

Large mass $q\bar{q}$ production from the Color Glass Condensate

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Abstract

We compute quark-antiquark pair production in the context of the Color Glass Condensate model for central heavy-ion collisions. The calculation is performed analytically to leading order in the density of hard sources present in the projectiles, and is applicable to quarks with a mass large compared to the saturation momentum. The formulas derived in this paper are compared to expressions derived in the framework of collinearly factorized perturbative QCD and in k_{\perp} factorization models. We comment on the breaking of k_{\perp} factorization which occurs beyond leading order in our approach.

1 Introduction

Heavy Quark pair production in the collinear factorization formalism of perturbative QCD is a well developed subject [1, 2, 3]. It is however not evident that this formalism is applicable in the kinematic regime where the heavy quark mass m is much smaller than the center of mass energy ($m \ll \sqrt{s}$). In this kinematic regime, small x effects may be important. An alternative formalism, the k_{\perp} factorization formalism, was developed to describe the physics in this region [4, 5]. A key feature of this formalism is that, unlike the collinear factorization formalism, the small x gluons that produce the heavy quark pair have intrinsic transverse momenta on the order of the hard scale of the process. At high k_{\perp} , the collinear factorization formalism is recovered. A comparison of the k_{\perp} factorized formalism to recent hadroproduction data has been performed in Ref. [6].

In Ref. [7], it was suggested that at small x , the intrinsic transverse momenta in the k_{\perp} factorized formalism are of the order of the saturation scale $Q_s(x)$ [8, 9]. Since the saturation scale grows with energy, so does the intrinsic transverse momentum of gluons, thereby significantly increasing the hadroproduction cross-section. The saturation inspired k_{\perp} factorized formalism and phenomenological applications including comparisons with the NLO perturbative QCD formalism are reviewed in Ref. [10]. A related approach is the color dipole approach which has also been applied to heavy quark hadroproduction [11, 12].

A systematic way to study high parton density effects in QCD is the Color Glass Condensate (CGC) [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Heavy quark production from the CGC was first discussed in the context of deep inelastic scattering [31] and subsequently in the context of inclusive and diffractive photoproduction [32, 33]. We will discuss here the problem of hadro-production of heavy quark pairs at very high energies. We will discuss specifically heavy quark production in the scattering of two large nuclei which has some simplifying features. Inclusive gluon production in the scattering of two nuclei has been studied previously, first to lowest order in α_s and lowest order in the parton density [34, 35, 36, 37] and later to all orders in the parton density (and lowest order in α_s) [38, 39, 40, 41].

The results for inclusive gluon production in the CGC approach, at lowest order in the parton density, can be expressed in a k_{\perp} factorized form [36, 37]. This k_{\perp} factorization is however broken when inclusive gluon production is computed to all orders in the parton density [38, 39, 40, 41]. We expect a similar breakdown of k_{\perp} factorization in heavy quark production at the small transverse momenta $\Lambda_{QCD} < k_{\perp} < Q_s$ when parton density effects are computed to all orders [42]. For a recent discussion of k_{\perp} factorization and the CGC, see Ref. [43].

We do not explicitly include the effects of quantum evolution in our approach – the energy dependence of our results is determined entirely by the x dependence of the saturation scale Q_s . An interesting consequence of quantum evolution in the CGC is the geometrical scaling [44] of distributions with Q_s . This persists in the kinematic window $Q_s < k_{\perp} < Q_s^2/\Lambda_{QCD}$ outside the saturation region $k_{\perp} < Q_s$ [45, 46, 47, 48]. A number of phenomenological models have argued that data at HERA for structure functions, vector mesons and even charm [44, 49, 50, 51] (as well as data from RHIC [52] and NMC [53]) exhibit geometrical scaling. The competition between k_{\perp} factorization preserving quantum evolution effects and multiple-scattering k_{\perp} factorization breaking effects immediately outside the saturation regime is a very interesting problem [54, 55, 56, 57, 58] but we will not address it further here.

This paper is organized as follows. We begin in section 2.1 by discussing the general formalism for pair production in a time-dependent external field. The pair production probability for a single pair is expressed in terms of the classical background field produced by the color charge sources in the two nuclei. The cross-section and the average number of pairs are obtained by averaging the pair production probability with the appropriate weight functional which governs the likelihood of different orientations of the color charge density. In

section 2.2, we argue that the leading order contribution to pair production is of order $O(\rho_1^2 \rho_2^2)$, where ρ_1 and ρ_2 are the color charge densities of the two sources. We write down explicit expressions, in covariant gauge, for the classical field that contributes to the probability at this order. These covariant gauge expressions were first derived by Kovchegov and Rischke [36]. For completeness, we reproduce their derivation in appendix A. With these expressions for the classical field, we write down an expression for the pair production amplitude in section 2.3. In the following sub-section, we show that this amplitude satisfies a Ward identity. The pair production probability is computed in section 3. Some details of the computation are given in appendix B. In section 4, we relate the results of our computation to the k_\perp factorization approach. The breakdown of k_\perp factorization is discussed in the following section. We conclude with a summary of our results.

2 Pair production amplitude

2.1 Basics

In order to compute the pair production amplitude in the Color Glass Condensate model, we first recall some results [59] about particle production in an external time-dependent¹ classical field. We denote $G_F(x, y)$ the Feynman (aka time-ordered) propagator of a quark from the point y to the point² x in the presence of the external color field, and $G_R(x, y)$ the retarded propagator of the quark between the same points. Both are Green's functions of the Dirac operator:

$$(i\cancel{\partial}_x - g\mathcal{A}(x) - m)G_{F,R}(x, y) = i\delta^{(4)}(x - y), \quad (1)$$

but with different boundary conditions. For instance, the boundary condition for the retarded propagator is simply:

$$\lim_{x_0 \rightarrow y_0^+} G_R(x, y) = \delta(\mathbf{x} - \mathbf{y})\gamma^0. \quad (2)$$

The boundary conditions for the Feynman propagator are awkward and not very illuminating.

From these propagators, it is customary to first perform a Fourier transform:

$$G_{F,R}(q, p) \equiv \int d^4x d^4y e^{iq \cdot x} e^{-ip \cdot y} G_{F,R}(x, y), \quad (3)$$

and then to extract the ‘‘scattering matrix’’ $\mathcal{T}_{F,R}$ via:

$$\underline{G_{F,R}(q, p)} = (2\pi)^4 \delta(q - p) G_{F,R}^0(p) + G_{F,R}^0(q) \mathcal{T}_{F,R}(q, p) G_{F,R}^0(p), \quad (4)$$

¹More exactly, one needs that the external field be time dependent *in any Lorentz frame*. For instance, a single moving nucleus, which generates a time dependent classical color field in the laboratory frame, cannot produce pairs.

²The order of the points may seem unnatural, but it is chosen so that the Dirac matrices (which should be read from the endpoint to the starting point of the propagator) appear with the correct ordering.

where $G_{F,R}^0$ is the *free* Feynman or retarded quark propagator. This definition removes the free term, and amputates the external legs of the propagator.

Several quantities related to pair production in the external classical field can then be expressed simply in terms of these scattering matrices. The probability for producing a single $q\bar{q}$ pair is given by³:

$$P_1 [A^\mu] = |\langle 0_{\text{in}} | 0_{\text{out}} \rangle|^2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}} \int \frac{d^3 \mathbf{q}}{(2\pi)^3 2\omega_{\mathbf{q}}} |\bar{u}(\mathbf{q}) \mathcal{T}_F(q, -p) v(\mathbf{p})|^2, \quad (5)$$

where the argument A^μ has been used to remind the reader that this is a quantity defined for one particular configuration of the external classical field. The field A^μ is a functional of the hard sources ρ_1 and ρ_2 (one for each nucleus) that generate this classical field. The prefactor⁴ $|\langle 0_{\text{in}} | 0_{\text{out}} \rangle|^2$ is the square of the overlap between the vacua at $x_0 = -\infty$ and $x_0 = +\infty$. This quantity is strictly smaller than unity in an external field that can produce pairs. It is necessary for unitarity to be preserved (for instance to ensure that the *probability* $P_1 [A^\mu]$ is always smaller than unity). The cross-section for single pair production can be obtained from here by:

$$\sigma_1 = \int d^2 \mathbf{b} \int [\mathcal{D}\rho_1 \mathcal{D}\rho_2] W[\rho_1, \rho_2; \mathbf{b}] P_1 [A^\mu[\rho_1, \rho_2]], \quad (6)$$

where $W[\rho_1, \rho_2; \mathbf{b}]$ is the functional weight that defines the statistical distribution of the hard sources at a given energy. This functional depends on the impact parameter \mathbf{b} between the two nuclei. The latter must be integrated out in order to convert the probability $P_1 [A^\mu]$ into a cross-section.

Similarly, the average number of produced pairs in a given external field configuration is given by:

$$\bar{n} [A^\mu] = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}} \int \frac{d^3 \mathbf{q}}{(2\pi)^3 2\omega_{\mathbf{q}}} |\bar{u}(\mathbf{q}) \mathcal{T}_R(q, -p) v(\mathbf{p})|^2. \quad (7)$$

The average number of pairs produced in a collision at a given impact parameter \mathbf{b} is given by:

$$\bar{n}(\mathbf{b}) = \int [\mathcal{D}\rho_1 \mathcal{D}\rho_2] W[\rho_1, \rho_2; \mathbf{b}] \bar{n} [A^\mu[\rho_1, \rho_2]]. \quad (8)$$

Note also that both P_1 and \bar{n} can be expressed as differential distributions by undoing the integrations over the momenta \mathbf{q} and \mathbf{p} of the quark and/or the antiquark.

³From eqs. (4) and (5), and given the fact that the canonical dimension of the spinors $\bar{u}(\mathbf{q})$ and $v(\mathbf{p})$ is (momentum)^{1/2}, one can check that P_1 is indeed dimensionless.

