

**Bosonization and Fermion Liquids in Dimensions Greater Than One**

A. Houghton

*Department of Physics  
Box 1843  
Brown University  
Providence, RI 02912 U.S.A.*

and

J. B. Marston

*Institute for Theoretical Physics  
University of California  
Santa Barbara, CA 93106-4030*

and

*Department of Physics  
Box 1843  
Brown University  
Providence, RI 02912 U.S.A.*

*October 21, 1992*

ABSTRACT: We develop and describe new approaches to the problem of interacting Fermions in spatial dimensions greater than one. These approaches are based on generalizations of powerful tools previously applied to problems in one spatial dimension. We begin with a review of one-dimensional interacting Fermions. We then introduce a simplified model in two spatial dimensions to study the role that spin and perfect nesting play in destabilizing Fermion liquids. The complicated functional renormalization group equations of the full problem are made tractable in our model by replacing the continuum of points that make up the closed Fermi line with four Fermi points. Despite this drastic approximation, the model exhibits physically reasonable behavior both at half-filling (where instabilities occur) and away from half-filling (where a Luttinger liquid arises). Next we implement the Bosonization of higher dimensional Fermi surfaces introduced by Luther and advocated most recently by Haldane. Bosonization incorporates the phase space and small-angle scattering processes neglected in our model (but does not, as yet, address questions of stability). The charge sector is equivalent to an exactly solvable Gaussian quantum field theory; the spin sector, however, must be solved semiclassically. Using the Luther-Haldane approach we recover the collective mode equation of Fermi-liquid theory and in three dimensions reproduce the  $T^3 \ln(T)$  contribution to the specific heat due to small angle scattering processes. We conclude with a discussion of our results and some speculation about future possibilities.

PACS: 03.65.Fd, 11.40.Dw, 71.10.+x, 71.45.-d

## I. INTRODUCTION AND REVIEW OF A 1-D MODEL WITH 2-FOLD U(1) SYMMETRY

Fermi liquid theory is now nearly forty years old. It is important to ascertain the range of its validity and determine whether more exotic generalizations, such as Luttinger liquids which exhibit spin-charge separation, exist. Shankar recently has emphasized the advantage of the renormalization-group (RG) approach over various types of mean-field approximations for answering these questions<sup>1</sup>. In short, mean-field descriptions prejudice the outcome of the analysis by assuming that one, or at most a few, type of instability dominate the physics. RG analysis, on the other hand, treats all possible instabilities on an equal footing. Unfortunately, in spatial dimensions greater than one, the RG flows are described by nearly intractable functional equations.

The approach we take in this paper is to slowly work up to the full problem by first reviewing rather well known one dimensional physics<sup>2</sup>. We then study a simplified model in two dimensions that incorporates some of the new physics that arises in higher spatial dimensions while still retaining the simplicity of one dimensional systems. Of course the price we pay for this simplicity is the drastic approximation to physical reality that we must make in order to arrive at the model: we completely neglect small angle scattering processes. Nevertheless, the model suggests a way to completely reformulate Fermi liquid theory. Following Haldane's suggestion<sup>3</sup> we now view Fermi and Luttinger liquids as zero temperature quantum critical fixed points characterized by infinite U(1) symmetry. The reformulation sheds light on how one might go beyond the drastic approximations of the model to include small angle scattering processes.

Spin-charge separation occurs automatically in one spatial dimension, at least in the weak coupling limit and at long length scales. Consider the following low energy effective theory for excitations near the two Fermi points depicted in Figure [1]. The action in the non-interacting limit is given by:

$$S_0 = \int dx dt \{ \psi_L^{\dagger\alpha} \partial_- \psi_{L\alpha} + \psi_R^{\dagger\alpha} \partial_+ \psi_{R\alpha} \} . \quad (1.1)$$

Here L and R refer to the left and right Fermi points;  $\partial_{\pm} \equiv \partial_t \mp i v_f \partial_x$  where  $v_f$  is the Fermi velocity which we will set equal to one for now. The Lorentz symmetry of this action guarantees that the

left moving Fermi fields are purely functions of the combination  $(x + iv_f t)$  whereas the right fields are functions of  $(x - iv_f t)$ . The electron destruction fields  $c_\alpha(x, t)$ , where  $\alpha = \uparrow$  or  $\downarrow$  for up and down spins (with summation convention assumed), are related to the slowly varying continuum fields  $\psi_{L,R}$  by:

$$c_\alpha(x, t) \equiv \frac{1}{\sqrt{2}} \{e^{-ik_f x} \psi_{L\alpha}(x, t) + e^{ik_f x} \psi_{R\alpha}(x, t)\}. \quad (1.2)$$

Upon substituting this form into any given microscopic Hamiltonian the many-body interactions take the form of quartic and higher powers of the continuum fields. Most of the interactions are irrelevant in the renormalization group sense (see below) and the most general *marginal* interaction takes the form:

$$S_{int} = \int dx dt \left\{ \frac{\pi}{2} \delta v_c (J_L^2 + J_R^2) + \frac{\pi}{6} \delta v_s (J_{L\alpha}^\beta J_{L\beta}^\alpha + J_{R\alpha}^\beta J_{R\beta}^\alpha) \right. \\ \left. + \lambda_c J_L J_R + \lambda_s J_{L\alpha}^\beta J_{R\beta}^\alpha \right\}. \quad (1.3)$$

Here for instance the charge current at the left point is defined by  $J_L \equiv: \psi_L^{\dagger\alpha} \psi_{L\alpha} :$  where the normal ordering symbols “:” indicate that we have subtracted the constant background charge density from the current to make  $\langle J_L(x, t) \rangle = 0$ . The spin current is most conveniently expressed in matrix form:  $J_{L\beta}^\alpha(x) \equiv \psi_L^{\dagger\alpha}(x) \psi_{L\beta}(x) - \frac{1}{2} \delta_\beta^\alpha \psi_L^{\dagger\gamma}(x) \psi_{L\gamma}(x)$ . The matrix form can always be converted into the more familiar vector form with the identity:  $J_L^a = \frac{1}{2} (\sigma^a)_\alpha^\beta J_{L\beta}^\alpha$  where  $a = x, y, \text{ or } z$ . Note that the spin current has no charge current component because it is traceless. It also has zero vacuum expectation value because in 1+1 dimensions the vacuum cannot break the continuous SU(2) spin rotational invariance by a quantum generalization of the Mermin-Wagner theorem. In the Fermion action, spin-charge separation is apparent even before Bosonization. That is, the interaction term involves only products of either pure spin or pure charge currents. The Gaussian part of the action,  $S_0$ , also can be expressed purely in terms of separate products of the charge and spin currents (see below).

Omitted from the action are terms that oscillate rapidly with wavevectors of order  $k_f$ , interactions involving derivatives that arise from Taylor expansions of non-local interactions, and terms with more than four Fermion fields. Many of these terms break spin-charge separation; however,

each is irrelevant in the renormalization group sense and the coupling constants flow rapidly to zero in the low-energy limit. To show the irrelevance, consider the scale transformation  $x \rightarrow sx$  and  $t \rightarrow st$  where  $s > 1$ . The Gaussian part of the action,  $S_0$ , remains invariant if we rescale the fields  $(\psi^\dagger, \psi) \rightarrow s^{-1/2}(\psi^\dagger, \psi)$ . (Note that any non-linearities in the dispersion relation due to band structure are smoothed out as  $s \rightarrow \infty$ .) Similarly,  $S_{int}$  remains invariant, showing that it is a marginal interaction. All other terms will, however, scale away at least as fast as an inverse power of  $s$  when  $s \rightarrow \infty$ . Thus, in one dimension, spin-charge separation occurs in the low-energy effective theory regardless of how the marginal interactions flow. Note that non-zero temperature acts as an infrared cutoff (since the time direction has a finite extent  $\beta \equiv \frac{1}{k_B T}$ ) that stops scaling towards the low-energy region beyond this scale. Irrelevant terms therefore persist at non-zero temperature so the phenomenon of spin-charge separation must be construed as a zero-temperature critical property of the theory.

