# QUASI-CLASSICAL LIE-SUPER ALGEBRA AND LIE-SUPER TRIPLE SYSTEMS

by

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**Abstract**

Notions of quasi-classical Lie-super algebra as well as Lie-super triple systems have been given and studied with some examples. Its application to Yang-Baxter equation has also been given.

1991 Mathematics Subject Classification Primary 17A40, 17B60

## 1. Quasi-classical Lie-super Algebra

In this note, we will first introduce the notion of quasi-classical Lie-super algebra as well as quasi-classical Lie-super triple system with some examples. We will apply them to obtain some new solutions of Yang-Baxter equation in section 3. Algebras in this note are assumed to be finite dimensional over a field of characteristic not two.

Let $L$ be a Lie-super algebra, i.e. it is first a direct sum

$$L = V_0 \oplus V_1 \tag{1.1}$$

of bosonic ($V_0$) and fermionic ($V_1$) spaces. We denote the grade by

$$\sigma(x) = \begin{cases} 0\ , & \text{if} \quad x \in V_0 \\ 1\ , & \text{if} \quad x \in V_1 \end{cases} \tag{1.2}$$

and write

$$(-1)^{\sigma(x)\sigma(y)} = (-1)^{xy} \quad . \tag{1.3}$$

Then, the Lie-product $[x, y]$ satisfies the following conditions:

(i) $\sigma([x, y]) = \{\sigma(x) + \sigma(y)\} \bmod 2$ (1.4a)

(ii) $[y, x] = -(-1)^{xy}[x, y]$ (1.4b)

(iii) $(-1)^{xy}[[x, z], y] + (-1)^{yz}[[y, x], z] + (-1)^{zx}[[z, y], x] = 0 \quad .$ (1.4c)

Suppose now that $L$ possesses a bilinear non-degenerate form $< .|. >$ satisfying conditions:

(i) $< x|y > = 0$, unless $\sigma(x) = \sigma(y)$ (1.5a)

(ii) $< y|x > = (-1)^{xy} < x|y >$ (1.5b)

(iii) $< [x, y]|z > = < x|[y, z] > \quad .$ (1.5c)

We will then call $L$ quasi-classical. If $L$ is a simple Lie-super algebra with non-zero Killing form [1], we may then set

$$< x|y > = \ \mathrm{Tr}(adx\ ady)$$

where Tr hereafter stands for the super-trace. However, the converse is not necessarily true as we will soon see.

Let $e_1, e_2, \ldots, e_N$ with $N = \dim L$ be a basis of $L$ with

$$\sigma(e_j) = \sigma_j \tag{1.6a}$$

$$[e_j, e_k] = \sum_{\ell=1}^{N} C_{jk}^{\ell} e_\ell \quad . \tag{1.6b}$$

Suppose that $L$ possesses a Casimir invariant $I_2$ given by

$$I_2 = \sum_{j,k=1}^{N} g^{jk} e_j e_k \tag{1.7a}$$

$$g^{jk} = (-1)^{\sigma_j \sigma_k} g^{kj} \tag{1.7b}$$

$$g^{jk} = 0 \quad \text{if} \quad \sigma_j \neq \sigma_k \quad . \tag{1.7c}$$

The condition $[I_2, e_\ell] = 0$ is equivalent to the validity of

$$\sum_{m=1}^{N} g^{jm} C_{m\ell}^{k} = \sum_{m=1}^{N} C_{\ell m}^{j} g^{mk} \quad . \tag{1.8}$$

**Proposition 1.1**

A necessary and sufficient condition of a Lie-super algebra $L$ being quasi-classical is the existence of the Casimir invariant $I_2$ such that $g^{jk}$ is non-degenerate with its inverse $g_{jk}$, i.e.

$$\sum_{\ell=1}^{N} g^{k\ell} g_{\ell j} = \sum_{\ell=1}^{N} g_{j\ell} g^{\ell k} = \delta_j^k \tag{1.9a}$$

$$g_{jk} = (-1)^{\sigma_j \sigma_k} g_{kj} \tag{1.9b}$$

$$g_{jk} = 0 \quad \text{unless} \quad \sigma_j = \sigma_k \quad . \tag{1.9c}$$

**Proof**

Suppose that $L$ is quasi-classical. Setting

$$g_{jk} = < e_j | e_k > \quad ,$$

it has its inverse $g^{jk}$. The relation

$$< [e_j, e_k] | e_\ell > = < e_j | [e_k, e_\ell] >$$

can easily be shown to be equivalent to Eq. (1.8) so that $I_2$ defined by Eq. (1.7a) is the Casimir invariant. Conversely, let us assume that the Casimir invariant $I_2$ exists. We introduce the bilinear form $< .|. >$ in $L$ by

$$< e_j|e_k > = g_{jk}$$

which defines the desired bilinear non-degenerate supersymmetric form satisfying Eqs. (1.5). ■

**Remark 1.1**

This proposition is a straightforward generalization of the result of [2].

We will now give some examples of quasi-classical Lie and Lie-super algebras below.

**Example 1.1**

Let $L = V_0 = \{e, f, x_1, \ldots, x_n, y_1, \ldots, y_n\}$ with $V_1 = 0$. Only non-zero Lie products are assumed to be given by

$$[x_j, f] = -[f, x_j] = x_j$$

$$[y_j, f] = -[f, y_j] = -y_j$$

$$[x_j, y_k] = -[y_k, x_j] = \delta_{jk} e$$

for $j, k = 1, 2, \ldots, n$. It is easy to verify that $L$ is a Lie algebra with the Casimir invariant

$$I_2 = \lambda e^2 + ef + fe - \sum_{j=1}^{n} (x_j y_j + y_j x_j)$$

for arbitrary constant $\lambda$. Note that $e$ is a center element of $L$. We can now introduce the inner product by

$$< e|f > = < f|e > = 1 \quad , \quad < f|f > = -\lambda \quad ,$$

$$< x_j|y_k > = < y_k|x_j > = -\delta_{jk} \quad ,$$

while all other inner products are assumed to be zero. We can readily verify that $L$ is quasi-classical.