⁴This prefactor is also the probability that no pair is produced during the collision, which explains why it has to be smaller than 1. Diagrammatically, this quantity is the sum of the “vacuum-vacuum” diagrams, that have no external legs besides those that connect to the external field. More precisely, if we denote V the sum of all *connected* vacuum-vacuum diagrams, then $|\langle 0_{\text{in}} | 0_{\text{out}} \rangle|^2 = \exp(-2\text{Im} V)$ [59, 32, 33]. The expansion of $\text{Im} V$ in powers of the hard color sources starts at the order $\mathcal{O}(\rho_1^2 \rho_2^2)$.

When we give explicit expressions for the average over the distribution of the hard color source charge densities, ρ_1 and ρ_2 , we use the McLerran-Venugopalan Gaussian model for the functional $W[\rho_1, \rho_2; \mathbf{b}]$, which reads:

$$W[\rho_1, \rho_2; \mathbf{b}] = \exp - \int d^2 \mathbf{x}_\perp \left[\frac{\rho_{1,a}(\mathbf{x}_\perp) \rho_{1,a}(\mathbf{x}_\perp)}{2\mu_1^2(\mathbf{x}_\perp)} + \frac{\rho_{2,a}(\mathbf{x}_\perp) \rho_{2,a}(\mathbf{x}_\perp)}{2\mu_2^2(\mathbf{x}_\perp - \mathbf{b})} \right], \quad (9)$$

where the functions $\mu_1^2(\mathbf{x}_\perp)$ and $\mu_2^2(\mathbf{x}_\perp)$ correspond to the color charges squared per unit area of the two nuclei. They are functions of the transverse coordinate that describe the number of hard color sources per unit area at the transverse coordinate \mathbf{x}_\perp . (The origin of the transverse coordinates is taken at the center of the first nucleus, hence the argument $\mathbf{x}_\perp - \mathbf{b}$ for the function that describes the second nucleus.) The canonical dimension of $\mu_{1,2}^2$ is (momentum)². In a model of a nucleus with sharp edges, these functions would be almost flat inside the nuclear disc and zero outside.

In the MV model, μ^2 is energy (or x) independent. The x dependence comes in through quantum evolution [16, 17, 18, 19, 20, 21, 22, 24, 26, 27]. In the MV model, one can relate μ^2 to the saturation scale Q_s ⁵:

$$Q_s^2(x, x_\perp) = \alpha_s N_c \mu^2(x, x_\perp) \ln \left(\frac{g^2 \mu^2}{\Lambda_{QCD}^2} \right). \quad (10)$$

The relation between the two scales is more non-trivial under quantum evolution since the color screening scale is itself of order Q_s [45, 46, 47, 48, 61]. We will see later in Section 3 that, in our formalism, the x dependence of pair-production enters only through the x dependence of the saturation scale.

2.2 Leading order approximation

The relations of eqs. (5) and (7) are completely general and are valid to all orders in the sources ρ_1 and ρ_2 . In QCD, the external field $A^\mu[\rho_1, \rho_2]$ is a nonlinear functional of the sources ρ_1 and ρ_2 , that receives contributions to all orders in ρ_1 and ρ_2 . In addition, the scattering matrices $\mathcal{T}_{F,R}$ are themselves functionals of the external field A^μ , receiving contributions to all orders in A^μ . These facts make the quantities $P_1[A^\mu]$ and $\bar{n}[A^\mu]$ extremely complicated functionals of ρ_1 and ρ_2 that cannot be calculated analytically. A complete solution would likely require a numerical solution [42].

The leading term in the hard sources can however be calculated in closed form. This approximation is justified if there is a scale in the problem (the quark mass or the quark transverse momentum) which is large compared to the scale set by the density of hard color sources (namely, the saturation momentum Q_s). We will assume that the strong coupling constant α_s is sufficiently small to justify keeping only the lowest order in α_s . However, all corrections of order $\alpha_s \ln(s)$ are included since they are resummed via the quantum evolution of the functional $W[\rho_1, \rho_2; \mathbf{b}]$.

⁵For a discussion of different conventions for μ^2 , see Ref. [60].

One can convince oneself that both $P_1[A^\mu]$ and $\bar{n}[A^\mu]$ start at the order $\mathcal{O}(\rho_1^2\rho_2^2)$ in the hard sources. This is so because these quantities are squares, and because the quark line must be attached at least once to each nucleus if the pair is to be produced on-shell. At this order in the hard color sources, there are several important simplifications that are worth mentioning. Firstly, the prefactor $|\langle 0_{\text{in}}|0_{\text{out}}\rangle|^2$ that appears in P_1 can be replaced by 1. Indeed, one has $|\langle 0_{\text{in}}|0_{\text{out}}\rangle|^2 = 1 + \mathcal{O}(\rho_1^2\rho_2^2)$. Keeping higher order corrections in this prefactor would only produce terms of order higher than $\mathcal{O}(\rho_1^2\rho_2^2)$ in P_1 . The second simplification occurs because there is no difference between the time-ordered and retarded amplitudes. Indeed, since the probabilities P_2, P_3, \dots to produce two or more pairs in a collision are of higher order, we find

$$\begin{aligned}\bar{n}[A^\mu] &\equiv P_1[A^\mu] + 2P_2[A^\mu] + 3P_3[A^\mu] + \dots \\ &\approx P_1[A^\mu].\end{aligned}\tag{11}$$

The time-ordered and retarded amplitudes are the same at this order because, in the language of Feynman diagrams, none of the intermediate state quark propagators can be on their mass-shell (namely, the $i\epsilon$ prescription of the quark propagators does not matter).

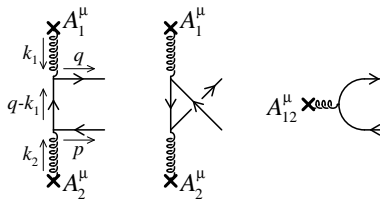


Figure 1: The leading contributions to the pair production amplitude in terms of A_1^μ , A_2^μ and A_{12}^μ . The gluon line terminated by a cross represents an insertion $-igA^\mu(x)$ of the external field.

Our first task is therefore to obtain the classical field $A^\mu[\rho_1, \rho_2]$ up to the order $\mathcal{O}(\rho_1\rho_2)$. We can formally write:

$$A^\mu(x) \equiv A_1^\mu(x) + A_2^\mu(x) + A_{12}^\mu(x) + \dots,\tag{12}$$

where $A_1^\mu(x)$ is the contribution of order $\mathcal{O}(\rho_1)$ to the classical field, $A_2^\mu(x)$ the term of order $\mathcal{O}(\rho_2)$, and $A_{12}^\mu(x)$ the term of order $\mathcal{O}(\rho_1\rho_2)$. In terms of these contributions to the classical field, we have to evaluate the three terms in figure 1. The derivation of the classical color field in the covariant gauge up to the order $\mathcal{O}(\rho_1\rho_2)$ can be found in [36]. We merely quote the results in momentum space here, and refer the reader to the original literature or to appendix A of

the present paper where this derivation has been reproduced in our notation⁶:

$$\begin{aligned} A_{1,a}^+(k) &= 2\pi g \delta(k^-) \frac{1}{\mathbf{k}_\perp^2} \rho_{1,a}(\mathbf{k}_\perp), & A_{1,a}^-(k) &= A_{1,a}^i(k) = 0, \\ A_{2,a}^-(k) &= 2\pi g \delta(k^+) \frac{1}{\mathbf{k}_\perp^2} \rho_{2,a}(\mathbf{k}_\perp), & A_{2,a}^+(k) &= A_{2,a}^i(k) = 0, \end{aligned} \quad (13)$$

$$\begin{aligned} A_{12,a}^+(k) &= -ig \frac{f^{abc}}{k^2} \int \frac{d^4 k_1}{(2\pi)^4} \left\{ k_1^+ + \frac{k_1^2}{k_2^-} \right\} A_{1,b}^+(k_1) A_{2,c}^-(k_2), \\ A_{12,a}^-(k) &= ig \frac{f^{abc}}{k^2} \int \frac{d^4 k_1}{(2\pi)^4} \left\{ k_2^- + \frac{k_2^2}{k_1^+} \right\} A_{1,b}^+(k_1) A_{2,c}^-(k_2), \\ A_{12,a}^i(k) &= ig \frac{f^{abc}}{k^2} \int \frac{d^4 k_1}{(2\pi)^4} \{ k_2^i - k_1^i \} A_{1,b}^+(k_1) A_{2,c}^-(k_2), \end{aligned} \quad (14)$$

where $k_2 \equiv k - k_1$.

2.3 Pair production amplitude

Following figure 1, we can write the leading order pair production amplitude as:

$$\bar{u}(\mathbf{q}) \mathcal{T}_{F,R} v(\mathbf{p}) \equiv \mathcal{M}(\mathbf{q}, \mathbf{p}) = \mathcal{M}_{1+2}(\mathbf{q}, \mathbf{p}) + \mathcal{M}_{12}(\mathbf{q}, \mathbf{p}), \quad (15)$$

where we have defined

$$\begin{aligned} \mathcal{M}_{1+2}(\mathbf{q}, \mathbf{p}) &\equiv \int \frac{d^4 k_1}{(2\pi)^4} (-ig A_{1,a}^\mu(k_1)) (-ig A_{2,b}^\nu(k_2)) \\ &\quad \times \bar{u}(\mathbf{q}) \{ \gamma_\mu t_a G^0(q - k_1) \gamma_\nu t_b + \gamma_\nu t_b G^0(k_1 - p) \gamma_\mu t_a \} v(\mathbf{p}) \end{aligned} \quad (16)$$

and

$$\mathcal{M}_{12}(\mathbf{q}, \mathbf{p}) \equiv (-ig A_{12,a}^\mu(p + q)) \bar{u}(\mathbf{q}) \gamma_\mu t_a v(\mathbf{p}), \quad (17)$$

where the $t_{a,b}$ are color matrices in the fundamental representation. $G^0(k)$ is the free quark propagator⁷:

$$G^0(k) \equiv i \frac{\not{k} + m}{k^2 - m^2 + i\epsilon}, \quad (18)$$

and $k_2 \equiv p + q - k_1$.