Apart from the observation of spin-charge separation, the low-energy theory can be classified in terms of the symmetries that it obeys. In addition to the global SU(2) spin rotational symmetry, there exist two separate U(1) symmetries: one for each Fermi point. This  $U(1)_L \otimes U(1)_R$  symmetry may be exhibited by considering the effect of separate left and right phase rotations by angles  $\Gamma_L$  and  $\Gamma_R$  on the Fermion variables:

$$\begin{aligned}\psi_{L\alpha}(x, t) &\rightarrow e^{i\Gamma_L} \psi_{L,\alpha}(x, t) \\ \psi_{R\alpha}(x, t) &\rightarrow e^{i\Gamma_R} \psi_{R,\alpha}(x, t) .\end{aligned}\tag{1.4}$$

All of the currents are clearly invariant under this transformation, as the  $\psi^\dagger$  fields transform with opposite phases. The physical meaning of the invariance is clear: the action, as it stands, conserves separately the number of left and right particles. We shall see that this special property has a natural generalization to higher spatial dimensions. Actually, one other marginal four-Fermi interaction can appear. The Umklapp term

$$\lambda_3 [(\psi_R^\dagger \psi_{L\alpha})^2 + H.c.] \tag{1.5}$$

is permitted at half-filling in a periodic one-dimensional solid and it breaks the  $U(1)_L \otimes U(1)_R$  symmetry down to the diagonal subgroup of ordinary  $U(1)$  transformations with  $\Gamma_L = \Gamma_R$ . It violates

the separate left and right U(1) symmetries because it transports two particles from one Fermi point to the other. Of course, total particle number remains conserved, and this conservation is reflected in the remaining diagonal U(1) symmetry. Like the other terms in the action, the Umklapp term preserves spin-charge separation because it transports charge, not spin, from one Fermi point to the other. To see this, note that it may be rewritten as:  $\frac{1}{2}\lambda_3 (\epsilon_{\alpha\beta}\psi_R^{\dagger\alpha}\psi_R^{\dagger\beta})(\epsilon^{\gamma\delta}\psi_{L\gamma}\psi_{L\delta}) + H.c.$  where  $\epsilon_{\alpha\beta}$  is the totally antisymmetric tensor with  $\epsilon_{12} = 1$ . Thus, only spin-singlet objects move from one Fermi point to the other.

We now return to the problem without the Umklapp term and determine the RG flows and the nature of the fixed points. Bosonization of the Fermion fields is a powerful tool for addressing these questions. For now we use Abelian Bosonization<sup>4</sup> and choose the spin quantization axis in the  $\hat{\mathbf{z}}$  direction. The current algebra will provide the vital link between the Fermion and Boson representations. We start by defining the normal-ordering operation carefully:

$$\begin{aligned} J_{L\alpha}(x, t) &\equiv : \psi_{\alpha}^{\dagger}(x, t) \psi_{\alpha}(x + \epsilon, t) : \\ &\equiv \psi_{\alpha}^{\dagger}(x, t) \psi_{\alpha}(x + \epsilon, t) - \langle \psi_{\alpha}^{\dagger}(x, t) \psi_{\alpha}(x + \epsilon, t) \rangle . \end{aligned} \tag{1.6}$$

Here we place the spin index as a subscript on the  $\psi^{\dagger}$  field to emphasize that we are no longer summing over it, and we imagine taking the  $\epsilon \rightarrow 0$  limit at the end of our calculations. This “point-splitting” procedure regularizes ultraviolet divergences in our calculation. We choose real-space regularization because the connection between Bosons and Fermions occurs most naturally in real space. Momentum space regularization will be introduced later to permit the evaluation of momentum space integrals; differences between the two regularization procedures do not change the low-energy results. Currents for the right moving sector are obtained by making the replacement  $L \rightarrow R$  and a simple calculation shows that left currents commute with right currents, while two left or two right currents at equal times obey the Kac-Moody algebra:

$$\begin{aligned} [J_{L\alpha}(x) , J_{L\beta}(y)] &= -\frac{i}{2\pi} \delta_{\alpha\beta} \delta'(x - y) . \\ [J_{R\alpha}(x) , J_{R\beta}(y)] &= +\frac{i}{2\pi} \delta_{\alpha\beta} \delta'(x - y) . \end{aligned} \tag{1.7}$$

(To derive these relations, use the equal-time propagators for the Fermions  $\langle \psi_L^{\dagger\alpha}(x)\psi_{L\beta}(0) \rangle = \delta_{\beta}^{\alpha} \frac{-i}{2\pi x}$  and  $\langle \psi_R^{\dagger\alpha}(x)\psi_{R\beta}(0) \rangle = \delta_{\beta}^{\alpha} \frac{+i}{2\pi x}$ .) The coefficient of the derivative of the Dirac  $\delta$ -function is known as the quantum anomaly. Note that it has the opposite sign for the left versus the right movers. The charge current defined previously may now be expressed in terms of these currents as:  $J_L(x) = J_{L\uparrow}(x) + J_{L\downarrow}(x)$  and the  $z$  component of the spin current is simply:  $J_{Lz}(x) = J_{L\uparrow}(x) - J_{L\downarrow}(x)$ . (However, the other two components of the spin current  $J_{Lx}$  and  $J_{Ly}$  are not so simply related to  $J_{L\uparrow}$  and  $J_{L\downarrow}$ .) The charge and spin currents also obey the Kac-Moody algebra, but with twice the anomaly.

We now introduce real-valued left and right moving free Boson fields  $\phi_{L\alpha}$  and  $\phi_{R\alpha}$  which satisfy the commutation relations:

$$\begin{aligned} [\phi_{L\alpha}(x) , \phi_{L\beta}(y)] &= -\frac{i}{4} \epsilon(x-y) , \\ [\phi_{R\alpha}(x) , \phi_{R\beta}(y)] &= +\frac{i}{4} \epsilon(x-y) , \end{aligned} \tag{1.8}$$

and

$$[\phi_{R\alpha}(x) , \phi_{L\beta}(y)] = i/4 , \tag{1.9}$$

where  $\epsilon(x) = 1$  for  $x > 0$  and  $= -1$  for  $x < 0$ . We also define canonical Boson currents

$$J_{L\alpha}(x, t) \equiv -\frac{1}{\sqrt{\pi}} \frac{\partial \phi_{L\alpha}(x, t)}{\partial x}$$

and

$$J_{R\alpha}(x, t) \equiv +\frac{1}{\sqrt{\pi}} \frac{\partial \phi_{R\alpha}(x, t)}{\partial x} . \tag{1.10}$$

It is a remarkable fact that these Boson currents obey the same Kac-Moody algebra as the Fermion currents defined previously in Eq. [1.6]. To check the current commutation relation, take spatial derivatives of the Free Boson propagators:

$$\begin{aligned} \langle \phi_{L\alpha}(x)\phi_{L\alpha}(0) - \phi_{L\alpha}^2(0) \rangle &= \frac{1}{4\pi} \ln \frac{a}{a+ix} \\ \langle \phi_{R\alpha}(x)\phi_{R\alpha}(0) - \phi_{R\alpha}^2(0) \rangle &= \frac{1}{4\pi} \ln \frac{a}{a-ix} \end{aligned} \tag{1.11}$$

(no sum over  $\alpha$  in these expressions) to form the expectation value of the commutator. Here we see the first indication of equivalence between the Fermion and Boson representations. A further connection is revealed by examining the spectra of the free Fermion and Boson theories: both of which are linear and are in fact identical. To go further, we rewrite the quadratic Boson Hamiltonian in terms of the Boson currents defined in Eq. [1.10]:

$$H = \pi \sum_{\alpha=\uparrow,\downarrow} \int dx \{J_{L\alpha}^2(x) + J_{R\alpha}^2(x)\} . \quad (1.12)$$

The key point to be made here is that the currents appearing in this Hamiltonian could just as well be the Fermion currents Eq. [1.6] since the algebras are identical. Although in this representation the Hamiltonian is a quartic function of the Fermion fields and as such would appear to be intractable, remarkably, as we have seen, it is equivalent to a free Fermion theory.