**Example 1.2**

Let $L = V_0 \oplus V_1$ with

$$V_0 = \{e, f\} \quad , \quad V_1 = \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \quad ,$$

where only non-zero products are assumed to be given by

$$[x_j, f] = -[f, x_j] = x_j \quad ,$$
$$[y_j, f] = -[f, y_j] = -y_j \quad ,$$
$$[x_j, y_k] = [y_k, x_j] = \epsilon_{jk} e \quad .$$

Here, $\epsilon_{jk} = -\epsilon_{kj}$ is antisymmetric with its inverse $\epsilon^{jk}$. Especially, $n$ must now be even. The Casimir invariant is found to be

$$I_2 = \lambda e^2 + ef + fe + \sum_{j,k=1}^{n} \epsilon^{jk} \{x_j y_k - y_k x_j\} \quad .$$

We introduce inner products by

$$< f|f > = -\lambda \quad , \quad < e|f > = < f|e > = 1 \quad ,$$

$$< x_j|y_k > = - < y_k|x_j > = -\epsilon_{jk} \quad ,$$

while all other $< .|. >$ are zero. Here, $\lambda$ is again an arbitrary constant. We can verify that $L$ is quasi-classical.

**Remark 1.2**

Both examples 1.1 and 1.2 given above are not simple but solvable, since they satisfy the identity

$$[L, [[L, L], [L, L]]] = 0 \quad . \tag{1.10}$$

However, they are not nilpotent since $[L, [L, L]] = [L, L] \neq 0$. We will next give examples of nilpotent quasi-classical Lie and Lie-super algebras.

**Example 1.3**

$$L = V_0 = \{x_j, u_j, y_A, v_A, Y_{jA}\} \quad \text{with} \quad V_1 = 0 \quad ,$$

where indices $j$ and $A$ assumes $j = 1, 2, \ldots, n$ and $A = 1, 2, \ldots, m$. Only non-zero commutators are given by

$$[x_j, Y_{kA}] = -[Y_{kA}, x_j] = \delta_{jk} v_A$$

$$[y_A, Y_{jB}] = -[Y_{jB}, y_A] = -\delta_{AB} u_j$$

$$[x_j, y_A] = -[y_A, x_j] = -Y_{jA}$$

for $j, k = 1, 2, \ldots, n$ and $A, B = 1, 2, \ldots, m$. $L$ can be verified to be a Lie algebra with center elements $\{u_j, v_A\}$. The Casimir invariants is found to be

$$I_2 = \sum_{j=1}^{n} (x_j u_j + u_j x_j) + \sum_{A=1}^{m} (v_A y_A + y_A v_A) + \sum_{j=1}^{n} \sum_{A=1}^{m} \Lambda_{jA} \Lambda_{jA} \quad .$$

Actually, we can add bilinear terms involving center elements $u_j$ and $v_A$ to this expression. However, we will not do so here for simplicity. The corresponding inner products are calculated to be

$$< Y_{jA} | Y_{kB} > = \delta_{jk} \delta_{AB}$$

$$< x_j | u_k > = < u_k | x_j > = \delta_{jk}$$

$$< v_A | y_B > = < y_B | v_A > = \delta_{AB}$$

while all other $< . | . >$ are zero.

**Example 1.4**

$$L_0 = V_0 \oplus V_1 \quad \text{with} \quad V_0 = \{x_j, u_j\} \quad , \quad V_1 = \{y_A, v_A, Y_{jA}\}$$

as in Example 1.3, except for the fact that we replace relations for $[y_A, Y_{jB}]$, $< v_A | y_B >$ etc. by

$$[y_A, Y_{jB}] = [Y_{jB}, y_A] = -\epsilon_{AB} u_j \quad , \quad < Y_{jA} | Y_{kB} > = \delta_{jk} \epsilon_{AB} \quad ,$$

$$< v_A | y_B > = - < y_B | v_A > = \epsilon_{AB}$$

for a symplectic form $\epsilon_{AB} = -\epsilon_{BA}$ with its inverse $\epsilon^{AB}$. The Casimir invariant $I_2$ will now be given by

$$I_2 = \sum_{j=1}^{n}(x_j u_j + u_j x_j) + \sum_{A,B=1}^{m} \epsilon^{AB}(v_A y_B - y_B v_A)$$
$$+ \sum_{j=1}^{n} \sum_{A,B=1}^{m} \epsilon^{AB} \Lambda_{jA} \Lambda_{jB} \quad .$$

**Remark 1.3**

Let us define $L_n (n = 1, 2, \ldots)$ by $L_1 = L$, and $L_{n+1} = [L, L_n]$ inductively. If we have $L_{n+1} = 0$ but $L_n \neq 0$, then we say that the Lie super-algebra $L$ is nilpotent of the length $n$. The examples 1.3 and 1.4 satisfy $L_3 \neq 0$ but $L_4 = 0$ so that both are nilpotent with length 3.

**Remark 1.4**

The non-degenerate bilinear form $< x|y >$ is not unique. Note that the examples 1.1 and 1.2 contain an arbitrary parameter $\lambda$. This is due to the existence of the center element $e$, as the following proposition will show. Some other examples of quasi-classical Lie algebras which are not super algebra are also found in ref. [3].

**Proposition 1.2**

Let a Lie-super algebra $L$ possess two bilinear forms $< x|y >_j$ $(j = 1, 2)$ satisfying conditions Eqs. (1.5). Suppose that the adjoint representation of $L$ is irreducible i.e., that if $A \epsilon End\ L$ is grade-preserving and satisfies $[adx, A] = 0$ for all $x \epsilon L$, then $A = \lambda Id$ for a constant $\lambda$. Here $Id$ is the identity mapping in $L$. Then, if $< x|y >_1$ is non-degenerate, we have

$$< x|y >_2 = \lambda < x|y >_1$$

for a constant $\lambda$. We note that we need not assume the non-degeneracy of $< x|y >_2$.