⁶The light-cone coordinates are defined as $x^\pm \equiv (x^0 \pm x^3)/\sqrt{2}$. With this convention, the invariant scalar product of two 4-vectors is $a \cdot b = a^+ b^- + a^- b^+ - \mathbf{a}_\perp \cdot \mathbf{b}_\perp$ and the element of 4-volume is $d^4 x = dx^+ dx^- d^2 \mathbf{x}_\perp$.

⁷The prefactor i in this propagator is purely conventional. However, it cannot be chosen independently of the $-i$ in the field insertion $-ig A^\mu$.

Using the fact that the momenta p and q are on-shell, that all but one component of A_1^μ and A_2^μ are zero, and finally $k_1^- = k_2^+ = 0$, we can obtain easily:

$$\begin{aligned} \mathcal{M}_{1+2}(\mathbf{q}, \mathbf{p}) &\equiv ig^2 \int \frac{d^4 k_1}{(2\pi)^4} A_{1,a}^+(k_1) A_{2,b}^-(k_2) \\ &\quad \times \bar{u}(\mathbf{q}) \left\{ \gamma^- \frac{m - \boldsymbol{\gamma}_\perp \cdot (\mathbf{q}_\perp - \mathbf{k}_{1\perp})}{2q^- p^+ + (\mathbf{q}_\perp - \mathbf{k}_{1\perp})^2 + m^2} \gamma^+ t_a t_b \right. \\ &\quad \left. + \gamma^+ \frac{m + \boldsymbol{\gamma}_\perp \cdot (\mathbf{p}_\perp - \mathbf{k}_{1\perp})}{2q^+ p^- + (\mathbf{p}_\perp - \mathbf{k}_{1\perp})^2 + m^2} \gamma^- t_b t_a \right\} v(\mathbf{p}) . \end{aligned} \quad (19)$$

We readily see that the denominators in this expression are positive definite, which justifies a posteriori the fact that the $i\epsilon$ prescription in the free quark propagator is irrelevant at this order. Using eqs. (14), we can similarly write the term $\mathcal{M}_{12}(\mathbf{q}, \mathbf{p})$ as:

$$\begin{aligned} \mathcal{M}_{12}(\mathbf{q}, \mathbf{p}) &= ig^2 \frac{[t_a, t_b]}{(p+q)^2} \int \frac{d^4 k_1}{(2\pi)^4} A_{1,a}^+(k_1) A_{2,b}^-(k_2) \\ &\quad \times \bar{u}(\mathbf{q}) \left\{ \gamma^- \left(k_1^+ - \frac{\mathbf{k}_{1\perp}^2}{k_2^-} \right) - \gamma^+ \left(k_2^- - \frac{\mathbf{k}_{2\perp}^2}{k_1^+} \right) + \boldsymbol{\gamma}_\perp \cdot (\mathbf{k}_{1\perp} - \mathbf{k}_{2\perp}) \right\} v(\mathbf{p}) . \end{aligned} \quad (20)$$

Here again, all the denominators are strictly positive, making the $i\epsilon$ prescriptions irrelevant. Note that we can write the expression of \mathcal{M}_{12} as follows:

$$\mathcal{M}_{12}(\mathbf{q}, \mathbf{p}) = ig^2 \frac{[t_a, t_b]}{(p+q)^2} \int \frac{d^4 k_1}{(2\pi)^4} A_{1,a}^+(k_1) A_{2,b}^-(k_2) \bar{u}(\mathbf{q}) \not{c} v(\mathbf{p}) , \quad (21)$$

where c^μ is the following 4-vector:

$$c \equiv \left(c^+ = p^+ + q^+ - \frac{\mathbf{k}_{1\perp}^2}{p^- + q^-}, c^- = \frac{\mathbf{k}_{2\perp}^2}{p^+ + q^+} - p^- - q^-, \mathbf{c}_\perp = \mathbf{k}_{2\perp} - \mathbf{k}_{1\perp} \right) . \quad (22)$$

We note that the vector c^μ is indeed nothing else but the standard effective Lipatov vertex (see eq. (80) of [62] for instance). In particular, one may check that it obeys the transversality relation:

$$(p+q) \cdot c = 0 . \quad (23)$$

At this stage, it is trivial to replace A_1^+ and A_2^- by their expression in terms of the source densities (eqs. (13)) in order to obtain the leading order pair production amplitude as a functional of the hard color sources ρ_1 and ρ_2 .

2.4 Ward identity verification

One can write the pair production amplitude $\mathcal{M}(\mathbf{q}, \mathbf{p})$ as follows:

$$\mathcal{M}(\mathbf{q}, \mathbf{p}) = \int \frac{d^4 k_1}{(2\pi)^4} A_{1,a}^\mu(k_1) A_{2,b}^\nu(k_2) m_{\mu\nu}^{ab}(k_1, k_2; \mathbf{q}, \mathbf{p}) . \quad (24)$$

The expressions of the previous subsection tell us that:

$$\begin{aligned}
m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p}) &= ig^2 \\
&\times \bar{u}(\mathbf{q}) \left\{ \frac{\gamma^-(m - \gamma_\perp \cdot (\mathbf{q}_\perp - \mathbf{k}_{1\perp})) \gamma^+ t_a t_b}{2q^- p^+ + (\mathbf{q}_\perp - \mathbf{k}_{1\perp})^2 + m^2} + \frac{\gamma^+(m + \gamma_\perp \cdot (\mathbf{p}_\perp - \mathbf{k}_{1\perp})) \gamma^- t_b t_a}{2q^+ p^- + (\mathbf{p}_\perp - \mathbf{k}_{1\perp})^2 + m^2} \right. \\
&\quad \left. + \frac{[t_a, t_b]}{(p+q)^2} \left[\gamma^- \left[k_1^+ - \frac{\mathbf{k}_{1\perp}^2}{k_2^-} \right] - \gamma^+ \left[k_2^- - \frac{\mathbf{k}_{2\perp}^2}{k_1^+} \right] + \gamma_\perp \cdot (\mathbf{k}_{1\perp} - \mathbf{k}_{2\perp}) \right] \right\} v(\mathbf{p}), \tag{25}
\end{aligned}$$

while the other components of $m_{ab}^{\mu\nu}$ do not appear in the expression of the pair production amplitude. One notes that $m_{ab}^{\mu\nu}(k_1, k_2; \mathbf{q}, \mathbf{p})$ is *not* the amplitude for the process $gg \rightarrow q\bar{q}$. Indeed, contracting the $gg \rightarrow q\bar{q}$ amplitude with the classical fields $A_1^\mu(k_1)$ and $A_2^\nu(k_2)$ would only give the first three diagrams of figure 2, and leave out the bremsstrahlung diagrams.

It is interesting to investigate the limit $\mathbf{k}_{1\perp} = \mathbf{0}$ of $m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p})$. We have:

$$\begin{aligned}
\lim_{\mathbf{k}_{1\perp} \rightarrow 0} m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p}) &= \\
&= ig^2 \bar{u}(\mathbf{q}) \left\{ \frac{\gamma^-(m - \gamma_\perp \cdot \mathbf{q}_\perp) \gamma^+ t_a t_b}{2q^- p^+ + \mathbf{q}_\perp^2 + m^2} + \frac{\gamma^+(m + \gamma_\perp \cdot \mathbf{p}_\perp) \gamma^- t_b t_a}{2q^+ p^- + \mathbf{p}_\perp^2 + m^2} \right. \\
&\quad \left. + \frac{[t_a, t_b]}{(p+q)^2} \left[\gamma^- k_1^+ - \gamma^+ \left[k_2^- - \frac{\mathbf{k}_{2\perp}^2}{k_1^+} \right] - \gamma_\perp \cdot \mathbf{k}_{2\perp} \right] \right\} v(\mathbf{p}). \tag{26}
\end{aligned}$$

Using the Dirac equations obeyed by the spinors $\bar{u}(\mathbf{q})$ and $v(\mathbf{p})$, as well as the following relations:

$$\begin{aligned}
\mathbf{q}_\perp^2 + m^2 &= 2q^+ q^-, \quad \mathbf{p}_\perp^2 + m^2 = 2p^+ p^-, \\
k_1^+ &= p^+ + q^+, \quad k_2^- = p^- + q^-, \quad \mathbf{k}_{2\perp} = \mathbf{p}_\perp + \mathbf{q}_\perp \text{ (if } \mathbf{k}_{1\perp} = \mathbf{0}) \\
\gamma^+ \gamma^- \gamma^+ &= 2\gamma^+, \tag{27}
\end{aligned}$$

we can simplify the above limit into:

$$\begin{aligned}
\lim_{\mathbf{k}_{1\perp} \rightarrow 0} m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p}) &= \\
&= ig^2 \bar{u}(\mathbf{q}) \gamma^+ v(\mathbf{p}) \frac{[t_a, t_b]}{p^+ + q^+} \left\{ 1 + \left[\frac{(\mathbf{p}_\perp + \mathbf{q}_\perp)^2}{p^+ + q^+} - 2(p^- + q^-) \right] \frac{p^+ + q^+}{(p+q)^2} \right\} \\
&= 0. \tag{28}
\end{aligned}$$

Similarly, one can check that:

$$\lim_{\mathbf{k}_{2\perp} \rightarrow 0} m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p}) = 0. \tag{29}$$

These simple limits are consequences of the following Ward identities⁸:

$$k_{1\mu} m_{ab}^{\mu\nu}(k_1, k_2; \mathbf{q}, \mathbf{p}) = k_{2\nu} m_{ab}^{\mu\nu}(k_1, k_2; \mathbf{q}, \mathbf{p}) = 0. \quad (30)$$

Indeed, since $k_1^- = 0$, the first one implies for instance:

$$k_1^+ m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p}) = k_1^i m_{ab}^{i+}(k_1, k_2; \mathbf{q}, \mathbf{p}) \underset{\mathbf{k}_{1\perp} \rightarrow 0}{=} \mathcal{O}(k_1^i). \quad (31)$$

In other words, $m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p})$ must vanish linearly with $\mathbf{k}_{1\perp}$ when $\mathbf{k}_{1\perp} \rightarrow 0$ in order to fulfill the Ward identity. The physical meaning of these vanishing limits is that in order to produce the quark-antiquark pair on-shell, some transverse momentum has to come from both nuclei. As we shall see later, this property is also essential in order to soften some collinear singularities.