We now consider the effect of the three types of interactions appearing in the action Eq. [1.3]. First, the current bilinears proportional to  $\delta v_c$  and  $\delta v_s$  simply renormalize the coefficients of the quadratic Boson Hamiltonian. [Although all three spin components of the spin current appear in Eq. [1.3], SU(2) invariance means that  $J_{L\beta}^\alpha(x)J_{L\alpha}^\beta(x)$  can be replaced by  $3J_{Lz}^2(x)$  so in this case it suffices to consider only the z-component of the spin current.] Using:

$$J_{L\uparrow}^2 + J_{L\downarrow}^2 = \frac{1}{2}\{J_L^2 + J_{Lz}^2\} \quad (1.13)$$

we see that so far spin-charge separation is explicit as the renormalized Boson Hamiltonian may now be expressed as the sum of two pieces,  $H = H_c + H_s$ , that separately describe charge and spin excitations propagating at different velocities:

$$H_c = \frac{\pi}{2} v_c \int dx \{J_L^2(x) + J_R^2(x)\} \quad (1.14)$$

and

$$H_s = \frac{\pi}{2} v_s \int dx \{J_{Lz}^2(x) + J_{Rz}^2(x)\} \quad (1.15)$$

where  $v_c = 1 + \delta v_c$  and  $v_s = 1 + \delta v_s$ . It should be emphasized that the coefficients  $\delta v_c$  and  $\delta v_s$  do not flow under RG transformations but instead simply renormalize the bare velocities. It may



seem strange to have two different “velocities of light” – indeed, Lorentz invariance is broken. But since the charge and spin sectors are separate, Lorentz invariance is now manifest separately in the two sectors.

Similarly, the charge current coupling,  $\lambda_c$ , in the action Eq. [1.3] remains fixed as  $s \rightarrow \infty$ . To see this, note that it may be incorporated by adding another term quadratic in the Boson currents to the charge Hamiltonian:

$$H_c = \int dx \left\{ \frac{\pi}{2} [J_L^2(x) + J_R^2(x)] + \lambda_c J_L(x) J_R(x) \right\}. \quad (1.16)$$

To determine the effect of  $\lambda_c$  on the spectrum, we must diagonalize this Hamiltonian. Diagonalization is accomplished via a Bogoliubov transformation that respects the Kac-Moody algebra. Let:

$$J_L(x) = \cosh(\eta) J'_L(x) + \sinh(\eta) J'_R(x)$$

and

$$J_R(x) = \sinh(\eta) J'_L(x) + \cosh(\eta) J'_R(x). \quad (1.17)$$

Then the primed charge currents obey the same algebra as the original (unprimed) charge currents. Upon substituting these currents into the Hamiltonian Eq. [1.16] we find that the choice

$$\tanh(2\eta) = -\frac{\lambda_c}{\pi} \quad (1.18)$$

eliminates cross terms of the form  $J'_L(x) J'_R(x)$ . We now introduce primed Boson fields  $\phi'_{Lc}(x) \equiv \frac{1}{\sqrt{2}} (\phi'_{L\uparrow} + \phi'_{L\downarrow})$  and  $\phi'_{Rc}$  associated with the primed charge currents. A factor of  $\sqrt{2}$  is needed to reproduce the correct anomaly:  $J'_L(x) \equiv \sqrt{2/\pi} \partial_x \phi'_{Lc}(x)$  with a similar formula for the right sector.

The Hamiltonian written in terms of these fields is simply:

$$H_c = \left(1 - \frac{\lambda_c^2}{\pi^2}\right)^{\frac{1}{2}} \int dx \left\{ \left(\frac{\partial \phi'_{Lc}}{\partial x}\right)^2 + \left(\frac{\partial \phi'_{Rc}}{\partial x}\right)^2 \right\}. \quad (1.19)$$

Thus the Bosonic theory remains Gaussian, even for  $\lambda_c \neq 0$ .

It might be expected that the spin current coupling  $\lambda_s$  could be incorporated in a similar fashion. However, the spin interaction  $\lambda_s J_{L\beta}^\alpha(x) J_{R\alpha}^\beta(x)$  differs in a fundamental way from the charge

interaction  $\lambda_c J_L(x) J_R(x)$ . Only the  $J_{Lz} J_{Lz}$  part of the interaction has a quadratic representation in terms of the Boson field  $\phi_s$ . The other two components are rather more complicated. Note that SU(2) invariance may be employed only when both currents in the bilinear are of the left or right type; for example, when we replace  $J_{L\beta}^\alpha(x) J_{L\alpha}^\beta(x) \rightarrow 3J_{Lz}^2(x)$ . The non-trivial nature of this term is apparent in the Fermion basis:  $\lambda_s$  is the only interaction coefficient in  $S_{int}$  that renormalizes. To second order in weak-coupling perturbation theory (see below) it flows as:

$$\frac{d\lambda_s}{d(\ln(s))} = 2\pi(\lambda_s)^2. \quad (1.20)$$

Because no fixed points intervene at intermediate coupling in the original lattice Hubbard model (which was exactly solved by Lieb and Wu via the Bethe ansatz<sup>5</sup>) the flow for the continuum problem described by Eq. [1.20] is likely to be qualitatively correct at *all*  $\lambda_s$ . The  $\lambda_s = 0$  fixed point of Eq. [1.20] (which is stable when approached from the  $\lambda_s < 0$  side) exhibits an enlarged  $SU(2)_L \otimes SU(2)_R$  symmetry because the left and right spin currents are decoupled. In other words, separate SU(2) rotations on the left and right currents leave the fixed point action invariant. The strong coupling fixed point, with  $\lambda_s \rightarrow \infty$  shows no such symmetry; instead a spin gap opens and the electrons pair into singlets that require energy to break apart.

To understand better the role of spin-charge separation in the one-dimensional problem, we examine the single-particle Green's function along the fixed line  $\lambda_s = 0$ . Following Shankar<sup>4</sup>, we introduce momentum space regularization by including a convergence factor  $e^{-\frac{1}{2}a|p|}$   $dp$  along with the integration measure, and we take the  $a \rightarrow 0$  limit at the end. Now the (equal time) correlation functions for the  $\phi'_{Lc} \equiv \frac{1}{\sqrt{2}} [\phi'_{L\uparrow} + \phi'_{L\downarrow}]$  fields are given by:

$$\begin{aligned} G_{Lc}(x) &\equiv \langle \phi'_{Lc}(x) \phi'_{Lc}(0) - \phi_{Lc}^{\prime 2}(0) \rangle \\ &= \frac{1}{4\pi} \ln \frac{a}{a + ix} \\ G_{Rc}(x) &\equiv \langle \phi'_{Rc}(x) \phi'_{Rc}(0) - \phi_{Rc}^{\prime 2}(0) \rangle \\ &= \frac{1}{4\pi} \ln \frac{a}{a - ix}. \end{aligned} \quad (1.21)$$

The spin Bosons  $\phi_s$  exhibit identical correlations. It is easy to restore the time-dependence of these correlation functions by using Lorentz invariance (of course with different velocities in the charge

and spin sectors). The Bosonization procedure is completed with the observation that Fermion operators are equivalent to exponentials of the original unprimed Boson fields:

$$\begin{aligned}\psi_{L\alpha} &= \frac{1}{\sqrt{2\pi a}} \exp[-i\sqrt{4\pi} \phi_{L\alpha}] \\ \psi_{R\alpha} &= \frac{1}{\sqrt{2\pi a}} \exp[i\sqrt{4\pi} \phi_{R\alpha}].\end{aligned}\tag{1.22}$$

One way to prove relations Eq. [1.22] is by constructing the Fermion currents with the point splitting procedure described above. Then spatial derivatives of the Boson fields appear, and the currents defined in Eq. [1.10] are obtained. Since  $\phi_{Lc} = \cosh(\eta)\phi'_{Lc} + \sinh(\eta)\phi'_{Rc}$  and  $\phi_{L\uparrow} = \frac{1}{\sqrt{2}}(\phi_{Lc} + \phi_{Ls})$  etc. we can combine the Bosonization formulas to find the Fermion two-point functions.

$$\begin{aligned}\langle \psi_L^{\dagger\beta}(x, t) \psi_{L\beta}(0, 0) \rangle &= 2 \times \frac{1}{2\pi a} \exp\{2\pi[\cosh^2(\eta) G_{Lc}(x, t) + \sinh^2(\eta) G_{Rc}(x, t)]\} \\ &\quad \times \exp\{2\pi G_{Ls}(x, t)\} \\ &= \frac{1}{\pi} (x - iv_s t)^{-1/2} (x - iv_c t)^{-1/2} (x^2 + v_c^2 t^2)^{-\alpha}\end{aligned}$$

and

$$\langle \psi_R^{\dagger\beta}(x, t) \psi_{R\beta}(0, 0) \rangle = \frac{1}{\pi} (x + iv_s t)^{-1/2} (x + iv_c t)^{-1/2} (x^2 + v_c^2 t^2)^{-\alpha}.\tag{1.23}$$

Here the anomalous exponent  $\alpha \equiv \sinh^2(\eta)$ . The explicit separation of charge and spin reflects both the different velocities of the two types of excitations and the remaining interaction  $\lambda_c$  in the charge sector.