**Proof**

Since $L$ is finite dimensional and since $< x|y >_1$ is assumed to be non-degenerate, the standard reasoning implies the existence of $A \epsilon End\ L$ such that

$$< x|y >_2 = < Ax|y >_1 \quad .$$

Moreover, $A$ is grade-preserving, i.e. $\sigma(Ax) = \sigma(x)$. Now, the condition $< [y,x]|z >_j = < y|[x,z] >_j$ is then rewritten as

$$< [A, adx]y|z >_1 = 0$$

which leads to $[A, adx] = 0$ because of the non-degeneracy of $< y|z >_1$. The irreducibility assumption leads to the desired result $A = \lambda Id$ and hence $< x|y >_2 = \lambda < x|y >_1$.

**Remark 1.5**

The adjoint representation is irreducible, if $L$ is simple and, if the underlying field is algebraically closed. ■

Applying a theorem due to Dieudonné (see [4] p. 24) on an algebra possessing an associative bilinear form, we have also the following proposition.

**Proposition 1.3**

Suppose that we have $[B, B] \neq 0$ for every ideal $B$ of a quasi- classical Lie-super algebra $L$. Then, $L$ is uniquely expressible as a direct sum

$$L = B_1 \oplus B_2 \oplus \ldots \oplus B_t$$

of simple ideals $B_j$.

## 2. Quasi-classical Lie-super Triple System

A $Z_2$-graded vector space $V$ is called a $\delta$ Lie-super triple system for $\delta = \pm 1$, if it possesses a triple linear product $V \otimes V \otimes V \to V$ satisfying

(0) $\sigma([x,y,z]) = (\sigma(x) + \sigma(y) + \sigma(z)) \pmod 2$ (2.1a)

(1) $[y,x,z] = -\delta(-1)^{xy}[x,y,z]$ (2.1b)

(2) $(-1)^{xz}[x,y,z] + (-1)^{yx}[y,z,x] + (-1)^{zy}[z,x,y] = 0$ (2.1c)

(3) $$[u,v,[x,y,z]] = [[u,v,x],y,z] + (-1)^{(u+v)x}[x,[u,v,y],z] + (-1)^{(u+v)(x+y)}[x,y,[u,v,z]] \quad . \tag{2.1d}$$

Especially, the case of $\delta = 1$ defines a Lie-super triple system while the other case of $\delta = -1$ may be termed an anti-Lie-super triple system as in [5].

Moreover, suppose that there exists a non-degenerate bilinear form $< .|. >$ in $V$ obeying conditions:

$$(1)\ < x|y > = 0 \quad \text{unless} \quad \sigma(x) = \sigma(y) \tag{2.2a}$$

$$(2)\ < y|x > = \delta(-1)^{xy} < x|y > \tag{2.2b}$$

$$(3)\ < [x, y, u]|v > = -(-1)^{(x+y)u} < u|[x, y, v] > \quad . \tag{2.2c}$$

We then call the $\delta$ Lie-super triple system $V$ quasi-classical.

We will first prove the following:

**Proposition 2.1**

Let $V$ be a $\delta$ Lie-super triple system with a possible exception of the validity of Eq. (2.1d). Moreover assume the validity of Eq. (2.2b). The following 3 conditions are then equivalent to each other:

$$(1)\ < [x, y, u]|v > = -(-1)^{(x+y)u} < u|[x, y, v] > \tag{2.3a}$$

$$(2)\ < [x, y, u]|v > = -(-1)^{(u+v)y} < x|[u, v, y] > \tag{2.3b}$$

$$(3)\ < x|[y, u, v] > = (-1)^{xy+uv} < y|[x, v, u] > \quad . \tag{2.3c}$$

**Proof**

(i) <u>(2) → (1)</u>

Letting $u \leftrightarrow v$ in (2), it gives

$$\begin{aligned} < [x, y, u]|v > &= -\delta(-1)^{uv} < [x, y, v]|u > \\ &= -(-1)^{uv}(-1)^{(x+y+v)u} < u|[x, y, v] > \\ &= -(-1)^{(x+y)u} < u|[x, y, v] > \end{aligned}$$

which is (1).

(ii) <u>(3) → (2)</u>

$$\begin{aligned} < [x, y, u]|v > &= \delta(-1)^{v(x+y+u)} < v|[x, y, u] > \\ &= \delta(-1)^{v(x+y+u)}(-1)^{vx+yu} < x|[v, u, y] > \\ &= -(-1)^{y(u+v)} < x|[u, v, y] > \end{aligned}$$

which is (2).

(iii) <u>(2) → (3)</u>

Because of (i), we may assume the validity of both (1) and (2). Then

$$< u|[x,y,v] > = -(-1)^{(x+y)u} < [x,y,u]|v >$$

by (1). However, $< [x,y,u]|v > = -(-1)^{y(u+v)} < x|[u,v,y] >$ by (2). Combining both, we obtain

$$< u|[x,y,v] > = (-1)^{xu+yv} < x|[u,v,y] > \quad .$$

Interchanging $x \to y \to u \to x$, this leads to (3).

(iv) <u>(1) → (2)</u>

We first note that (1) implies

$$< [x,y,u]|v > = -\delta(-1)^{uv} < [x,y,v]|u > \quad . \tag{2.4}$$

Using Eq. (2.1c), we calculate

$$\begin{aligned} < [x,y,u]|v > &= -(-1)^{(x+y)u} < u|[x,y,v] > \\ &= (-1)^{(x+y)u}\big\{(-1)^{xv+yx} < u|[y,v,x] > \\ &\qquad + (-1)^{xv+vy} < u|[v,x,y] > \big\} \\ &= -(-1)^{x(u+v+y)+uv} < [y,v,u]|x > \\ &\qquad - (-1)^{v(x+y+u)+yu} < [v,x,u]|y > \quad . \end{aligned}$$

