/(xy, x^2 + y^2))^{\mu_2}.$$

Using $R_0/(u, v)R_0 = \mathbb{C}$, one can easily see that

$$\mathcal{R}/(zw, u, v)\mathcal{R} = \mathbb{C}[y_1, y_2, y_4]/(y_1y_2, y_1^2 + y_2^2),$$

where $y_1 = xw, y_2 = yw, y_4 = w^2$. Put $y_3 := zw$. Then similarly to [Mor75, pp. 631–633] we obtain

$$\mathcal{R} \simeq \mathcal{O}_Z[y_1, y_2, y_3, y_4]/\mathcal{I},$$

where \mathcal{I} is generated by the following regular sequence

$$\begin{aligned} y_1y_2 + y_3\ell_1(y_1, \dots, y_3) &+ \psi_1(y_1, \dots, y_4; u, v), \\ y_1^2 + y_2^2 + y_3\ell_2(y_1, \dots, y_3) &+ \psi_2(y_1, \dots, y_4; u, v) \end{aligned}$$

with $\psi_i(y_1, \dots, y_4; 0, 0) = 0$. \square

As is seen in Theorem (1.2), a \mathbb{Q} -conic bundle is often constructed as a quotient of one of index two by a cyclic group. Theorem (12.1) is useful in

such a context. Finally we provide facts which are used in the study of \mathbb{Q} -conic bundles with imprimitive points (cf. Proposition (7.6)).

(12.1.10) Proposition. *Assume that in the notation of Theorem (12.1) a cyclic group μ_d acts on X and Z so that f is μ_d -equivariant. Then the diagram (12.1.1) can be chosen to be μ_d -equivariant.*

Proof. The sheaf $\mathcal{O}_X(-K_X)$ has a natural μ_d -linearization. Hence the embedding $X = \text{Proj } \mathcal{R} \hookrightarrow \mathbb{P}(1, 1, 1, 2)$ is μ_d -equivariant. \square

The following is obvious.

(12.1.11) Lemma. *In notation of Theorem (12.1) assume that f has an equivariant μ_d -action and that $f^{-1}(o)$ has two irreducible components, both of multiplicity 2 (i.e., we are in case (12.1.6)), which are permuted by some element of μ_d . Then the coordinates y_1, \dots, y_4, u, v can be chosen so that they and the equations (12.1.2) are semi-invariant.*

Proof. Indeed, by (12.1.10) the action of μ_d preserves the pencil $\lambda_1(q_1 - \psi_1) + \lambda_2(q_2 - \psi_2)$. It remains to note that in (12.1.6) q_1 and q_2 are the only degenerate quadratic forms in this pencil and they cannot be interchanged. \square

(12.1.12) Lemma. *In notation and assumptions of Theorem (12.1) and Lemma (12.1.11) assume additionally that $d = 2$. Furthermore assume that μ_2 acts on X and Z so that the action is free in codimension one, has a unique fixed point $P = (0, 0, 0, 1; 0, 0)$, and the quotient $(X, P)/\mu_2$ is a terminal singularity. Then modulo change of coordinates, we are in case (12.1.6) with the action written as follows:*

$$y_1 \mapsto y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto y_3, \quad y_4 \mapsto -y_4, \quad u \mapsto -u, \quad v \mapsto -v.$$

Proof. By Lemma (12.1.11) we can choose the coordinates y_1, \dots, y_4, u, v so that they and the equations (12.1.2) are semi-invariant. Since the action of μ_2 on $Z \simeq \mathbb{C}^2$ is free outside of o , this action is given by $u \mapsto -u, v \mapsto -v$. Modulo multiplication of ± 1 on the μ_2 -linearization of $\mathcal{O}(-K_X)$, we may assume also that $y_3 \mapsto y_3$. Then $y_i \mapsto \pm y_i$ for all i .

Recall that we are in the case (12.1.6) with $X \subset \mathbb{P}(1, 1, 1, 2)_{y_1, y_2, y_3, y_4} \times \mathbb{C}_{u, v}^2$. Since $(1, 1, 0, 0; 0, 0) \in X$, the point is not μ_2 -fixed by the assumption. Hence $y_1 y_2 \mapsto -y_1 y_2$. Modulo permutation of y_1, y_2 , we have $y_1 \mapsto y_1$ and $y_2 \mapsto -y_2$. It remains to show only that $y_4 \mapsto -y_4$. Assume to the contrary that $y_4 \mapsto y_4$.

Let $U \subset \mathbb{P}(1, 1, 1, 2)$ be the chart $y_4 \neq 0$. Then $U \simeq \mathbb{C}_{z_1, z_2, z_3}^3 / \mu_2(1, 1, 1)$. Let X^\sharp be the pull-back of $X \cap (U \times \mathbb{C}_{u, v}^2)$ on $\mathbb{C}_{z_1, z_2, z_3}^3 \times \mathbb{C}_{u, v}^2$ and let $P^\sharp \in X^\sharp$ be the preimage of P .

Since the induced map $X^\sharp \rightarrow X$ is étale in codimension one, $(X^\sharp, P^\sharp) \rightarrow (X, P)$ is the index-one cover. Hence $(X^\sharp, P^\sharp) \rightarrow (X, P)/\mu_2$ is also the index-one cover of the terminal point $(X, P)/\mu_2$ of index 4 (the last is true because the action of μ_2 is free in codimension one). Hence the morphism is a μ_4 -covering by the structure of terminal singularities. However $(X, P)/\mu_2$ is the quotient of (X^\sharp, P^\sharp) by commuting μ_2 -actions:

$$(z_1, z_2, z_3, u, v) \mapsto (-z_1, -z_2, -z_3, u, v), (z_1, -z_2, z_3, -u, -v)$$

This is a contradiction, and we have $y_4 \mapsto -y_4$ as claimed. \square

Acknowledgments

The work was carried out at Research Institute for Mathematical Sciences (RIMS), Kyoto University. The second author would like to thank RIMS for invitations to work there in 2005-2006, for hospitality and wonderful conditions of work.

The research of the first author was supported in part by JSPS Grant-in-Aid for Scientific Research (B)(2), No. 16340004. The second author was partially supported by Grant CRDF-RUM1-2692-MO-05.

Finally we are extremely grateful to the referee, who has pointed out so many mistakes and helped us improve the presentation.

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