The path integral picture yields the following free Lagrangian densities<sup>6</sup>:

$$\begin{aligned}L_c[\phi_c] &= \frac{1}{2} \left(1 - \frac{2\lambda_c}{\pi}\right) \partial_\mu \phi_c \partial^\mu \phi_c \\ L_s[\phi_s] &= \frac{1}{2} \partial_\mu \phi_s \partial^\mu \phi_s\end{aligned}\tag{1.24}$$

where the spatial derivatives in these two expressions implicitly include the different velocities factors,  $v_c$  and  $v_s$ . The Bosonization formulas Eq. [1.22] imply that the  $U(1)_L \otimes U(1)_R$  symmetry operation is effected simply by shifting the left and right Bosons by, in general, different constants. Thus,  $\phi_c$  is shifted ( $\phi_{Lc} \rightarrow \phi_{Lc} + \frac{\Gamma_L}{\sqrt{2\pi}}$  and  $\phi_{Rc} \rightarrow \phi_{Rc} + \frac{\Gamma_R}{\sqrt{2\pi}}$ ), but  $\phi_s$  remains invariant, reflecting the fact that the symmetry operation acts only on the charge sector. The Lagrangian density Eq. [1.24] remains invariant because  $\partial_\mu \phi_c$  is unaffected by the shift. The Umklapp term, as expected,

breaks the symmetry because it is equivalent to adding the term  $\lambda_3 \cos(\sqrt{8\pi} \phi_c)$  to  $L_c[\phi_c]$  and the cosine clearly changes under a shift of  $\phi_c$  by a constant.

At this point we might question whether the strange form of the propagator somehow eliminates the logarithmic divergences that give rise to the RG flow described by Eq. [1.20]. The answer, which is no, may be seen easily in the Boson basis where spin-charge decoupling is explicit. There the interaction appears as a term proportional to  $\lambda_s \cos(\sqrt{8\pi} \phi_s)$  added to  $L_s[\phi_s]$  of Eq. [1.24] which drives the RG flow. [Note that the  $SU(2)_L \otimes SU(2)_R$  symmetry at  $\lambda_s = 0$  now manifests itself in the Boson basis as separate shifts by a constant in  $\phi_{L_s}$  and  $\phi_{R_s}$ . Actually, full  $SU(2)$  symmetry would exhibit invariance under three types of rotations corresponding to the three generators of  $SU(2)$ . Abelian Bosonization, however, forces us to choose a spin quantization axis; consequently the theory only exhibits explicit symmetry under rotations about that axis. Invariance under rotations in the other two directions remains hidden.] In the Fermion basis, the problem is slightly more complicated. The logarithmic divergence that drives the flow described by Eq. [1.20] comes from a single loop diagram with four external fields that contains two propagators: one for left moving fields and one for right movers (see Figure [2]). (Diagrams with two left or two right propagators do not yield logarithmic divergences.) The diagram is most easily evaluated in position space. The integral to be evaluated is:

$$I(s) = (\lambda_s)^2 \int_1^s dx \int_{-\infty}^{\infty} dt (x^2 + v_c^2 t^2)^{-\frac{1}{2}-2\alpha} (x^2 + v_s^2 t^2)^{-1/2}. \quad (1.25)$$

Since we are performing a perturbative expansion to order  $(\lambda_s)^2$  it is sufficient to set  $\alpha = (\frac{\lambda_c}{2\pi})^2 + O(\lambda_c^4)$  equal to zero and study whether the logarithmic singularity persists when  $v_c \neq v_s$ . With  $\alpha = 0$  the spatial integral yields a hypergeometric function which, when integrated over time, indeed produces the desired logarithm.

Fermi liquid behavior arises only in the special case  $v_c = v_s$  and  $\alpha = 0$  as the two-point function Eq. [1.23] must have a single simple pole. However, the discontinuity at the Fermi surface remains even when  $v_c \neq v_s$  so long as  $\alpha = 0$ . To see this, note that the momentum space occupancy is found by taking the Fourier transform of the equal time propagator (which we define to be the average

of the correlation function evaluated at times  $t = 0+$  and  $t = 0-$  in order to specify definite time orderings which also cancel out the imaginary component). Use of the Fermion anticommutation relations then shows:

$$2n_L(k) - 1 = \frac{1}{2} \int dx e^{i(k+k_f)x} \{ \langle \psi_L^{\dagger\beta}(x, t = 0+) \psi_{L\beta}(0, 0) \rangle + \langle \psi_L^{\dagger\beta}(x, t = 0-) \psi_{L\beta}(0, 0) \rangle \} . \quad (1.26)$$

But as  $t \rightarrow 0\pm$  the two velocities disappear from the correlation function, which equals  $\frac{|x|^{-2\alpha}}{\pi x}$  and yields a step function in momentum space only for  $\alpha = 0$ . Apparently spin-charge separation and the destruction of the Fermi discontinuity are separate issues.<sup>7</sup> Both are characteristic properties of Luttinger liquids<sup>8</sup>. In the following we continue to speak of Fermi points and Fermi surfaces. Clearly these points or surfaces should now be defined more generally as manifolds of points in momentum space at which the zero-temperature occupancy shows non-analytic behavior characterized by an exponent  $\alpha$  instead of a discontinuity. In particular, near the Fermi momentum the occupancy varies as:  $n(k) \approx n(k_f) + C|k - k_f|^{2\alpha} \text{sgn}(k - k_f)$  where  $C$  is a non-universal constant that sets the momentum scale. It depends on the the momentum-energy cutoff in the interaction  $\lambda_c$ .

The outline for the rest of the paper is as follows. In section (II) we introduce a model with four Fermi points in two spatial dimensions that incorporates some key features of the higher dimension interacting Fermion problem. A renormalization group solution of the model away from half-filling finds a stable fixed line with four-fold  $U(1)$  symmetry that naturally generalizes the one-dimensional  $U(1)_L \otimes U(1)_R$  fixed point symmetry. Bosonization of the Fermions at these points suggests a new way to look at Fermions in higher dimension and in section (III) we follow this line of thought to arrive at full Fermi surface fixed point manifolds with infinite  $U(1)$  symmetry. In section (IV) we demonstrate that this way of looking at things yields concrete results by calculating non-analytic contributions to the specific heat. And in section (V) we rederive the collective mode equation in the new framework. Finally, section (VI) contains some discussion and speculations.

## II. A 2-D FIXED POINT WITH 4-FOLD U(1) SYMMETRY

We turn now to a simple model for two-dimensional interacting Fermions that illustrates how possibly spin-charge separation might occur in spatial dimensions greater than one. The Fermi surface of a nearest neighbor tight-binding model on a square lattice serves as inspiration for the model. Instead of treating the entire continuum of Fermi points that make up the Fermi line enclosing the occupied states, we make our problem tractable with the following drastic approximation: we treat each of the four sides of the Fermi surface as a single point labeled by  $\pm 1$  or  $\pm 2$  (see Figure [3]). At half-filling, these points lie respectively at momenta  $\pm(\pi/2, \pi/2)$  and  $\pm(\pi/2, -\pi/2)$  but away from half-filling the momentum is generally incommensurate with the reciprocal lattice vectors. With this simplification, the infinite set of renormalization group equations is reduced to a manageable finite set.

Note that this model differs from models studied earlier by Schulz<sup>9</sup> and Dzyaloshinskii<sup>10</sup> that focused on the Van Hove singularities at the four corners  $(0, \pm\pi)$  and  $(\pm\pi, 0)$  of the half-filled tight binding spectrum. Our model also is not equivalent to two coupled parallel Hubbard chains – a system studied by Anderson<sup>11</sup>, Finkel'stein<sup>12</sup>, and others. It differs in that excitations at points  $\pm 2$  propagate at right angles in momentum space with respect to excitations at points  $\pm 1$  whereas excitations in the two-chain system always propagate in parallel (or anti-parallel) directions. Consequently, different marginal interactions and renormalization group equations appear in our model. Anderson's analysis<sup>11</sup> of the two-chain problem led him to conclude that weak interchain coupling does not change the one-dimensional physics significantly but recent work by Finkel'stein<sup>12</sup> and Castellani, Di Castro and Metzner<sup>13</sup> suggests that interchain coupling is relevant and destabilizes the Luttinger liquid.