Now, we let $u \leftrightarrow v$ and note Eq. (2.4). We calculate then

$$\begin{aligned} 2 < [x,y,u]|v > &= < [x,y,u]|v > \; - \; \delta(-1)^{uv} < [x,y,v]|u > \\ &= (-1)^{x(u+v+y)+yv}\delta < (-1)^{uv}[v,y,u] + (-1)^{yv}[y,u,v]|x > \\ &\qquad - (-1)^{v(x+y)+yu} < (-1)^{uv}[v,x,u] + (-1)^{xv}[x,u,v]|y > \\ &= -\delta(-1)^{(u+v)(x+y)+xy} < [u,v,y]|x > \\ &\qquad + (-1)^{(u+v)(x+y)} < [u,v,x]|y > \quad . \end{aligned}$$

Now, interchanging $x \leftrightarrow u$ and $y \leftrightarrow v$ in Eq. (2.4), we have $< [u,v,x]|y > = -\delta(-1)^{xy} < [u,v,y]|x >$ so that

$$\begin{aligned} < [x,y,u]|v > &= -\delta(-1)^{(u+v)(x+y)+xy} < [u,v,y]|x > \\ &= -(-1)^{(u+v)y} < x|[u,v,y] > \end{aligned}$$

which is (2). This completes the proof. ∎

Next, we will define left and right multiplication operators $V \otimes V \to End\ V$ by

$$L(x,y)z = [x,y,z] \tag{2.5a}$$

$$R(x,y)z = (-1)^{z(x+y)}[z,x,y] \quad , \tag{2.5b}$$

and set

$$[L(u,v), R(x,y)] = L(u,v)R(x,y) - (-1)^{(x+y)(u+v)}R(x,y)L(u,v) \tag{2.6}$$

and similarly for $[L(u,v),\ L(x,y)]$.

**Lemma 2.1**

$$L(y,x) = -\delta(-1)^{xy}L(x,y) \tag{2.7a}$$

$$[L(u,v), L(x,y)] = L([u,v,x],y) + (-1)^{(u+v)x}L(x,[u,v,y]) \tag{2.7b}$$

$$[L(u,v), R(x,y)] = R([u,v,x],y) + (-1)^{(u+v)x}R(x,[u,v,y]) \quad . \tag{2.7c}$$

**Proof**

Eqs. (2.7a) and (2.7b) are immediate consequences of Eqs. (2.1b) and (2.1d). To show Eq. (2.7c), we calculate

$$\begin{aligned}
[L(u,v), R(x,y)]z &= (-1)^{(x+y)z}\{[u,v,[z,x,y]] - [[u,v,z],x,y]\} \\
&= (-1)^{(x+y+u+v)z}\{[z,[u,v,x],y] + (-1)^{x(u+v)}[z,x,[u,v,y]]\} \\
&= R([u,v,x],y)z + (-1)^{x(u+v)}R(x,[u,v,y])z
\end{aligned}$$

which proves (2.7c). ∎

**Proposition 2.2**

Let $V$ be a $\delta$ Lie-super triple system. If $< x|y >_1$ defined by

$$< x|y >_1 = \frac{1}{2}\ \mathrm{Tr}\{R(x,y) + \delta(-1)^{xy}R(y,x)\}$$

is non-degenerate, then $V$ is quasi-classical. Here Tr stands for the supertrace as before.

**Proof**

The conditions Eqs. (2.2a) and (2.2b) follow readily from the definition. Taking the supertrace of both sides, Eq. (2.7c) gives

$$\text{Tr } R([u,v,x],y) + (-1)^{(u+v)x} \text{ Tr } R(x,[u,v,y]) = 0$$

which leads to the validity of

$$< [u,v,x]|y >_1 = -(-1)^{(u+v)x} < x|[u,v,y] >_1 \quad .$$

**Remark 2.1**

We can prove contrarily Tr $L(x,y) = 0$ identically.

We shall now give some examples of quasi-classical $\delta$ Lie-super triple system.

**Example 2.1**

Let $V$ be a $Z_2$-graded vector space with a non-degenerate bilinear form $< x|y >$ satisfying

(i) $< x|y > = 0 \quad \text{unless} \quad \sigma(x) = \sigma(y)$

(ii) $< y|x > = \delta(-1)^{xy} < x|y > \quad .$

Then, the triple product

$$[x,y,z] = < y|z > x - \delta(-1)^{xy} < x|z > y$$

defines a quasi-classical $\delta$ Lie-super triple system.

**Example 2.2**

Let $V$ be as above, and let $P \epsilon End\ V$ satisfy conditions

(i) $\sigma(Px) = \sigma(x)$

(ii) $< x|Py > = < Px|y >$

(iii) $P^2 = cId$

for a constant $c$, where $Id$ stands for the identity mapping. The triple product defined by

$$[x,y,z] = < y|z > Px + \ < y|Pz > x - \delta(-1)^{xy}\{< x|z > Py + \ < x|Pz > y\} \qquad (2.8)$$

gives a quasi-classical $\delta$ Lie-super triple system. Moreover, we have

$$[Px, Py, Pz] = cP[x, y, z] \quad .$$

If $P = \frac{1}{2}Id$, then this case reduces to the example 2.1.

**Example 2.3**

Let $L$ be a quasi-classical Lie-super algebra ($\delta = 1$). If we introduce a triple product $[x, y, z]$ in $L$ by

$$[x, y, z] = [[x, y], z] \quad ,$$

then $L$ becomes a quasi-classical Lie-super triple system with $\delta = 1$. We may note that we then have

$$< [x, y, u]|v > = < [x, y]|[u, v] >$$

from which we can verify the validity of Eqs. (2.3).

**Remark 2.2**

We can calculate $< x|y >_1$ of the Proposition 2.2 for our various examples. First, the case of example 2.3 gives

$$< x|y >_1 = \ \mathrm{Tr}(adx\ ady)$$

i.e., the Killing form of the Lie-super algebra $L$. On the other side, we calculate

$$< x|y >_1 = (N_0 - 1) < x|y >$$

and

$$< x|y >_1 = (\mathrm{Tr}\ P) < x|y > \ + (N_0 - 2) < x|Py >$$

for examples (2.1) and (2.2), respectively. Here, we have set

$$N_0 = \ \mathrm{Tr}\ 1 = \dim V_0 - \dim V_1 \quad .$$

However, we find Tr $L(x, y) = 0$ for all cases in accordance with the Remark 2.1. ∎

Because of an intimate relationship between Lie-super algebra and Lie-super triple system for $\delta = 1$, we will hereafter restrict ourselves to consideration only of the case $\delta = 1$, unless it is stated otherwise.