3 Pair production probability

3.1 Average over the hard color sources

From eq. (24), we can write the pair production amplitude in terms of the hard color sources:

$$\mathcal{M}(\mathbf{q}, \mathbf{p}) = g^2 \int \frac{d^2 \mathbf{k}_{1\perp}}{(2\pi)^2} \frac{\rho_{1,a}(\mathbf{k}_{1\perp})}{\mathbf{k}_{1\perp}^2} \frac{\rho_{2,b}(\mathbf{k}_{2\perp})}{\mathbf{k}_{2\perp}^2} m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p}). \quad (32)$$

Squaring this amplitude, and averaging over the sources, we obtain:

$$\begin{aligned} P_1(\mathbf{b}) = \bar{n}(\mathbf{b}) &= g^4 \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}} \int \frac{d^3 \mathbf{q}}{(2\pi)^3 2\omega_{\mathbf{q}}} \int [\mathcal{D}\rho_1 \mathcal{D}\rho_2] W[\rho_1, \rho_2; \mathbf{b}] \\ &\times \int \frac{d^2 \mathbf{k}_{1\perp}}{(2\pi)^2} \frac{d^2 \mathbf{k}'_{1\perp}}{(2\pi)^2} \frac{\rho_{1,a}(\mathbf{k}_{1\perp})}{\mathbf{k}_{1\perp}^2} \frac{\rho_{2,b}(\mathbf{k}_{2\perp})}{\mathbf{k}_{2\perp}^2} \frac{\rho_{1,a'}^*(\mathbf{k}'_{1\perp})}{\mathbf{k}'_{1\perp}{}^2} \frac{\rho_{2,b'}^*(\mathbf{k}'_{2\perp})}{\mathbf{k}'_{2\perp}{}^2} \\ &\times \text{Tr} \left(m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p}) m_{a'b'}^{-+}(k'_1, k'_2; \mathbf{q}, \mathbf{p}) \right), \quad (33) \end{aligned}$$

where Tr denotes a trace of the color and Dirac matrices. In the McLerran-Venugopalan model where the functional $W[\rho_1, \rho_2; \mathbf{b}]$ is given by eq. (9), the source averages are given in coordinate space by:

$$\begin{aligned} &\int [\mathcal{D}\rho_1 \mathcal{D}\rho_2] W[\rho_1, \rho_2; \mathbf{b}] \rho_{1,a}(\mathbf{x}_{\perp}) \rho_{1,a'}(\mathbf{x}'_{\perp}) \rho_{2,b}(\mathbf{y}_{\perp}) \rho_{2,b'}(\mathbf{y}'_{\perp}) = \\ &= \delta^{aa'} \delta^{bb'} \mu_1^2(\mathbf{x}_{\perp}) \mu_2^2(\mathbf{y}_{\perp} - \mathbf{b}) \delta(\mathbf{x}_{\perp} - \mathbf{x}'_{\perp}) \delta(\mathbf{y}_{\perp} - \mathbf{y}'_{\perp}). \quad (34) \end{aligned}$$

⁸Because of eq. (24), the object $m_{ab}^{\mu\nu}(k_1, k_2; \mathbf{q}, \mathbf{p})$ may be seen as the lowest order value of the correlator $\langle J_1^\mu J_2^\nu \bar{\psi} \psi \rangle$ between the currents of the individual nuclei and a pair of fermions. The currents J_1^μ and J_2^ν behave like Abelian currents at the order we are considering (the non Abelian term in the current conservation law is of higher order in the sources $\rho_{1,2}$), which is the reason why we have Abelian-like Ward identities for $m_{ab}^{\mu\nu}(k_1, k_2; \mathbf{q}, \mathbf{p})$.

The Fourier transform of eq. (34) reads:

$$\begin{aligned} \int [\mathcal{D}\rho_1 \mathcal{D}\rho_2] W[\rho_1, \rho_2; \mathbf{b}] \rho_{1,a}(\mathbf{k}_{1\perp}) \rho_{2,b}(\mathbf{k}_{2\perp}) \rho_{1,a'}^*(\mathbf{k}'_{1\perp}) \rho_{2,b'}^*(\mathbf{k}'_{2\perp}) = \\ = \delta^{aa'} \delta^{bb'} e^{-i\mathbf{b}\cdot(\mathbf{k}_{1\perp} - \mathbf{k}'_{1\perp})} \widetilde{\mu}_1^2(\mathbf{k}_{1\perp} - \mathbf{k}'_{1\perp}) \widetilde{\mu}_2^2(\mathbf{k}'_{1\perp} - \mathbf{k}_{1\perp}), \end{aligned} \quad (35)$$

where $\widetilde{\mu}_{1,2}^2(\mathbf{k}_\perp)$ is the Fourier transform of $\mu_{1,2}^2(\mathbf{x}_\perp)$. We have used the fact that $\mathbf{k}_{2\perp} = \mathbf{p}_\perp + \mathbf{q}_\perp - \mathbf{k}_{1\perp}$.

3.2 Impact parameter and energy dependence

At this stage, we have the following expression for the pair production probability:

$$\begin{aligned} P_1(\mathbf{b}) = \bar{n}(\mathbf{b}) = g^4 \int \frac{d^3\mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}} \int \frac{d^3\mathbf{q}}{(2\pi)^3 2\omega_{\mathbf{q}}} \int \frac{d^2\mathbf{k}_{1\perp}}{(2\pi)^2} \frac{d^2\mathbf{k}'_{1\perp}}{(2\pi)^2} e^{-i\mathbf{b}\cdot(\mathbf{k}_{1\perp} - \mathbf{k}'_{1\perp})} \\ \times \widetilde{\mu}_1^2(\mathbf{k}_{1\perp} - \mathbf{k}'_{1\perp}) \widetilde{\mu}_2^2(\mathbf{k}'_{1\perp} - \mathbf{k}_{1\perp}) \frac{\text{Tr} \left(m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p}) m_{ab}^{-+*}(k'_1, k'_2; \mathbf{q}, \mathbf{p}) \right)}{\mathbf{k}_{1\perp}^2 \mathbf{k}'_{1\perp}{}^2 \mathbf{k}_{2\perp}^2 \mathbf{k}'_{2\perp}{}^2}. \end{aligned} \quad (36)$$

For large nuclei, these Fourier transforms are strongly peaked around $\mathbf{k}_\perp = 0$, with a typical width of the order of $1/R$ where R is the radius of the nucleus. For instance, for $R = 6$ fm, the spread in transverse momentum would be of order $\Delta\mathbf{k}_\perp \sim 1/R \sim 30$ MeV. The factors m_{ab}^{-+} contain a large momentum scale provided by the mass m of the heavy quarks, which we assume to be much larger than $1/R$. Therefore, we can safely neglect the difference $\mathbf{k}_{1\perp} - \mathbf{k}'_{1\perp}$ in these factors. Regarding the denominators $\mathbf{k}_{1\perp}^2 \mathbf{k}'_{1\perp}{}^2 \mathbf{k}_{2\perp}^2 \mathbf{k}'_{2\perp}{}^2$, the situation is more delicate. Indeed, they could be arbitrarily small because of the presence of a collinear singularity when $\mathbf{k}_{1\perp} \rightarrow 0$ or $\mathbf{k}_{1\perp} \rightarrow \mathbf{p}_\perp + \mathbf{q}_\perp$. We are going to assume that these collinear singularities are regularised by physics at the scale of the saturation momentum $Q_s \gg 1/R$, so that we can also neglect the difference $\mathbf{k}_{1\perp} - \mathbf{k}'_{1\perp}$ in the denominators. Therefore, we can easily integrate out the difference $\mathbf{k}_{1\perp} - \mathbf{k}'_{1\perp}$ in order to get the impact parameter dependence of the pair production probability:

$$\begin{aligned} P_1(\mathbf{b}) = \bar{n}(\mathbf{b}) = g^4 \left[\int d^2\mathbf{x}_\perp \mu_1^2(\mathbf{x}_\perp) \mu_2^2(\mathbf{x}_\perp - \mathbf{b}) \right] \\ \times \int \frac{d^3\mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}} \int \frac{d^3\mathbf{q}}{(2\pi)^3 2\omega_{\mathbf{q}}} \int \frac{d^2\mathbf{k}_{1\perp}}{(2\pi)^2} \frac{\text{Tr} \left(|m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p})|^2 \right)}{\mathbf{k}_{1\perp}^4 \mathbf{k}_{2\perp}^4}. \end{aligned} \quad (37)$$

The prefactor between the square brackets is the overlap of the density functions that describe the two colliding nuclei. In a crude model of ‘‘cylindrical’’ nuclei, it would be equal to $\mu_1^2 \mu_2^2 S(\mathbf{b})$ where $S(\mathbf{b})$ is the area of the overlap between the two nuclei. The explicit expression of the trace that appears in this formula is given in the appendix B.