We choose the model in part because it emphasizes the role that perfect nesting plays in destabilizing various fixed points (we elaborate on the nature of these fixed points below). Thus for example processes that transfer an electron across the Fermi surface from, say, point 1 to point -1 receive the same weight as processes that move an electron from point 1 to 2 because the density

of states is non-zero only at the four points. Note that the density of states at each point must be held constant regardless of the system size. One might be tempted to give each point the same weight as the entire line it replaces, but this choice proves uninteresting as quantum fluctuations would be suppressed in the thermodynamic limit. Photoemission experiments on the cuprate superconductors<sup>14</sup> provide another justification for our model<sup>15</sup>. These experiments show that hole pockets form around the momentum points  $\pm(\pi/2, \pi/2)$  and  $\pm(\pi/2, -\pi/2)$  as the compounds are doped away from the insulating limit. Low-energy excitations near these points may play an important role in the normal and superconducting phases.

Obviously van Hove singularities are ignored in our model. They break scale invariance in the Gaussian part of the action and therefore cannot be incorporated into the renormalization group scheme because the dispersion relation is not linear at those points. While the singularities may or may not be important at half-filling, the Umklapp terms drive various instabilities which the singularities are unlikely to prevent. In any case, here it is the problem away from half-filling that is of most concern to us. The other major limitation of our model – no small angle scattering processes – is a severe approximation to physical reality. Small angle scattering is clearly important for example in the formation of momentum-space Cooper pairs. We therefore expect, and indeed find, unphysical behavior in certain limits (see below). Nevertheless, our calculation suggests that stable fixed points describing whole Fermi surfaces, not just points, exist.

The electron operators  $c_{\mathbf{x}\alpha}$  may be written in terms of the continuum fields at the four points. We now allow  $\alpha$  to take on values 1, ..., n for the SU(n) case. We consider the general problem because it enables us to check our calculations more thoroughly for combinatorial errors. It also permits us to study the spinless case  $n = 1$ . Let  $u \equiv x + y$  and  $v \equiv x - y$  where  $x$  and  $y$  are integers labeling the coordinates of a point on a lattice with unit lattice constant and place the system in periodic box so that  $1 \leq u, v \leq L$ . Then at half-filling the lattice electron annihilation operator can be rewritten in terms of the continuum fields at the four points as:

$$c_{\mathbf{x}\alpha} = \frac{1}{2} \{ e^{i\pi u/2} \psi_{1\alpha}(u) + e^{-i\pi u/2} \psi_{-1\alpha}(u) + e^{i\pi v/2} \psi_{2\alpha}(v) + e^{-i\pi v/2} \psi_{-2\alpha}(v) \} . \quad (2.1)$$

Away from half-filling we simply replace  $\pi/2 \rightarrow k_f$  in this formula. With the replacement Eq. [2.1] we see that excitations are constrained to move in directions perpendicular to the Fermi “edges”. In other words, the Fermions cannot move in arbitrary directions, just forward and backwards along the lines depicted in Figure [3]. With this definition we can now break up any four Fermi interaction into two types of terms: marginal terms that vary smoothly in space and irrelevant terms that oscillate rapidly or contain derivatives. Keeping track of just the marginal terms a little algebra shows, for example, that at half-filling the Hubbard interaction  $\frac{U}{n} (c_{\mathbf{x}}^{\dagger\alpha} c_{\mathbf{x}\alpha} - n/2)^2$  generates the following four Fermi terms:

$$\begin{aligned}
& (\psi_{-2}^{\dagger\alpha}\psi_{-2\alpha} + \psi_{-1}^{\dagger\alpha}\psi_{-1\alpha} + \psi_1^{\dagger\alpha}\psi_{1\alpha} + \psi_2^{\dagger\alpha}\psi_{2\alpha})^2 \\
& + (\psi_2^{\dagger\alpha}\psi_{-2\alpha} + \psi_1^{\dagger\alpha}\psi_{-1\alpha} + \psi_{-1}^{\dagger\alpha}\psi_{1\alpha} + \psi_{-2}^{\dagger\alpha}\psi_{2\alpha})^2 \\
& + (\psi_{-2}^{\dagger\alpha}\psi_{-1\alpha} + \psi_{-1}^{\dagger\alpha}\psi_{-2\alpha} + \psi_2^{\dagger\alpha}\psi_{1\alpha} + \psi_1^{\dagger\alpha}\psi_{2\alpha})^2 \\
& + (\psi_1^{\dagger\alpha}\psi_{-2\alpha} + \psi_{-1}^{\dagger\alpha}\psi_{2\alpha} + \psi_2^{\dagger\alpha}\psi_{-1\alpha} + \psi_{-2}^{\dagger\alpha}\psi_{1\alpha})^2.
\end{aligned} \tag{2.2}$$

More generally, we can write down all possible four Fermi interactions consistent with the symmetries of the SU(n) spin group and the symmetry of the square Fermi surface. Thus the perturbation is:

$$\begin{aligned}
H_{int} = & \frac{1}{L} \int du dv \left\{ \lambda_{1c} (J_1 J_{-1} + J_2 J_{-2}) + \lambda_{1s} (J_{1\beta}^\alpha J_{-1\alpha}^\beta + J_{2\beta}^\alpha J_{-2\alpha}^\beta) \right. \\
& + \lambda_{2c} (J_1 + J_{-1})(J_2 + J_{-2}) + \lambda_{2s} (J_{1\beta}^\alpha + J_{-1\beta}^\alpha)(J_{2\alpha}^\beta + J_{-2\alpha}^\beta) \\
& + \lambda_3 [(\psi_1^{\dagger\alpha}\psi_{-1\alpha})^2 + (\psi_2^{\dagger\alpha}\psi_{-2\alpha})^2 + H.c.] \\
& + \lambda_4 (\psi_1^{\dagger\alpha}\psi_{-2\alpha}\psi_2^{\dagger\beta}\psi_{-1\beta} + \psi_{-2}^{\dagger\alpha}\psi_{-1\alpha}\psi_1^{\dagger\beta}\psi_{2\beta} + H.c.) \\
& + \lambda_5 (\psi_1^{\dagger\alpha}\psi_{-2\alpha}\psi_{-1}^{\dagger\beta}\psi_{2\beta} + \psi_{-2}^{\dagger\alpha}\psi_{-1\alpha}\psi_2^{\dagger\beta}\psi_{1\beta} + H.c.) \\
& + \lambda_6 (\psi_1^{\dagger\alpha}\psi_{-1\alpha}\psi_{-2}^{\dagger\beta}\psi_{2\beta} + \psi_1^{\dagger\alpha}\psi_{-1\alpha}\psi_2^{\dagger\beta}\psi_{-2\beta} + H.c.) \\
& \left. + \lambda_7 [(\psi_1^{\dagger\alpha}\psi_{2\alpha})^2 + (\psi_{-2}^{\dagger\alpha}\psi_{-1\alpha})^2 + (\psi_2^{\dagger\alpha}\psi_{-1\alpha})^2 + (\psi_{-2}^{\dagger\alpha}\psi_{1\alpha})^2 + H.c.] \right\}.
\end{aligned} \tag{2.3}$$

Here we have again introduced the charge and spin currents, now for each of the four points. Note that  $J_{\pm 1} = J_{\pm 1}(u)$ ,  $J_{\pm 2} = J_{\pm 2}(v)$ , and likewise for the spin currents. We rescale the interactions by a factor of  $\frac{1}{L}$  to keep the density of states constant. Not included in the above expression are terms like  $\frac{\pi}{2} \delta v_c (J_1)^2$  and  $\frac{\pi}{6} \delta v_s J_{1\alpha}^\beta J_{1\beta}^\alpha$  which simply renormalize the charge and spin velocities.



For the above on-site Hubbard interaction, the coupling constants take the following values:

$$\begin{aligned}
\lambda_{1c} &= \lambda_{2c} = (U/n)(2 - 2/n) \\
\lambda_{1s} &= \lambda_{2s} = -2U/n \\
\lambda_3 &= U/n \\
\lambda_4 &= 2U/n \\
\lambda_5 &= 2U/n \\
\lambda_6 &= 2U/n \\
\lambda_7 &= U/n
\end{aligned}
\tag{2.4}$$

but other bare interactions (ie. nearest-neighbor Coulomb or spin exchange) yield other values (see below).