**Remark 2.3**

Some connection exists between example 2.2 given above and example 1.1 or 1.2 of the previous section. Let $L$ be the quasi-classical Lie or Lie-super algebra of either 1.1 or 1.2. Let $P \epsilon End\ L$ be defined by

$$Pf = e \quad , \quad Pe = Px_j = Py_j = 0 \quad (j = 1, 2, \ldots, n)$$

which satisfies $P^2 = 0$ and $< Px|y > = < x|Py >$.

We can readily verify that $[[x, y], z]$ coincides with the expression $[x, y, z]$ given by Eq. (2.8) of the example 2.2 for the same $< x|y >$. ∎

As we stated in example 2.3, we can construct a quasi-classical Lie-super triple system from a quasi-classical Lie-super algebra. The converse is also true as we will see below. To see it, we first define $M$ to be a linear span of the left multiplication operator $L(x, y)$ defined by Eq. (2.5a), i.e.,

$$M = \left\{ Y | Y = \sum_{j,k} c_{jk} L(x_j, y_k) \right\} \tag{2.9}$$

for constants $c_{jk}$. Then, $M$ is a Lie-super algebra because of the lemma (2.1). A straightforward generalization of the well known canonical construction method enables us to go further as follows. Consider

$$L_0 = V \oplus M \tag{2.10}$$

for a Lie-super triple system $V$. We introduce a commutator in $L_0$, by

$$[x, y] = L(x, y) \ \epsilon \ M \quad , \tag{2.10a}$$

$$[L(x, y), z] = -(-1)^{(x+y)z} [z, L(x, y)] = [x, y, z] \ \epsilon \ V \quad . \tag{2.10b}$$

Then, $L_0$ can be readily verified to be a Lie-super algebra for grading of

$$\sigma(L(x, y)) = \{\sigma(x) + \sigma(y)\} \ (\text{mod } 2) \quad . \tag{2.11}$$

In order to make both $M$ and $L_0$ be quasi-classical, we introduce bilinear form in $M$ and $L_0$ by

$$< L(x, y)|L(u, v) > = < [x, y, u]|v > = -(-1)^{(u+v)y} < x|[u, v, y] > \tag{2.12}$$

$$< L(x, y)|z > = < z|L(x, y) > = 0 \tag{2.13}$$

in addition to $< x|y >$.

The second relation in Eq. (2.12) is the result of Proposition 2.1. Note that Eq. (2.12) is consistent with $L(x,y) = -(-1)^{xy}L(y,x)$ and $L(u,v) = -(-1)^{uv}L(v,u)$. However, we have to verify its well-definedness, i.e. we have to verify the validity of

$$< L(x',y')|L(u,v) > = < L(x,y)|L(u,v) > \quad ,$$

for all $u,\ v\ \epsilon\ V$ whenever we have $L(x',y') = L(x,y)$. This is trivially correct, since we will have $< [x',y',u]|v > = < [x,y,u]|v >$, if we note that $L(x',y') = L(x,y)$ implies $[x',y',u] = [x,y,u]$ for any $u\ \epsilon\ V$.

**<u>Proposition 2.3</u>**

The Lie-super algebras $M$ and $L_0$ constructed canonically from a quasi-classical Lie-super triple system $V$ are quasi-classical.

**<u>Proof</u>**

First we will show that $< L(x,y)|L(u,v) >$ defined by Eq. (2.12) is non-degenerate. Suppose that we have

$$< \sum_{j,k} c_{jk}L(x_j,y_k)|L(u,v) > = 0$$

for all $u,\ v\ \epsilon\ V$. This implies the validity of

$$< \sum_{j,k} c_{jk}[x_j,y_k,u]|v > = 0 \quad .$$

Because of non-degeneracy of $< .|. >$, this leads to

$$\sum_{j,k} c_{jk}[x_j,y_k,u] = 0$$

or equivalently $\sum_{j,k} c_{jk}L(x_j,y_k) = 0$, proving the non-degeneracy. Next, we note

$$< L(x,y)|L(u,v) > = < [x,y,u]|v > = 0 \quad ,$$

unless $\sigma(x) + \sigma(y) + \sigma(u) + \sigma(v) = 0 \pmod 2$ so that we find $< L(x,y)|L(u,v) > = 0$ unless we have $\sigma(L(u,v)) = \sigma(L(x,y))$. Similarly, we find the validity of

$$< L(x,y)|L(u,v) > = (-1)^{(u+v)(x+y)} < L(u,v)|L(x,y) > \quad .$$

Finally the proof for the validity of

$$< [L(x,y), L(z,w)]|L(u,v) > = < L(x,y)|[L(z,w), L(u,v)] > \tag{2.14}$$

goes as follows. In order to avoid unnecessary complications due to the sign factors $(-1)^{xy}$ etc., we will prove it only for non-super case. We can always supply sign factors for the super case to prove the same. Then, Eq. (2.14) is equivalent to

$$< [L(x,y), L(z,w)]|L(u,v) > = \ - < [L(u,v), L(z,w)]|L(x,y) > \quad . \tag{2.14$'$}$$

The left side of Eq. (2.14$'$) is computed to be

$$\begin{aligned} &< [L(x,y), L(z,w)]|L(u,v) > \\ &\qquad = < L([x,y,z],w) + L(z,[x,y,w])|L(u,v) > \\ &\qquad = < L([x,y,z],w) - L([x,y,w],z)|L(u,v) > \\ &\qquad = - < [x,y,z]|[u,v,w] > \ + \ < [x,y,w]|[u,v,z] > \quad . \end{aligned}$$

If we interchange $x \leftrightarrow u$, and $y \leftrightarrow v$ in this expression, we find the validity of Eq. (2.14$'$). This completes the proof, and the fact that $L_0$ is quasi-classical also can be similarly proved. ■

**<u>Remark 2.3</u>**

The canonical construction of an analogue of $L_0$ does <u>not</u> work for the case of $\delta = -1$.