Since there is no quantum evolution in the MV model, the functions $\mu_1^2(\mathbf{x}_\perp)$ and $\mu_2^2(\mathbf{x}_\perp)$ do not depend on the longitudinal momentum fraction of the gluons. As a consequence, the differential pair production probability is boost invariant, in the sense that it depends only on the rapidity difference $y_p - y_q$ of the quark and the antiquark. Strictly speaking, eq. (37) gives an infinite result because it contains also an integration over the mean rapidity $(y_p + y_q)/2$: this infinity should be cutoff by the fact that the rapidity of the produced quark and antiquark are limited by the rapidity of the projectiles. More formally, quantum evolution of the functional $W[\rho_1, \rho_2; \mathbf{b}]$ would introduce a dependence of the functions $\mu_1^2(\mathbf{x}_\perp)$ and $\mu_2^2(\mathbf{x}_\perp)$ on the gluon momentum fractions, such that they vanish when we reach the fragmentation region of the corresponding projectile.

4 Relation to k_\perp -factorization and Collinear factorization

In k_\perp -factorized perturbation theory, it is well known that the leading order diagrams contributing to the production of a quark-antiquark pair are those listed in figure 2. The first two diagrams are the analogs of the diagrams one would have in QED, except for the fact that the coupling of the gluon to the quark line involves a non-commuting SU(3) matrix. The third diagram involves the 3-gluon vertex, and the fourth diagram⁹ is a bremsstrahlung contribution required in order to ensure gauge invariance.

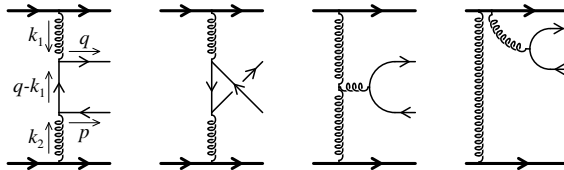


Figure 2: The Feynman diagrams that can contribute to pair production at the order $\mathcal{O}(\rho_1 \rho_2)$. The bold lines represent the classical sources ρ_1 and ρ_2 .

It is well known that the first two terms of figure 1 correspond to the first two Feynman diagrams of figure 2 in the limit where the transverse momentum of the two gluons is small in front of the longitudinal momentum of the hard sources. Moreover, it has been shown by Kovchegov and Rischke [36] that the term proportional to $A_{12}^\mu(x)$ contains both the diagram with the 3-gluon vertex and the bremsstrahlung diagrams. Therefore, it is very interesting to compare the result of our classical calculation to the standard result obtained in the framework of k_\perp -factorization [4, 5].

It turns out that the formula we have just derived has a fairly simple connection with the result one would obtain in the framework of k_\perp -factorization.

⁹There are four bremsstrahlung diagrams of this type, and only one has been represented.

In order to see this connection, one needs the following relationship between the quantities $\mu_{1,2}^2(\mathbf{x}_\perp)$ and the unintegrated gluon distribution¹⁰ $\varphi_{1,2}(k_\perp)$:

$$\frac{d\varphi(k_\perp, \mathbf{x}_\perp)}{d^2\mathbf{x}_\perp} = \pi g^2 d_A \frac{\mu^2(\mathbf{x}_\perp)}{\mathbf{k}_\perp^2}, \quad (39)$$

where $d\varphi(k_\perp, \mathbf{x}_\perp)/d^2\mathbf{x}_\perp$ is the number of gluons in a projectile, per unit of k_\perp^2 and per unit area, and where $d_A \equiv N_c^2 - 1$ is the dimension of the adjoint representation. In fact, this identification of the unintegrated gluon distribution is more general than the model of eq. (34), for one has in general:

$$\frac{d\varphi(k_\perp, \mathbf{x}_\perp)}{d^2\mathbf{x}_\perp} = \pi \frac{1}{\mathbf{k}_\perp^2} \int d^2\mathbf{r}_\perp e^{-i\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \langle \rho_a(\mathbf{x}_\perp + \mathbf{r}_\perp/2) \rho_a(\mathbf{x}_\perp - \mathbf{r}_\perp/2) \rangle_\rho, \quad (40)$$

where $\langle \dots \rangle_\rho$ denotes the average over the hard color sources. From there, using eqs. (6) and (37), it is easy to write the pair production cross-section as:

$$\begin{aligned} \frac{d\sigma_1}{dy_p dy_q d^2\mathbf{p}_\perp d^2\mathbf{q}_\perp} &= \frac{1}{(2\pi)^6 d_A^2} \int \frac{d^2\mathbf{k}_{1\perp}}{(2\pi)^2} \frac{d^2\mathbf{k}_{2\perp}}{(2\pi)^2} \delta(\mathbf{k}_{1\perp} + \mathbf{k}_{2\perp} - \mathbf{p}_\perp - \mathbf{q}_\perp) \\ &\times \int d^2\mathbf{b} d^2\mathbf{x}_\perp \frac{d\varphi_1(k_{1\perp}, \mathbf{x}_\perp)}{d^2\mathbf{x}_\perp} \frac{d\varphi_2(k_{2\perp}, \mathbf{x}_\perp - \mathbf{b})}{d^2\mathbf{x}_\perp} \frac{\text{Tr} \left(|m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p})|^2 \right)}{\mathbf{k}_{1\perp}^2 \mathbf{k}_{2\perp}^2}, \end{aligned} \quad (41)$$

where y_q and y_p are the rapidities of the quark and the antiquark. The integration over \mathbf{b} and \mathbf{x}_\perp is trivial to perform, and leads to the usual unintegrated gluon distributions:

$$\begin{aligned} \frac{d\sigma_1}{dy_p dy_q d^2\mathbf{p}_\perp d^2\mathbf{q}_\perp} &= \frac{1}{(2\pi)^6 d_A^2} \int \frac{d^2\mathbf{k}_{1\perp}}{(2\pi)^2} \frac{d^2\mathbf{k}_{2\perp}}{(2\pi)^2} \delta(\mathbf{k}_{1\perp} + \mathbf{k}_{2\perp} - \mathbf{p}_\perp - \mathbf{q}_\perp) \\ &\times \varphi_1(k_{1\perp}) \varphi_2(k_{2\perp}) \frac{\text{Tr} \left(|m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p})|^2 \right)}{\mathbf{k}_{1\perp}^2 \mathbf{k}_{2\perp}^2}. \end{aligned} \quad (42)$$

This expression has a structure which is very similar to what one would have in the k_\perp -factorization approach. Moreover, we have checked (see the appendix B) that the matrix element given by $\text{Tr} \left(|m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p})|^2 \right)$ is exactly identical to the matrix element obtained in the framework of k_\perp -factorization by Collins and Ellis [4]. Note also that the x dependence here comes in through the unintegrated gluon distribution. This x dependence of unintegrated gluon distribution is related to that of the correlator of ρ 's by eq. 40. The correlator in turn is determined by the weight function W in eq. 6 which, as discussed previously, satisfies a non-linear renormalization group equation in x .

¹⁰There are many definitions of the unintegrated gluon distribution in the literature. The one we are using here is:

$$xG(x, Q^2) \equiv \int_0^{Q^2} d(k_\perp^2) \varphi(x, k_\perp). \quad (38)$$

From the previous formula, it is well known how to recover the standard results of collinear factorization. One must take the limit $|\mathbf{k}_{1\perp}|, |\mathbf{k}_{2\perp}| \rightarrow 0$ in the quantity $\text{Tr}(|m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p})|^2)/k_1^2 k_2^2$ (but not in the unintegrated gluon distributions). This limit is perfectly defined thanks to the Ward identities discussed in section 2.4. Then, the integration over the azimuthal angles of the vectors $\mathbf{k}_{1\perp}$ and $\mathbf{k}_{2\perp}$ gives the expression of the matrix element $gg \rightarrow q\bar{q}$ in the limit of collinear factorization, while the integration of the non integrated gluon distributions over $k_{1\perp}^2$ and $k_{2\perp}^2$ reconstructs the integrated gluon distributions. After this procedure, we are left with a factor $\delta(\mathbf{p}_\perp + \mathbf{q}_\perp)$ which naturally corresponds to the fact that in this limit the quark and antiquark must be produced back-to-back in the transverse plane.

5 Beyond leading order

Corrections due to terms of higher order in the hard color sources, breaking the k_\perp factorization, are expected to become important in the soft regime, i.e. when the transverse mass of the quark or antiquark is of the order of the saturation momentum or smaller. In this section, we discuss the main issues that arise when one calculates corrections to the pair production amplitude that are of higher order in the hard color sources. In order to illustrate this discussion, let us first consider the example of corrections of order $\mathcal{O}(\rho_1^2 \rho_2)$ to the amplitude, which are represented in figure 3. The contributions at this order can be grouped in three

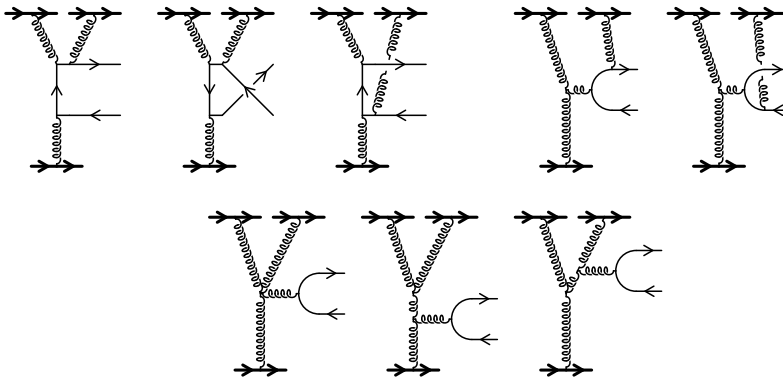


Figure 3: Contributions to the pair production amplitude at order $\mathcal{O}(\rho_1^2 \rho_2)$. The diagrams involving bremsstrahlung have been omitted. The bold lines represent the hard color sources.

sets, according to the number of classical field insertions along the quark line. The first set (the three diagrams in the upper left part of figure 3) involves only the Weizsäcker-Williams fields A_1^μ and A_2^μ produced by the individual nuclei. The only difference with the previous order is that there are now two powers of A_1^μ and one power of A_2^μ . Then, there are two diagrams (represented in the

upper right part of figure 3) involving one power of the classical field A_{12}^μ and one power of the field A_1^μ . Finally, we have three terms with a single insertion of the classical field A_{112}^μ , i.e. the term of order $\mathcal{O}(\rho_1^2\rho_2)$ in the solution of the classical Yang-Mills equation.