Each of the nine coupling constants in Eq. [2.3] corresponds to a particular process drawn in Figure [4]. Unlike the one dimensional case, we see that a number of these marginal processes break spin-charge separation. The current-current terms  $\lambda_{1c}$ ,  $\lambda_{2c}$ ,  $\lambda_{1s}$ , and  $\lambda_{2s}$  respect it, and so do the Umklapp terms  $\lambda_3$  and  $\lambda_7$  (at least for the physical SU(2) problem), but the other terms ( $\lambda_5$  and two of the Umklapp terms  $\lambda_4$  and  $\lambda_6$ ) are “mixed” processes that scatter both spin and charge. Away from half-filling, the term  $\lambda_5$  survives and it is this interaction that will draw our attention in the following renormalization group analysis. As in one dimension, the model possesses global SU(2) spin symmetry. However, it exhibits the four separate U(1) symmetries only if  $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$ . In other words, only current-current type interactions preserve  $U(1)_1 \otimes U(1)_2 \otimes U(1)_{-1} \otimes U(1)_{-2}$  symmetry. This behavior clearly differs from the one dimensional model, which automatically exhibits  $U(1)_L \otimes U(1)_R$  symmetry away from half-filling. Again  $\lambda_5$  is the sole offending term away from half-filling.

It is a straightforward, though lengthy, exercise to work out the RG flows to second order in the coupling strengths by evaluating one loop diagrams with four external Fermi field lines. The diagrams are essentially no different from the one we evaluated in one spatial dimension (Figure [2]). This is because at the one loop level only diagrams that contain both a 1 propagator

(ie.  $\langle \psi_1^{\dagger\alpha}(u, t) \psi_{1\alpha}(0, 0) \rangle$ ) and a -1 propagator (or 2 and -2 propagators) yield logarithmically divergent contributions. It follows that  $\lambda_{2c}$ ,  $\lambda_{2s}$ , and  $\lambda_7$  do not flow at this order because these interactions contain Fermions at points 1 and 2 (or -1 and 2, etc.) so the requisite propagators do not appear. Inspection of the diagrams in Figure [4] reveals the physical origin for this decoupling. The three interactions  $\lambda_{2c}$ ,  $\lambda_{2s}$ , and  $\lambda_7$  differ from the other terms in that exchange of momentum between the two points is forbidden because the two directions are perpendicular. For example, consider the momentum-space version of interactions  $\lambda_{1c}$  and  $\lambda_{2c}$ . Let  $p$  denote momentum in the  $u$ -direction and  $q$  be momentum in the  $v$ -direction. Then the interactions take the form:

$$\lambda_{1c} \left\{ \int \frac{dp}{2\pi} J_1(p) J_{-1}(-p) + \int \frac{dq}{2\pi} J_2(q) J_{-2}(-q) \right\}$$

and

$$\lambda_{2c} L [J_1(p=0) + J_{-1}(p=0)][J_2(q=0) + J_{-2}(q=0)] . \quad (2.5)$$

In contrast to  $\lambda_{1c}$  (and  $\lambda_{1s}$ ), only the zero-momentum component of the currents couple in the  $\lambda_{2c}$  (and  $\lambda_{2s}$ ) terms.

The remaining six couplings flow as follows [the prime denotes a derivative with respect to  $\pi \ln(s)$  ]:

$$\begin{aligned} \lambda'_{1c} &= 8(1 - 1/n)(\lambda_3)^2 + 2(\lambda_4)^2 - (2/n)\lambda_4\lambda_6 + 2(1/n - 1)(\lambda_5)^2 + 2(\lambda_6)^2 \\ \lambda'_{1s} &= n(\lambda_{1s})^2 + 4(n - 2)(\lambda_3)^2 - 2\lambda_4\lambda_6 + 2(\lambda_5)^2 + 2n(\lambda_6)^2 \\ \lambda'_3 &= 4\lambda_{1c}\lambda_3 + 2(n - 1 - 2/n)\lambda_{1s} \lambda_3 + (\lambda_4)^2 + 2\lambda_4\lambda_6 - n(\lambda_6)^2 \\ \lambda'_4 &= [2\lambda_{1c} - (2/n)\lambda_{1s} + 4\lambda_3]\lambda_4 \\ \lambda'_5 &= [-2\lambda_{1c} + 2(1 + 1/n)\lambda_{1s}]\lambda_5 \\ \lambda'_6 &= [2\lambda_{1c} + 2(n - 1/n)\lambda_{1s} + 4(1 - n)\lambda_3]\lambda_6 + [4\lambda_3 - 2\lambda_{1s}]\lambda_4 . \end{aligned} \quad (2.6)$$

We can perform several checks on these equations. First, the equations close: we have not forgotten any marginal operators. Second, if we consider only the pair of points 1 and -1, the equations must reduce to the known ones in one spatial dimension. By setting  $\lambda_4 = \lambda_5 = \lambda_6 = 0$  one can easily check that the remaining equations ( $\lambda_3$  is now the Umklapp term mentioned in the previous section) do agree with the known results in one dimension. As another test, note that two terms vanish in

the physical case  $n = 2$ . In particular,  $\lambda_3$  no longer couples to  $\lambda_{1s}$  because the Umklapp term is a  $SU(n)$  singlet operator *only* for the special  $SU(2)$  case. Finally, what happens when  $n = 1$ , the case of spinless Fermions? Many terms vanish because they do not exist for spinless Fermions. Thus, the spin singlet Umklapp terms  $\lambda_3 = \lambda_7 = 0$  by the Pauli exclusion principle and of course the spin current terms  $\lambda_{1s} = \lambda_{2s} = 0$  because there is no spin. Also,  $\lambda_6 = -\lambda_4$  because  $\lambda_4$  processes can no longer be distinguished from  $\lambda_6$  ones and  $\lambda_5 = 0$  due to internal cancellations present in Eq. [2.3] when the spin label is removed. Equations Eq. [2.6] respect this limit.

The RG flows described by Eq. [2.6] generically flow to large values. The flows are physically sensible: at half-filling Umklapp processes generate various instabilities and the system becomes gapped in the charge sector when the interactions are repulsive. Attractive interactions, on the other hand can lead to superconducting instabilities. The restricted phase space of our model obscures the interpretation of these instabilities. For example, the Goldstone mechanism tells us that phases of broken  $SU(n)$  symmetry exhibit gapless spin excitations. On the other hand, the Higgs mechanism suppresses gapless excitations in the charge sector if the  $U(1)$  symmetry breaks. But our model retains the character of 1+1 dimensional phase space which is not large enough to foster broken continuous symmetries. In any case, our failure to treat the van Hove singularities and small angle scattering processes is not as important as it might first seem: these processes are unlikely to inhibit the formation of instabilities.

The spinless case  $n = 1$  is an exception. As noted above, we can take  $\lambda_{1s} = \lambda_3 = \lambda_5 = 0$  and  $\lambda_4 = -\lambda_6$  in this case. The flows are described by the Kosterlitz - Thouless equations:

$$\begin{aligned}\lambda'_{1c} &= 6(\lambda_6)^2 \\ \lambda'_6 &= 2\lambda_{1c}\lambda_6 .\end{aligned}\tag{2.7}$$

Here the fixed line  $\lambda_6 = 0$  is stable for  $\lambda_{1c} \leq 0$ . We may interpret the instability at positive  $\lambda_{1c}$  as a tendency to form a charge-density wave. To see this, note that the next-nearest-neighbor Hubbard repulsion  $\frac{U_1}{n}(n_{\mathbf{x}+\hat{x}} + n_{\mathbf{x}+\hat{y}}) n_{\mathbf{x}}$ , where  $n_{\mathbf{x}} \equiv c_{\mathbf{x}\alpha}^\dagger c_{\mathbf{x}\alpha}$  is the electron occupancy at site  $\mathbf{x}$ , leads to the

following bare continuum couplings at half-filling:

$$\begin{aligned}
\lambda_{1c} &= \lambda_{2c} = (U_1/n)(2 + 2/n) \\
\lambda_{1s} &= \lambda_{2s} = 2U_1/n \\
\lambda_3 &= -U_1/n \\
\lambda_4 &= 0 \\
\lambda_5 &= 0 \\
\lambda_6 &= -2U_1/n \\
\lambda_7 &= 0 .
\end{aligned} \tag{2.8}$$

(For the case  $n = 1$ ,  $\lambda_3$  should be set equal to zero by the Pauli exclusion principle.) So repulsive nearest-neighbor interactions grow, a tendency towards the formation of a charge-density-wave sets in and sites on the even sublattice exhibit different charge density than those on the odd sublattice. This behavior is consistent with that found by Shankar<sup>1</sup> in his functional RG calculation for spinless Fermions and therefore lends credibility to our model. On the other hand, Shankar finds a superconducting instability for attractive interactions which contrasts with the stability shown by our model. We reconcile this difference by noting that the Cooper instability is driven by small angle scattering processes that scatter pairs of Fermions of opposite momentum around the Fermi surface. Again the phase space for such processes in our model is severely restricted by existence of only four Fermi points, and we should therefore not expect momentum-space Cooper pairs. Real-space Cooper pairs can arise, however, as we show below. Nevertheless, negative  $\lambda_{1c}$  corresponds to an attractive interaction and is indicative of a tendency towards the formation of superconducting pairs.