**<u>Def. 2.1</u>**

A non-zero sub-vector space $B$ of a $\delta$ Lie-super triple system $V$ is called an ideal of $V$, if we have

$$[B, V, V] \subseteq B \quad .$$

**<u>Proposition 2.4</u>**

If $B$ is a ideal of a quasi-classical Lie-super triple system $V$ ($\delta = 1$), then $L(B,V)$ and $B \oplus L(B,V)$ are ideals of quasi-classical Lie-super algebras $M$ and $L_0$, respectively.

**<u>Proof</u>**

It is straightforward.

**Proposition 2.5**

Suppose that every ideal $B$ of a quasi-classical $\delta$ Lie-super triple system $V$ satisfies the condition

$$[B, B, V] \neq 0 \quad .$$

Then, $V$ is a direct sum of simple ideals $B_j$:

$$V = B_1 \oplus B_2 \oplus \ldots \oplus B_t \quad .$$

Moreover, we have

(i) $< B_j | B_k > = 0 \quad \text{if} \quad j \neq k$

(ii) $[B_j, B_k, V] = 0 \quad \text{if} \quad j \neq k \quad .$

**Proof**

Let $B$ be a maximal ideal of $V$ and set

$$B' = < x | < x | B > = 0, \ x \ \epsilon \ V > \quad .$$

Then, $B'$ is a ideal of $V$, satisfying

$$\text{(i)} \quad < B | B' > = 0 \qquad \text{(ii)} \quad [B, B', V] = 0 \qquad \text{(iii)} \quad B \cap B' = 0 \quad .$$

The fact that $B'$ is an ideal of $V$ follows immediately from the Proposition 2.1, since

$$< [B', V, V] | B > = < B' | [V, B, V] > = 0 \quad .$$

Moreover,

$$< [B, B', V] | V > = < B | [V, V, B'] > = 0$$

also because of Eqs. (2.3b) and (2.1c). The non-degeneracy of $< .|. >$ then requires $[B, B', V] = 0$. Next, set $A = B \cap B'$. Suppose that $A \neq 0$. Then, $A$ is clearly an ideal of $V$. However, $[A, A, V] = 0$ which is a contradiction with the hypothesis. Since $B$ is assumed to be maximal, these imply

$$V = B \oplus B' \quad .$$

Moreover, $B$ and $B'$ satisfy the same conditions as $V$. Hence, repeating the same arguments for $B$ and $B'$, we reach at the conclusion of the Proposition.

**Remark 2.4**

It is plausible that $L_0$ corresponding to a simple quasi-classical Lie-super triple system will also be simple. However, the question will be discussed elsewhere. Note that $M$ may be semi-simple (rather than being simple) even when $L_0$ is simple. See ref. [6] for such an example.

**Remark 2.5**

The special case $\delta = 1$ in example 2.1 has been studied in [6] in connection with the para-statistics. It has been shown there that both $M$ and $L_0$ lead to simple Lie-super algebras of the type osp$(n|m)$ [1]. For other examples, see also ref. [6]. ∎

In ending this section, we would like to make some comments on Freudendal-Kantor triple systems, [7], since they are intimately connected with Lie-triple systems. Let $V$ be a $Z_2$-graded vector space with triple product $xyz$. If it satisfies

$$\begin{aligned} uv(xyz) = (uvx)yz + \epsilon(-1)^{(u+v)x+uv}x(vuy)z \\ + (-1)^{(u+v)(x+y)}xy(uvz) \end{aligned} \tag{2.15}$$

for $\epsilon = \pm 1$, $V$ is called a generalized Freudenthal-Kantor triple system. Especially, any Lie-super triple system is a generalized Freudenthal-Kantor triple system for $\epsilon = -1$ with $xyz = [x, y, z]$. On the other side, if we have

$$xyz = \delta(-1)^{xy+yz+zx}zyx \tag{2.16}$$

with $\epsilon = -\delta$ in addition, it defines a $\delta$ Jordan-super triple system. Returning to the general case, we introduce a linear multiplication operator $K(.,.) : V \otimes V \to End\ V$ by

$$K(x, y)z = (-1)^{yz}xzy - \delta(-1)^{x(y+z)}yzx \tag{2.17}$$

for $\delta = \pm 1$. When we have identity

$$K(xyz, w) + (-1)^{z(x+y)}K(z, xyw) + \delta(-1)^{y(z+w)}K(x, K(z, w)y) = 0 \quad , \tag{2.18}$$

$V$ is called a $(\epsilon, \delta)$ Freudenthal-Kantor triple system [8].

The special case $K(x, y) = 0$ with $\epsilon = -\delta$ will reproduce the $\delta$ Jordan-super triple system. We can construct Lie-super triple systems out of $(\epsilon, \delta)$ Freudenthal-Kantor systems. Here, we will present the following proposition.

**Def. 2.2**

Let $V$ be a $\delta$ Jordan-super triple system with bilinear non-degenerate form $< x|y >$ satisfying

(i) $< x|y > = 0 \quad \text{unless} \quad \sigma(x) = \sigma(y)$

(ii) $< y|x > = \delta(-1)^{xy} < x|y >$

(iii) $< xyu|v > = < x|yuv > \quad .$

Then, $V$ is called quasi-classical.

**Proposition 2.6**

Let $V$ be a quasi-classical $\delta$ Jordan triple system. We introduce the left multiplication operation

$$L \ : \ V \otimes V \rightarrow End\ V$$

by

$$L(x, y)z = xyz \tag{2.19}$$

with inner product

$$< L(x, y)|L(u, v) > = < xyu|v > = < x|yuv > \quad . \tag{2.20}$$

The resulting Lie-super algebra given by

$$[L(u, v), L(x, y)] = L(uvx, y) - \delta(-1)^{(u+v)x+uv} L(x, vuy) \tag{2.21}$$

is quasi-classical.