Beyond leading order, there also arises the issue of the vacuum-vacuum diagrams contained in the prefactor $|\langle 0_{\text{in}}|0_{\text{out}}\rangle|^2$, since this factor cannot be replaced by 1 anymore. Although this factor can be calculated order by order in an expansion in powers of the hard color sources, this makes the calculation of the single pair production probability much more difficult than the calculation of the average pair multiplicity. Alternatively, one could be less rigorous and simply discard this prefactor in the calculation of P_1 . This would of course violate unitarity (thereby resulting in a probability P_1 larger than unity), which could then be restored in an approximate way by the replacement $P_1 \rightarrow P_1 \exp(-P_1)$. This way of restoring unitarity is equivalent to assuming that the distribution of pair multiplicities is a Poisson distribution.

With this in mind, in order to perform the calculation of pair production beyond the leading order in the hard sources, one needs to go through the following steps:

- solve the classical Yang-Mills equation with two sources up to the required order. This can in principle be done analytically by a generalization of the method presented in appendix A. But for a non-perturbative calculation to all orders, one has to resort to a numerical resolution of the classical Yang-Mills equation with retarded boundary conditions. This step is under control after the work performed in [38, 39, 40, 41].
- compute the propagator of a quark in the previously obtained classical field. Only terms whose order is not higher than $\mathcal{O}(\rho_1^2\rho_2)$ need to be kept in the above example. For a systematic study of higher orders, one must rewrite the retarded amplitude $\bar{u}(\mathbf{q})\mathcal{T}_R v(\mathbf{p})$ in terms of retarded solutions of the Dirac equation, and then solve this equation numerically [42]. Here it is crucial to limit ourselves to the calculation of the pair multiplicity, which requires only the retarded quark propagator in the external field. Indeed, the numerical determination of the time-ordered propagator is a much harder problem, due to the great complexity of the boundary conditions obeyed by this type of propagator.
- study the quantum evolution of the functional $W[\rho_1, \rho_2; \mathbf{b}]$. Contrary to the leading order where only the correlators $\langle \rho_{1,2}\rho_{1,2} \rangle$ were necessary, we need now higher order correlators. For instance, in the calculation at order $\mathcal{O}(\rho_1^2\rho_2)$ considered as an example, we would need the correlators $\langle \rho_1\rho_1\rho_1 \rangle$ and $\langle \rho_1\rho_1\rho_1\rho_1 \rangle$. Indeed, strictly speaking quantum evolution does not preserve the Gaussian structure (unless in certain specific limits [46]). There are now ways to evaluate numerically these correlators [63]. Alternatively, at collision energies that are not too large, this step can be skipped since a Gaussian weight is a good approximation.

6 Conclusions

We have discussed in this paper heavy quark pair production in heavy ion collisions, in the framework of the Color Glass Condensate model. We have computed explicitly the leading order contribution to the heavy quark pair production cross-section. This computation is valid in the kinematic region where $k_{\perp} \gg Q_s$. We show explicitly that the classical result in this approximation is identical to the k_{\perp} factorization results derived previously. We discuss how k_{\perp} factorization may be violated for smaller k_{\perp} as one enters the saturation regime. Numerical computations addressing this issue are in progress and will be reported separately [42].

We have not attempted here to relate our results to phenomenology. There have already been several phenomenological works applying k_{\perp} factorization to various aspects of heavy quark production in Deeply Inelastic Scattering and in hadron-hadron scattering experiments. k_{\perp} factorization ideas are also being studied in the framework of relativistic heavy ion collisions ¹¹.

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A Classical color field to order $\mathcal{O}(\rho_1\rho_2)$

We derive the classical field in the covariant gauge, and we follow closely the method of Kovchegov and Rischke. The gauge field is a solution of the classical Yang-Mills equation:

$$[D_{\mu}, F^{\mu\nu}] = J^{\nu} , \quad (43)$$

where J^{ν} is the classical current which must be covariantly conserved:

$$[D_{\nu}, J^{\nu}] = 0 , \quad (44)$$

and the gauge condition is:

$$\partial_{\mu} A^{\mu} = 0 . \quad (45)$$

Making use of the gauge condition, the Yang-Mills equation can be rewritten as:

$$\square A^{\nu} = J^{\nu} + ig [A_{\mu}, F^{\mu\nu} + \partial^{\mu} A^{\nu}] . \quad (46)$$

Let us first start with the terms A_1^{μ} and A_2^{μ} , which are linear in the hard sources. Since the commutator $[A_{\mu}, F^{\mu\nu} + \partial^{\mu} A^{\nu}]$ is at least quadratic in the sources, we can drop it at this order, and we have:

$$\square A_{1,2}^{\mu} = J_{1,2}^{\mu} , \quad (47)$$

¹¹Private communication from D. Kharzeev and K. Tuchin.

where $J_{1,2}^\mu$ are the color currents associated to the individual nuclei. If we assume that the nucleus 1 is moving in the $+z$ direction at the speed of light, while the nucleus 2 is moving in the $-z$ direction, the currents are simply:

$$\begin{aligned} J_{1,a}^\mu &= g\delta^{\mu+}\delta(x^-)\rho_{1,a}(\mathbf{x}_\perp) , \\ J_{2,a}^\mu &= g\delta^{\mu-}\delta(x^+)\rho_{2,a}(\mathbf{x}_\perp) . \end{aligned} \quad (48)$$

The index a is the color index carried by the classical sources and currents. This means in particular that at this order, the sources do not have any transverse components, i.e. that they describe recoilless objects.

Eq. (47) then simply becomes a Poisson equation in the transverse plane, and the only non-zero components of A_1^μ and A_2^μ can be written formally as:

$$\begin{aligned} A_{1,a}^+(x) &= -g\delta(x^-)\frac{1}{\nabla_\perp^2}\rho_{1,a}(\mathbf{x}_\perp) , \\ A_{2,a}^-(x) &= -g\delta(x^+)\frac{1}{\nabla_\perp^2}\rho_{2,a}(\mathbf{x}_\perp) . \end{aligned} \quad (49)$$

In fact, this solution will be needed in momentum space when we include it in the pair production amplitude:

$$\begin{aligned} A_{1,a}^+(k) &= 2\pi g\delta(k^-)\frac{1}{\mathbf{k}_\perp^2}\rho_{1,a}(\mathbf{k}_\perp) , \\ A_{2,a}^-(k) &= 2\pi g\delta(k^+)\frac{1}{\mathbf{k}_\perp^2}\rho_{2,a}(\mathbf{k}_\perp) . \end{aligned} \quad (50)$$

In these formulas, k can be seen as the momentum flowing from the hard source to the quark line on which we insert the classical field.

Let us now determine the contribution A_{12}^μ to the classical field. If we isolate the terms of Eq. (46) that are quadratic in the hard sources, we have:

$$\square A_{12}^\nu = J_{12}^\nu + ig[A_{\mu 1} + A_{\mu 2}, F_1^{\mu\nu} + F_2^{\mu\nu} + \partial^\mu(A_1^\nu + A_2^\nu)] , \quad (51)$$

where J_{12}^μ is the correction of order $\mathcal{O}(\rho_1\rho_2)$ to the color current. It can be determined by the current conservation which, including terms that are quadratic in the sources, reads:

$$\partial_\nu J_1^\nu + \partial_\nu J_2^\nu + \partial_\nu J_{12}^\nu - ig[A_{\nu 1} + A_{\nu 2}, J_1^\nu + J_2^\nu] = 0 . \quad (52)$$

The current conservation at linear order in the sources tells us that $\partial_\nu J_{1,2}^\nu = 0$, so that we have simply:

$$\partial_\nu J_{12}^\nu = ig[A_{\nu 1} + A_{\nu 2}, \square(A_{\nu 1} + A_{\nu 2})] . \quad (53)$$

In addition to this equation, we know the following properties about J_{12}^μ :

$$\begin{aligned} J_{12}^+ &\propto \delta(x^-)\theta(x^+) , \\ J_{12}^- &\propto \delta(x^+)\theta(x^-) , \\ J_{12}^i &= 0 . \end{aligned} \quad (54)$$

The first two conditions mean that the + (resp. -) component of the current has to move along with the nucleus going in the + (resp. -) direction, and that effects of the second nucleus cannot start before the nuclei actually collide (hence the step functions). The third condition is simply a statement that the classical sources do not recoil. Eq. (53) can be made more explicit by writing¹²:

$$\partial^+ J_{12}^- + \partial^- J_{12}^+ = ig [T_a, T_b] \left\{ A_{1,a}^+ \square A_{2,b}^- - A_{2,b}^- \square A_{1,a}^+ \right\} , \quad (55)$$

where $T_{a,b}$ are color matrices. Moreover, the current J_{12}^+ , which follows the motion of nucleus 1, must depend locally on the source ρ_1 and receive a non-local correction from the second nucleus. This means that in eq. (55) the term in $A_{1,a}^+ \square A_{2,b}^-$ (which is non-local in ρ_1 and local in ρ_2) must go into J_{12}^- , and vice versa. At this point, we can write formally:

$$\begin{aligned} J_{12}^+ &= -ig [T_a, T_b] \frac{1}{\partial^-} A_{2,b}^- \square A_{1,a}^+ , \\ J_{12}^- &= ig [T_a, T_b] \frac{1}{\partial^+} A_{1,a}^+ \square A_{2,b}^- . \end{aligned} \quad (56)$$

Note that the inverses $1/\partial^\pm$, which are a priori not uniquely defined, are made unambiguous by the step functions in eqs. (54).