The problem of spinning Fermions away from half-filling is rather more interesting. Setting the Umklapp terms  $\lambda_3 = \lambda_4 = \lambda_6 = \lambda_7 = 0$  we obtain the reduced set of flow equations:

$$\begin{aligned}
\lambda'_{1c} &= 2(1/n - 1)(\lambda_5)^2 \\
\lambda'_{1s} &= n(\lambda_{1s})^2 + 2(\lambda_5)^2 \\
\lambda'_5 &= 2[(1 + 1/n)\lambda_{1s} - \lambda_{1c}] \lambda_5 .
\end{aligned} \tag{2.9}$$

Here we find a stable fixed line defined by  $\lambda_{1s} = \lambda_5 = 0$  and  $\lambda_{1c} \geq 0$ . In fact numerical integration shows that it attracts flows starting from the repulsive Hubbard couplings given by Eq. [2.4] (see Figure [5]). Again,  $\lambda_{2c}$  and  $\lambda_{2s}$  do not renormalize at second order. Like  $\lambda_{1c}$ , these couplings can be non-zero along the fixed line. The instability at negative  $\lambda_{1c}$  can be interpreted as a tendency to form real-space Cooper pairs something like those proposed in the original resonating-valence-bond (RVB) theory of Anderson<sup>16</sup> and collaborators. Thus the Fermion spin permits the formation of singlet pairs.

What is the nature of the stable region of the fixed line? First note that it exhibits both spin-charge separation and four-fold U(1) symmetry because only current-current type interactions remain. The fixed line thus represents a natural generalization of the one-dimensional Luttinger liquid and as such motivates the approach to the continuum Fermi surface problem we describe in the next section. Let us first look more closely at our solution by transforming to Boson variables. At first it seems strange to contemplate Bosonization in spacetime dimensions greater than two. But the problem remains essentially one-dimensional; the Fermions at each of the four points are restricted to move along lines. In fact it is convenient to introduce complex space-time coordinates analogous to those in 1 + 1 dimensions. Let  $u_{\pm} \equiv u \pm it$  and  $v_{\pm} \equiv v \pm it$  where we remember that the velocities, here set equal to one, are in general different in the spin and charge sectors of the theory. Obviously the group of conformal transformations in 2 + 1 dimensions is finite. Consequently, the model does not possess the infinite symmetries of a true 1 + 1 dimensional conformal field theory. But it is the current algebra that concerns us most here. It is the essential ingredient that permits us to map Fermions onto Bosons and vice versa at each of the Fermi points. Again either Abelian or non-Abelian Bosonization works. In this case we choose non-Abelian Bosonization<sup>17</sup> for the spin sector by introducing the Wess-Zumino-Witten (WZW) field  $g_{\beta}^{\alpha}$  (the charge sector is still described by an Abelian Boson  $\phi$ ). Non-Abelian Bosonization is superior in the sense that it explicitly exhibits global SU(n) invariance. With it we can readily classify all SU(n) invariant operators. (Abelian Bosonization hides SU(n) invariance because an explicit choice for the spin

quantization direction must be made.)

The Bosonization dictionary again translates currents defined in terms of the Fermi fields into Bosonic operators. For example at point 1 we have:

$$\begin{aligned} J_1(u) &= \sqrt{\frac{2}{\pi}} \partial_u \phi_1(u) \\ J_{1\alpha}^\beta(u) &= \frac{i}{4\pi} [\partial_u g_{1\gamma}^\beta(u)] [g_{1\alpha}^\gamma(u)]. \end{aligned} \quad (2.10)$$

Here the free fixed point theory with all the  $\lambda$ 's equal to zero has its charge sector described by a free Lagrangian density:

$$L_c = \frac{1}{2L} \{(\partial_{u_+} \phi_1)^2 + (\partial_{u_-} \phi_{-1})^2 + (\partial_{v_+} \phi_2)^2 + (\partial_{v_-} \phi_{-2})^2\}. \quad (2.11a)$$

The spin sector consists of a  $k = 1$  WZW action given by  $S_s[g_1, g_2, g_{-1}, g_{-2}] \equiv S_1[g_1 + g_{-1}] + S_2[g_2 + g_{-2}]$  where:

$$\begin{aligned} S_1[g] &= \frac{1}{8\pi} \int_{\partial V} du dt Tr \{ \partial_u g \partial_u g^\dagger + \partial_t g \partial_t g^\dagger \} \\ &+ \frac{1}{12\pi} \int_V du dt dz \epsilon^{\mu\nu\lambda} Tr (g^\dagger \partial_\mu g g^\dagger \partial_\nu g g^\dagger \partial_\lambda g). \end{aligned} \quad (2.11b)$$

Here the second integral, the topological Wess-Zumino term, is defined by extending the domain of the  $g$ -field from physical two-dimensional  $(u, t)$  space-time to a three dimensional volume  $V$  with space-time coordinates  $x_\mu = (u, t, z)$ . The boundary  $\partial V$  of  $V$  is taken to be the  $(u, t)$  space-time. Of course  $S_2[g]$  is similar in form to  $S_1[g]$  but the spatial variable  $v$  replaces  $u$ . The spin sector of the free theory displays  $SU(n)_1 \otimes SU(n)_2 \otimes SU(n)_{-1} \otimes SU(n)_{-2}$  invariance because the spin currents at the four points are decoupled.

Now the residual fixed line interactions  $\lambda_{1c}$ ,  $\lambda_{2c}$  and  $\lambda_{2s}$  can be included by using the Bosonization rules of Eq. [2.10]. In the Boson language, the 4-fold  $U(1)$  symmetry operation amounts to a shift in each of the charged Boson fields by a constant:  $\phi_1(u, t) \rightarrow \phi_1(u, t) + \Gamma_1$ , etc. Since only derivatives of the Boson field appear in the action, it continues to manifest four-fold  $U(1)$  invariance as expected. On the other hand, local  $SU(n)_1 \otimes SU(n)_2 \otimes SU(n)_{-1} \otimes SU(n)_{-2}$  invariance is at least partly broken by non-zero  $\lambda_{2s}$  which couples together the zero-momentum components of the spin currents at the four points.

Two issues remain to be investigated in our model: First, how are Fermi statistics maintained in the Bosonization scheme, now that there are two spatial directions? And second, how do the residual fixed point interactions change the character of the Fermi points? We answer these questions by constructing a more general framework in the next section.

### III. LUTHER-HALDANE BOSONIZATION: INFINITE U(1) SYMMETRY

Encouraged by the renormalization group flows in our model, we now take a leap of faith, advocated most recently by Haldane<sup>3</sup>, and postulate the existence of a similar fixed point, not just for the four Fermi points, but rather for a continuum of Fermi points, in other words, a Fermi surface. We outline the construction of the currents and the Hamiltonian first and later [in sections (IV) and (V)] demonstrate that the framework reproduces well known results.