**Proof**

We first prove the validity of

$$< xyu|v > = (-1)^{(x+y)(u+v)} < uvx|y > \tag{2.22}$$

since

$$
\begin{aligned}
< xyu|v > &= \delta(-1)^{xy+(x+y)u} < uyx|v > = \delta(-1)^{xy+(x+y)u} < u|yxv > \\
&= \delta(-1)^{xy+(x+y)u} \cdot \delta(-1)^{v(x+y)+xy} < u|vxy > \\
&= (-1)^{(x+y)(u+v)} < uvx|y > \quad .
\end{aligned}
$$

We will have then

$$< L(u,v)|L(x,y) > = (-1)^{(u+v)(x+y)} < L(x,y)|L(u,v) > \quad .$$

It is easy to see then that it defines a non-degenerate super-symmetric bilinear form. Finally, the validity of

$$< [L(u,v), L(z,w)]|L(x,y) > = < L(u,v)|[L(z,w), L(x,y)] >$$

can be similarly shown just as in the proof of Eq. (2.14′), if we note Eq. (2.21) and (2.22) to calculate

$$< [L(x,y), L(z,w)]|L(u,v) > = < xyz|wuv > \; - (-1)^{w(x+y+z)+z(u+v)} < wxy|uvz > \quad . \quad \blacksquare$$

**<u>Proposition 2.7</u>**

Let $xyz$ be a quasi-classical $\delta$ Jordan-super triple product. Then,

$$[x,y,z] = xyz - \delta(-1)^{xy} yxz$$

defines a quasi-classical $\delta$ Lie-super triple system.

**<u>Proof</u>**

It is straightforward.

**<u>Example 2.4</u>**

Suppose that $< x|y >$ and $P \; \epsilon \; End \; V$ satisfy conditions of the example 2.2. Then, the product

$$xyz = < x|y > Pz + \; < x|Py > z + \; < y|Pz > x + \; < y|z > Px$$

defines a quasi-classical $\delta$ Jordan triple product. Further $[x, y, z]$ constructed in Proposition 2.7 reproduces the example 2.2. ∎

**Example 2.5**

Let $L$ be a nilpotent Lie-super algebra of length at most 4, i.e., $L_5 = 0$. Especially, the examples 1.3 and 1.4 of section 1 satisfy the condition. For any two constants $c_1$ and $c_2$, we introduce a triple product by

$$xyz = c_1[x, [y, z]] + c_2[[x, y], z]$$

which defines a $(\epsilon, \delta)$ Freudenthal-Kantor system trivially. This is because we have

$$uv(xyz) = u(xyz)v = (xyz)uv = 0$$

in view of $L_5 = 0$. Moreover, if we choose $c_1 = c_2$, it gives a quasi-classical Jordan-super triple system for $\delta = -\epsilon = 1$.

**Example 2.6**

Let $< x|y >$ satisfy

$$< x|y > = -\epsilon(-1)^{xy} < y|x > \quad .$$

Moreover suppose that $P \ \epsilon \ End \ V$ obeys the condition

$$< Px|y > = < x|Py > \quad .$$

When we set

$$xyz = < y|Pz > x \quad ,$$

we can verify the fact that it defines a $(\epsilon, \delta)$ Freudenthal-Kantor triple system.

## 3. Application to Yang-Baxter Equation

Let $R(\theta)$ be an element of $End\ (V \otimes V)$ for a parameter $\theta$ which is called the spectral parameter. We introduce $R_{jk}(\theta) \ \epsilon \ End\ (V \otimes V \otimes V)$ for $j < k,\ j, k = 1, 2, 3$ to be exactly like the operation of $R(\theta)$ operating only in $j$th and $k$th copies of $V$ in $V \otimes V \otimes V$. If we have

$$R_{12}(\theta)R_{13}(\theta')R_{23}(\theta'') = R_{23}(\theta'')R_{13}(\theta')R_{12}(\theta) \tag{3.1}$$

for parameter $\theta$, $\theta'$, and $\theta''$ satisfying

$$\theta' = \theta + \theta'' \quad , \tag{3.2}$$

then the relation is called Yang-Baxter equation (e.g. see [9]). Although we can generalize our result to the case of super space, we will consider here only non-super case for simplicity. Suppose that $V$ possesses a non-degenerate bilinear symmetric inner product $< .|. >$ so that we have $< y|x > = < x|y >$. We can then introduce ([10] and [11]) two $\theta$-dependent triple products $[x, y, z]_\theta$ and $[x, y, z]^*_\theta$ satisfying

$$\text{(i)} \ < x|[y, u, v]_\theta > = < y|[x, v, u]^*_\theta > \tag{3.3}$$

$$\text{(ii)} \ R(\theta)(x \otimes y) = \sum_{j=1}^N [e^j, y, x]^*_\theta \otimes e_j = \sum_{j=1}^N e_j \otimes [e^j, x, y]_\theta \quad . \tag{3.4}$$

Here, $e_j$ and $e^j$ $(j = 1, 2, \ldots, N)$ are basis and its dual basis of $V$, respectively. Then, the Yang-Baxter equation (hereafter abbreviated as YBE) can be rewritten as a triple product relation

$$\begin{aligned} &\sum_{j=1}^N [v, [u, e_j, z]_{\theta'}, [e^j, x, y]_\theta]^*_{\theta''} \\ &\qquad = \sum_{j=1}^N [u, [v, e_j, x]^*_{\theta'}, [e^j, z, y]^*_{\theta''}]_\theta \quad . \end{aligned} \tag{3.5}$$

We are hereafter interested only in the case when we have

$$[x, y, z]^*_\theta = [x, y, z]_\theta \tag{3.6a}$$

or equivalently

$$< y|[x, v, u]_\theta > = < x|[y, u, v]_\theta > \quad . \tag{3.6b}$$

Note that Eq. (3.6b) has the same form as Eq. (2.3c). Under these assumptions, we will first show:

**<u>Lemma 3.1</u>**

A necessary and sufficient condition to have

$$[R_{ij}(\theta), R_{k\ell}(\theta')] = 0 \tag{3.7}$$

for all $i, j, k, \ell = 1, 2, 3$ is the validity of

$$[u, v, [x, y, z]_\theta]_{\theta'} = [x, y, [u, v, z]_{\theta'}]_\theta \quad . \tag{3.8}$$

**Remark 3.1**

The validity of Eq. (3.7) implies that the YBE (3.1) as well as classical Yang-Baxter equation [9]

$$[R_{12}(\theta), R_{13}(\theta')] + [R_{12}(\theta), R_{23}(\theta'')] + [R_{13}(\theta'), R_{23}(\theta'')] = 0$$

hold valid without assuming the constraint Eq. (3.2).

**Proof**

We calculate for example

$$\begin{aligned} R_{13}(\theta')R_{12}(\theta)x \otimes y \otimes z &= R_{13}(\theta') \sum_{j=1}^{N} [e^j, y, x]_\theta \otimes e_j \otimes z \\ &= \sum_{j,k=1}^{N} [e^k, z, [e^j, y, x]_\theta]_{\theta'} \otimes e_j \otimes e_k \quad , \\ R_{12}(\theta)R_{13}(\theta')x \otimes y \otimes z &= R_{12}(\theta) \sum_{k=1}^{N} [e^k, z, x]_{\theta'} \otimes y \otimes e_k \\ &= \sum_{j,k=1}^{N} [e^j, y, [e^k, z, x]_{\theta'}]_\theta \otimes e_j \otimes e_k \quad , \end{aligned}$$

from Eqs. (3.4) and (3.6a). Comparing both, we find $R_{12}(\theta)R_{13}(\theta') = R_{13}(\theta')R_{12}(\theta)$ if we have Eq. (3.8). Similarly, we can prove the rest of relations. ■

**Lemma 3.2**

Let $L$ be a Lie algebra satisfying

$$[L, [[L, L], [L, L]]] = 0$$

as in the example 1.1. Then, the triple product defined by

$$[x, y, z] = [[x, y], z]$$

satisfies

$$[u, v, [x, y, z]] = [x, y, [u, v, z]] \tag{3.9a}$$

or

$$[L(u, v), L(x, y)] = 0 \quad . \tag{3.9b}$$

**Proof**

By a straightforward computation, we calculate

$$\begin{aligned} &[u, v, [x, y, z]] - [x, y, [u, v, z]] \\ &\qquad = [[u, v], [[x, y], z]] - [[x, y], [[u, v], z]] \\ &\qquad = [z, [[x, y], [u, v]]] = 0 \quad . \quad \blacksquare \end{aligned}$$

**Proposition 3.1**

Let $V$ be a quasi-classical Lie-triple systems satisfying

$$[u, v, [x, y, z]] = [x, y, [u, v, z]] \quad .$$

Then, $\theta$-dependent triple product

$$[x, y, z]_\theta = f(\theta)[x, y, z] + g(\theta) < x|y > z$$

for arbitrary functions $f(\theta)$ and $g(\theta)$ of $\theta$ gives a solution of Eq. (3.7), and hence of YBE.

**Proof**

The condition Eq. (3.6b) follows readily from Proposition (2.1), while we can easily verify the validity of Eq. (3.8). $\blacksquare$

**Proposition 3.2**

Let $L$ be a nilpotent quasi-classical Lie algebra of length at most 4, i.e., $L_5 = 0$. Then,

$$[x, y, z]_\theta = f_1(\theta)[[x, y], z] + f_2(\theta)[x, [y, z]] + g(\theta) < x|y > z$$

for arbitrary functions $f_1(\theta)$, $f_2(\theta)$ and $g(\theta)$ of $\theta$ is a solution of YBE.

**Proof**

If we set

$$< x, y, z >_\theta = f_1(\theta)[[x, y], z] + f_2(\theta)[x, [y, z]] \quad ,$$

it satisfies

$$< u, v, < x, y, z >_\theta >_{\theta'} = < u, < x, y, z >_\theta, v >_{\theta'} = << x, yz >_\theta, u, v >_{\theta'} = 0$$

as well as

$$< y| < x, v, u >_\theta > = < x| < y, u, v >_\theta > \quad .$$

Then, it is easy to check the validity of the required conditions Eqs. (3.6b) and (3.8). ∎

**Remark 3.2**

Examples 1.3 of section 1 satisfies $L_4 = 0$ and hence $L_5 = 0$ of the condition. Another example satisfying Eq. (3.7) can be obtained as follows, although it does not correspond to a Lie triple system. Let $J_\mu \ \epsilon \ End \ V$ for $\mu = 1, 2, \ldots, m$ satisfy

$$[J_\mu, J_\nu] = 0 \quad .$$

Then,

$$R(\theta) = \sum_{\mu,\nu=1}^{m} f_{\mu\nu}(\theta) J_\mu \otimes J_\nu$$

for arbitrary functions $f_{\mu\nu}(\theta)$ of $\theta$ clearly satisfy Eq. (3.7). Such an example has been used elsewhere [12] to construct a rather curious link invariant.

**Remark 3.3**

We can find a solution of the YBE (3.5) but not necessarily of Eq. (3.7) as follows. Let $L$ be a nilpotent quasi-classical Lie algebra of length at most 6, i.e. $L_7 = 0$. Then,

$$[x, y, z]_\theta = f_1(\theta)[x, [y, z]] + f_2(\theta)[[x, y], z]$$

is a solution of the YBE, which may not necessarily satisfy now Eq. (3.7). We can verify indeed that both sides of Eq. (3.5) vanish identically in view of $L_7 = 0$. ∎

**Acknowledgement**

The work of one of the authors (S.O.) is supported in part by U.S. Department of Energy Grant No. DE-FG02-91ER40685.

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