If we now realize that the only non-zero components of the first order strength tensor are:

$$\begin{aligned} F_1^{i+} &= -F_1^{+i} = \partial^i A_1^+ , \\ F_2^{i-} &= -F_2^{-i} = \partial^i A_2^- , \end{aligned} \quad (57)$$

the evolution equations for A_{12}^μ read:

$$\begin{aligned} \square A_{12}^+ &= J_{12}^+ + ig [A_2^-, \partial^+ A_1^+] , \\ \square A_{12}^- &= J_{12}^- + ig [A_1^+, \partial^- A_2^-] , \\ \square A_{12}^i &= -ig [A_1^+, \partial^i A_2^-] - ig [A_2^-, \partial^i A_1^+] . \end{aligned} \quad (58)$$

Using the explicit form of the current J_{12} , we can solve this as follows:

$$\begin{aligned} A_{12}^+ &= -ig [T_a, T_b] \frac{1}{\square} \left\{ (\square A_{1,a}^+) \frac{1}{\partial^-} A_{2,b}^- + (\partial^+ A_{1,a}^+) A_{2,b}^- \right\} , \\ A_{12}^- &= ig [T_a, T_b] \frac{1}{\square} \left\{ (\square A_{2,b}^-) \frac{1}{\partial^+} A_{1,a}^+ + (\partial^- A_{2,b}^-) A_{1,a}^+ \right\} , \\ A_{12}^i &= ig [T_a, T_b] \frac{1}{\square} \left\{ (\partial^i A_{1,a}^+) A_{2,b}^- - (\partial^i A_{2,b}^-) A_{1,a}^+ \right\} . \end{aligned} \quad (59)$$

Going to momentum space is trivial and leads to:

$$A_{12}^+(k) = -ig \frac{[T_a, T_b]}{-k^2} \int \frac{d^4 k_1}{(2\pi)^4} \left\{ ik_1^+ - \frac{k_1^2}{ik_2^-} \right\} A_{1,a}^+(k_1) A_{2,b}^-(k_2) ,$$

¹²We denote $\partial^+ \equiv \partial/\partial x^-$ and $\partial^- \equiv \partial/\partial x^+$.

$$\begin{aligned}
A_{12}^-(k) &= ig \frac{[T_a, T_b]}{-k^2} \int \frac{d^4 k_1}{(2\pi)^4} \left\{ ik_2^- - \frac{k_2^2}{ik_1^+} \right\} A_{1,a}^+(k_1) A_{2,b}^-(k_2), \\
A_{12}^i(k) &= ig \frac{[T_a, T_b]}{-k^2} \int \frac{d^4 k_1}{(2\pi)^4} \{ ik_2^i - ik_1^i \} A_{1,a}^+(k_1) A_{2,b}^-(k_2), \quad (60)
\end{aligned}$$

where we use the shorthand $k_2 \equiv k - k_1$. One can note that the order $\mathcal{O}(\rho_1 \rho_2)$ correction to the classical gauge field is proportional to a commutator of color matrices. This was to be expected as it is known that in an Abelian gauge theory there would not be such correction: the gauge field would simply be the sum of the gauge fields created by the individual nuclei.

B Calculation of the traces

In this appendix, we provide the result of the calculation of $\text{Tr} |m_{ab}^{-+}|^2$. In order to make this computation more compact, let us first introduce the following 4-vectors:

$$\begin{aligned}
a &\equiv (a^+ = 0, a^- = 0, \mathbf{a}_\perp = \mathbf{q}_\perp - \mathbf{k}_{1\perp}), \\
b &\equiv (b^+ = 0, b^- = 0, \mathbf{b}_\perp = \mathbf{k}_{1\perp} - \mathbf{p}_\perp), \\
c &\equiv \left(c^+ = p^+ + q^+ - \frac{\mathbf{k}_{1\perp}^2}{p^- + q^-}, c^- = \frac{\mathbf{k}_{2\perp}^2}{p^+ + q^+} - p^- - q^-, \mathbf{c}_\perp = \mathbf{k}_{2\perp} - \mathbf{k}_{1\perp} \right), \quad (61)
\end{aligned}$$

and the following notations for the denominators:

$$\begin{aligned}
2q^- p^+ + (\mathbf{q}_\perp - \mathbf{k}_{1\perp})^2 + m^2 &\equiv m^2 - \hat{t}, \\
2q^+ p^- + (\mathbf{p}_\perp - \mathbf{k}_{1\perp})^2 + m^2 &\equiv m^2 - \hat{u}, \\
(p + q)^2 &\equiv \hat{s}, \quad (62)
\end{aligned}$$

where \hat{s} , \hat{t} and \hat{u} are the standard Mandelstam variables for the $gg \rightarrow q\bar{q}$ subprocess. We can rewrite the amplitude as follows:

$$\begin{aligned}
m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p}) &= \\
&= ig^2 \bar{u}(\mathbf{q}) \left\{ t_a t_b \left[\frac{\gamma^-(m + \not{d}) \gamma^+}{m^2 - \hat{t}} + \frac{\not{c}}{\hat{s}} \right] + t_b t_a \left[\frac{\gamma^+(m + \not{b}) \gamma^-}{m^2 - \hat{u}} - \frac{\not{c}}{\hat{s}} \right] \right\} v(\mathbf{p}). \quad (63)
\end{aligned}$$

Depending on how terms are paired when squaring the amplitude, there are two kinds of color traces:

$$\begin{aligned}
\text{tr}_c (t_a t_a t_b t_b) &= N_c C_F^2, \\
\text{tr}_c (t_a t_b t_a t_b) &= -\frac{1}{2} C_F, \quad (64)
\end{aligned}$$

where $C_F \equiv (N_c^2 - 1)/(2N_c)$ is the Casimir in the fundamental representation of $SU(N_c)$. In the large N_c limit, the first trace scales like N_c^3 , while the second

trace scales only like N_c . We can therefore write the trace of the squared amplitude as:

$$\text{Tr} \left(|m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p})|^2 \right) = g^4 C_F \left[N_c C_F T_3 - \frac{1}{2} T_1 \right], \quad (65)$$

where we denote:

$$\begin{aligned} T_3 \equiv \text{tr} \left\{ (\not{q} + m) \left[\frac{\gamma^-(m + \not{q})\gamma^+}{m^2 - \hat{t}} + \frac{\not{q}}{\hat{s}} \right] (\not{p} - m) \left[\frac{\gamma^+(m + \not{q})\gamma^-}{m^2 - \hat{t}} + \frac{\not{q}}{\hat{s}} \right] \right. \\ \left. + (\not{q} + m) \left[\frac{\gamma^+(m + \not{p})\gamma^-}{m^2 - \hat{u}} - \frac{\not{q}}{\hat{s}} \right] (\not{p} - m) \left[\frac{\gamma^-(m + \not{p})\gamma^+}{m^2 - \hat{u}} - \frac{\not{q}}{\hat{s}} \right] \right\} \quad (66) \end{aligned}$$

and

$$\begin{aligned} T_1 \equiv \text{tr} \left\{ (\not{q} + m) \left[\frac{\gamma^-(m + \not{q})\gamma^+}{m^2 - \hat{t}} + \frac{\not{q}}{\hat{s}} \right] (\not{p} - m) \left[\frac{\gamma^-(m + \not{p})\gamma^+}{m^2 - \hat{u}} - \frac{\not{q}}{\hat{s}} \right] \right. \\ \left. + (\not{q} + m) \left[\frac{\gamma^+(m + \not{p})\gamma^-}{m^2 - \hat{u}} - \frac{\not{q}}{\hat{s}} \right] (\not{p} - m) \left[\frac{\gamma^+(m + \not{q})\gamma^-}{m^2 - \hat{t}} + \frac{\not{q}}{\hat{s}} \right] \right\} \quad (67) \end{aligned}$$

where the subscripts 1, 3 refer to the order of the corresponding terms when $N_c \rightarrow \infty$. It is in fact a bit simpler to write

$$\text{Tr} \left(|m_{ab}^{-+}(k_1, k_2; \mathbf{q}, \mathbf{p})|^2 \right) = \frac{1}{2} g^4 C_F \left[N_c^2 T_3 - T_1' \right], \quad (68)$$

with $T_1' \equiv T_1 + T_3$, because it turns out that T_1' has a more compact expression than T_1 . A direct calculation using FORM[64] leads after some rearrangement of the terms to the following expressions:

$$\begin{aligned} T_1' = 16 \left\{ \frac{(m^2 - a^2)p^+q^-}{(m^2 - \hat{t})^2} + \frac{(m^2 - b^2)p^-q^+}{(m^2 - \hat{u})^2} \right. \\ \left. + \frac{(a \cdot b - m^2)(\mathbf{q}_\perp \cdot \mathbf{p}_\perp - m^2) - m^2(\mathbf{q}_\perp - \mathbf{p}_\perp)^2 + (p \cdot a)(q \cdot b) + (p \cdot b)(q \cdot a) - 2m^4}{(m^2 - \hat{t})(m^2 - \hat{u})} \right\} \quad (69) \end{aligned}$$

and

$$\begin{aligned} T_3 = \frac{16(m^2 - a^2)p^+q^-}{(m^2 - \hat{t})^2} + \frac{16(m^2 - b^2)p^-q^+}{(m^2 - \hat{u})^2} - \frac{16(q \cdot c)^2 + 4c^2\hat{s}}{\hat{s}^2} \\ + \frac{8}{\hat{s}(m^2 - \hat{t})} \left[(a \cdot c)(p^+q^- - p^-q^+ + \hat{s}/2) \right. \\ \left. + (2p^+c^- - \mathbf{p}_\perp \cdot \mathbf{c}_\perp)(m^2 - q \cdot a) - (2q^-c^+ - \mathbf{q}_\perp \cdot \mathbf{c}_\perp)(m^2 + p \cdot a) \right] \\ + \frac{8}{\hat{s}(m^2 - \hat{u})} \left[(b \cdot c)(p^+q^- - p^-q^+ - \hat{s}/2) \right. \end{aligned}$$

$$+(2q^+c^- - \mathbf{q}_\perp \cdot \mathbf{c}_\perp)(m^2 + p \cdot b) - (2p^-c^+ - \mathbf{p}_\perp \cdot \mathbf{c}_\perp)(m^2 - q \cdot b) \Big]. \quad (70)$$

It is straightforward to verify that T'_1 and T_3 vanish in the limits $\mathbf{k}_{1\perp} \rightarrow 0$ and $\mathbf{k}_{1\perp} \rightarrow \mathbf{p}_\perp + \mathbf{q}_\perp$.

One can also verify that our expressions, derived by solving perturbatively the classical Yang-Mills equation, are strictly equivalent to the formulas obtained in the framework of \mathbf{k}_\perp -factorization by Collins and Ellis [4] (also derived independently by Catani et al in [5]). In order to perform this comparison, one needs the following dictionary (our notations are on the left, and theirs are on the right):

$$\begin{aligned} q &\iff p_3, \\ p &\iff p_4, \end{aligned} \quad (71)$$

(the notation for the momenta $k_{1,2}$ of the incoming gluons is the same). The relationship between our amplitude squared and the formulas given in eqs. (5.7) and (5.9) of [4] is then summarized by:

$$\begin{aligned} T'_1 &= 8Y_2, \\ T_3 &= 4(Y_1 + Y_2). \end{aligned} \quad (72)$$

However, we do not agree with the amplitude squared given in the appendix of [7]. A private communication from Levin acknowledged the presence of a mistake in the formulas quoted in [7], but the corrected expressions were unavailable to us in order to compare them with our result.

References

- [1] P. Nason, S. Dawson, R.K. Ellis, Nucl. Phys. **B 303**, 607 (1988).
- [2] P. Nason, S. Dawson, R.K. Ellis, Nucl. Phys. **B 327**, 49 (1989), Erratum-
ibid. **B 335**, 260 (1990).
- [3] S. Frixione, M.L. Mangano, P. Nason, G. Ridolfi, hep-ph/9702287.
- [4] J.C. Collins, R.K. Ellis, Nucl. Phys. **B 360**, 3 (1991).
- [5] S. Catani, M. Ciafaloni, F. Hautmann, Nucl. Phys. **B 366**, 135 (1991).
- [6] P. Hagler, R. Kirschner, A. Schafer, L. Szymanowski, O. Teryaev, Phys. Rev. **D 62**, 071502 (2000).
- [7] E.M. Levin, M.G. Ryskin, Yu. M. Shabelsky, A.G. Shuvaev, Sov. J. Nucl. Phys. **53**, 657 (1991).
- [8] L.V. Gribov, E.M. Levin, M.G. Ryskin, Phys. Rept. **100**, 1 (1983).

- [9] A.H. Mueller, J-W. Qiu, Nucl. Phys. **B 268**, 427 (1986).
- [10] M.G. Ryskin, Yu.M. Shabelski, A.G. Shuvaev, Lecture given at 34th Annual Winter School on Nuclear and Particle Physics (PNPI 2000), Gatchina, Russia, 14-20 Feb 2000, hep-ph/0011111 .
- [11] J. Raufeisen, J.C. Peng, Phys. Rev. **D 67**, 054008 (2003).
- [12] B.Z. Kopeliovich, A.V. Tarasov, Nucl. Phys. **A 710**, 180 (2002).
- [13] L.D. McLerran, R. Venugopalan, Phys. Rev. **D 49**, 2233 (1994).
- [14] L.D. McLerran, R. Venugopalan, Phys. Rev. **D 49**, 3352 (1994).
- [15] L.D. McLerran, R. Venugopalan, Phys. Rev. **D 50**, 2225 (1994).
- [16] J. Jalilian-Marian, A. Kovner, A. Leonidov, H. Weigert, Nucl. Phys. **B 504**, 415 (1997).
- [17] J. Jalilian-Marian, A. Kovner, A. Leonidov, H. Weigert, Phys. Rev. **D 59**, 014014 (1999).
- [18] J. Jalilian-Marian, A. Kovner, A. Leonidov, H. Weigert, Phys. Rev. **D 59**, 034007 (1999).
- [19] J. Jalilian-Marian, A. Kovner, A. Leonidov, H. Weigert, Erratum. Phys. Rev. **D 59**, 099903 (1999).
- [20] A. Kovner, G. Milhano, Phys. Rev. **D 61**, 014012 (2000).
- [21] A. Kovner, G. Milhano, H. Weigert, Phys. Rev. **D 62**, 114005 (2000).
- [22] I. Balitsky, Nucl. Phys. **B 463**, 99 (1996).
- [23] Yu.V. Kovchegov, Phys. Rev. **D 54**, 5463 (1996).
- [24] Yu.V. Kovchegov, Phys. Rev. **D 61**, 074018 (2000).
- [25] J. Jalilian-Marian, A. Kovner, L.D. McLerran, H. Weigert, Phys. Rev. **D 55**, 5414 (1997).
- [26] E. Iancu, A. Leonidov, L.D. McLerran, Nucl. Phys. **A 692**, 583 (2001).
- [27] E. Iancu, A. Leonidov, L.D. McLerran, Phys. Lett. **B 510**, 133 (2001).
- [28] E. Iancu, R. Venugopalan, hep-ph/0303204 .
- [29] E. Iancu, A. Leonidov, L.D. McLerran, Lectures given at Cargese Summer School on QCD Perspectives on Hot and Dense Matter, Cargese, France, 6-18 Aug 2001, hep-ph/0202270 .
- [30] A.H. Mueller, Lectures given at the International Summer School on Particle Production Spanning MeV and TeV Energies (Nijmegen 99), Nijmegen, Netherlands, 8-20, Aug 1999, hep-ph/9911289 .

- [31] L.D. McLerran, R. Venugopalan, Phys. Rev. **D 59**, 094002 (1999).
- [32] F. Gelis, A. Peshier, Nucl. Phys. **A 697**, 879 (2002).
- [33] F. Gelis, A. Peshier, Nucl. Phys. **A 707**, 175 (2002).
- [34] A. Kovner, L.D. McLerran, H. Weigert, Phys. Rev. **D 52**, 3809 (1995).
- [35] A. Kovner, L.D. McLerran, H. Weigert, Phys. Rev. **D 52**, 6231 (1995).
- [36] Yu.V. Kovchegov, D.H. Rischke, Phys. Rev. **C 56**, 1084 (1997).
- [37] M. Gyulassy, L.D. McLerran, Phys. Rev. **C 56**, 2219 (1997).
- [38] A. Krasnitz, R. Venugopalan, Phys. Rev. Lett. **84**, 4309 (2000).
- [39] A. Krasnitz, R. Venugopalan, Phys. Rev. Lett. **86**, 1717 (2001).
- [40] A. Krasnitz, Y. Nara, R. Venugopalan, Phys. Rev. Lett. **87**, 192302 (2001).
- [41] T. Lappi, Phys. Rev. **C 67**, 054903 (2003).
- [42] F. Gelis, K. Kajantie, T. Lappi, Work in progress .
- [43] V.P. Goncalves, M.V.T. Machado, hep-ph/0307129 .
- [44] A.M. Stasto, K. Golec-Biernat, J. Kwiecinski, Phys. Rev. Lett. **86**, 596 (2001).
- [45] E. Iancu, K. Itakura, L.D. McLerran, Nucl. Phys. **A 708**, 327 (2002).
- [46] E. Iancu, K. Itakura, L.D. McLerran, Nucl. Phys. **A 724**, 181 (2003).
- [47] A.H. Mueller, D.N. Triantafyllopoulos, Nucl. Phys. **B 640**, 331 (2002).
- [48] D.N. Triantafyllopoulos, Nucl. Phys. **B 648**, 293 (2003).
- [49] H. Kowalski, D. Teaney, hep-ph/0304189 .
- [50] S. Munier, A.M. Stasto, A.H. Mueller, Nucl. Phys. **B 603**, 427 (2001).
- [51] V.P. Goncalves, M.V.T. Machado, hep-ph/0307090 .
- [52] J. Schaffner-Bielich, D. Kharzeev, L.D. McLerran, R. Venugopalan, Nucl. Phys. **A 705**, 494 (2002).
- [53] A. Freund, K. Rummukainen, H. Weigert, A. Schafer, Phys. Rev. Lett. **90**, 222002 (2003).
- [54] D. Kharzeev, Yu. Kovchegov, K. Tuchin, hep-ph/0307037 .
- [55] D. Kharzeev, E. Levin, L.D. McLerran, Phys. Lett. **B 561**, 93 (2003).
- [56] J. Jalilian-Marian, Y. Nara, R. Venugopalan, nucl-th/0307022 .

- [57] R. Baier, A. Kovner, U.A. Wiedemann, Phys. Rev. **D 68**, 054009 (2003).
- [58] J.L. Albacete, N. Armesto, A. Kovner, C.A. Salgado, U.A. Wiedemann, hep-ph/0307179 .
- [59] A.J. Baltz, F. Gelis, L.D. McLerran, A. Peshier, Nucl. Phys. **A 695**, 395 (2001).
- [60] A. Krasnitz, Y. Nara, R. Venugopalan, hep-ph/0305112, to appear in Nucl. Phys. **A** .
- [61] E. Iancu, A.H. Mueller, hep-ph/0308315 .
- [62] V. Del Duca, hep-ph/9503226 .
- [63] K. Rummukainen, H. Weigert, hep-ph/0309306 .
- [64] J.A.M. Vermaseren, math-ph/0010025 .