To be definite, we study the case of three spatial dimensions; generalizations to other dimensions are straightforward. We begin with the charge sector and study a smooth Fermi surface parameterized by radial vectors  $\mathbf{S}$  and  $\mathbf{T}$  that label a fine, locally flat (and rectangular) mesh of grid points on the Fermi surface with spacing  $\Lambda \ll k_f$  between the points. We also place the system in a cubic box with sides of length  $L$  and use periodic boundary conditions so that the momenta are quantized as  $\mathbf{p}_m = \frac{2\pi}{L} \mathbf{m}$  where  $\mathbf{m}$  is a vector with integer components. The most general charge Hamiltonian possessing infinite U(1) symmetry may then be written as:

$$H_c = \frac{1}{2} \sum_{\mathbf{S}, \mathbf{T}} \sum_{\mathbf{q}} V_c(\mathbf{S}, \mathbf{T}; \mathbf{q}) J(\mathbf{S}; \mathbf{q}) J(\mathbf{T}; -\mathbf{q}). \quad (3.1)$$

The prefactor of  $\frac{1}{2}$  compensates for over-counting due to the symmetry of the summand under  $\mathbf{q} \rightarrow -\mathbf{q}$  with  $\mathbf{S} \leftrightarrow \mathbf{T}$ . The function  $V_c(\mathbf{S}, \mathbf{T}; \mathbf{q})$  encapsulates not only the Fermi velocity  $v_f$  of the non-interacting system but also the residual Fermi-liquid type interactions  $V_c'$  between quasiparticles. The current at each point  $J(\mathbf{S}; \mathbf{q})$  is now defined in momentum space as:

$$J(\mathbf{S}; \mathbf{q}) \equiv \sum_{\mathbf{k}} \theta(\mathbf{S}; \mathbf{k} + \mathbf{q}) \theta(\mathbf{S}; \mathbf{k}) \{ \psi_{\mathbf{k}+\mathbf{q}}^{\dagger\alpha} \psi_{\alpha\mathbf{k}} - \delta_{\mathbf{q},\mathbf{0}}^3 n_{\mathbf{k}} \}. \quad (3.2)$$

The subtraction of the vacuum charge expectation value  $n_{\mathbf{k}} \equiv \langle \psi_{\mathbf{k}}^{\dagger\beta} \psi_{\beta\mathbf{k}} \rangle$  in Eq. [3.2] amounts to normal ordering. Our geometric construction of the currents involves tiling the Fermi surface with squat rectangular pill boxes at each grid point  $\mathbf{S}$ . The boxes have dimensions  $\Lambda \times \Lambda$  along the surface and extend in height  $\pm\lambda/2$  above and below the Fermi surface (see Figure [6]). The function  $\theta(\mathbf{S}; \mathbf{k}) = 1$  if  $\mathbf{k}$  lies inside the box; otherwise it is zero. Thus  $v_f\lambda$  functions as an ultraviolet energy cutoff.



The current commutation relations may now be found by direct computation with the use of the canonical anticommutation relation  $\{ \psi_{\mathbf{q}}^{\dagger\alpha} , \psi_{\beta\mathbf{p}} \} = \delta_{\beta}^{\alpha} \delta_{\mathbf{q},\mathbf{p}}^3$ . First note that currents in different patches commute because the Fermion operators that make up the currents are also located in different patches. So:

$$\begin{aligned}
[J(\mathbf{S}; \mathbf{q}) , J(\mathbf{T}; \mathbf{p})] &= \delta_{\mathbf{S},\mathbf{T}}^2 \left\{ \sum_{\mathbf{k}} \theta(\mathbf{S}; \mathbf{k} + \mathbf{q} + \mathbf{p}) \theta(\mathbf{S}; \mathbf{k}) [\theta(\mathbf{S}; \mathbf{k} + \mathbf{q}) - \theta(\mathbf{S}; \mathbf{k} + \mathbf{p})] \delta_{\mathbf{q}+\mathbf{p},\mathbf{0}}^3 n_{\mathbf{k}} \right. \\
&\quad + \sum_{\mathbf{k}} \theta(\mathbf{S}; \mathbf{k} + \mathbf{q} + \mathbf{p}) \theta(\mathbf{S}; \mathbf{k}) [\theta(\mathbf{S}; \mathbf{k} + \mathbf{q}) - \theta(\mathbf{S}; \mathbf{k} + \mathbf{p})] \\
&\quad \left. \times (\psi_{\mathbf{k}+\mathbf{q}+\mathbf{p}}^{\dagger\alpha} \psi_{\mathbf{k}\alpha} - \delta_{\mathbf{q}+\mathbf{p},\mathbf{0}}^3 n_{\mathbf{k}}) \right\} .
\end{aligned} \tag{3.3}$$

In one dimension, the index  $\mathbf{S}$  just labels the left and right Fermi points and the first sum in Eq. [3.3] is the usual quantum anomaly. The second sum vanishes in the limit of infinite bandwidth ( $\lambda \rightarrow \infty$ ) because in that case  $\theta(q) = 1$  except very deep inside or outside the Fermi sea. In this limit, matrix elements of the operator  $(\psi_{\mathbf{k}+\mathbf{q}+\mathbf{p}}^{\dagger\alpha} \psi_{\mathbf{k}\alpha} - \delta_{\mathbf{q}+\mathbf{p},\mathbf{0}}^3 n_{\mathbf{k}})$  vanish and we recover the usual one-dimensional Kac-Moody algebra. Thus, for the right movers,

$$[J(R; q) , J(R; p)] = 2 \frac{qL}{2\pi} \delta_{q+p,0} \tag{3.4}$$

where the prefactor of 2 comes from the two spins. We recognize this algebra as the momentum space version of Eq. [1.7].

One might expect that the natural generalization of the currents to two or three spatial dimensions would take the fields to be organized along narrow rays of vanishing thickness radiating outward from the center of the Fermi sea. In fact this approach was adopted by Luther in his pioneering work on the Bosonization of free Fermions in higher dimension<sup>18</sup> since it reduces the higher dimensional problem to a set of simple decoupled 1 + 1 dimensional systems. However it is clear that the procedure breaks down when interactions of the Fermi liquid type are included. The charge Hamiltonian, Eq. [3.1], couples charge currents in different boxes at positions  $\mathbf{S}$  and  $\mathbf{T}$ . As the Fermi surface must have non-zero curvature, any wavevector  $\mathbf{q}$  that lies inside a tube at position  $\mathbf{S}$ , no matter how small, will be accompanied by a wavevector  $-\mathbf{q}$  that in general does *not* fit inside the tube at a different point  $\mathbf{T}$ .

The problem is avoided with the use of the squat boxes. The price we pay for this new geometrical construction is the introduction of several limits which must be carefully taken in order to arrive at the correct commutation relations. This delicate series of limits in fact correspond to the Fermi liquid theory limits of  $\omega \rightarrow 0$  and  $|\mathbf{q}| \rightarrow 0$  such that  $\frac{v_f|\mathbf{q}|}{\omega} \rightarrow 0$ , the so-called  $\omega$ -limit which pertains to collective modes rather than quasi-particle scattering<sup>19</sup>. First we require the wavevectors  $\mathbf{q}$  and  $\mathbf{p}$  in Eq. [3.3] to be small:  $|\mathbf{q}| < \frac{\Lambda}{N}$  and  $|\mathbf{p}| < \frac{\Lambda}{N}$  where we take  $N \rightarrow \infty$ . Thus we may think of  $\mathbf{q}$  as lying within a small sphere inside the squat box – see Figure [6]. The limit insures that only the component of  $\mathbf{q}$  normal to the surface appears in the quantum anomaly. As we shall see, it is the normal component that is needed to reproduce the spectrum of low-lying excitations in the free Fermion problem. This geometrical result may be obtained by using the fact that  $n_{\mathbf{k}} = 2$  for momenta  $\mathbf{k}$  lying deep inside the Fermi sea and zero far outside. With the limits  $|\mathbf{q}| < \frac{\Lambda}{N}$  and  $|\mathbf{p}| < \frac{\Lambda}{N}$  we have  $\theta(\mathbf{S}; \mathbf{k} + \mathbf{q} + \mathbf{p}) \approx \theta(\mathbf{S}; \mathbf{k})$ . The sum in the first term of Eq. [3.3] can be done and we have:

$$[J(\mathbf{S}; \mathbf{q}) , J(\mathbf{T}; \mathbf{p})] = 2 \delta_{\mathbf{S},\mathbf{T}} [ \delta_{\mathbf{q}+\mathbf{p},\mathbf{0}}^3 \Lambda^2 \left(\frac{L}{2\pi}\right)^3 \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} + \text{error term} ]. \quad (3.5a)$$

Here  $\hat{\mathbf{n}}_{\mathbf{S}}$  is the normal vector pointing outward at point  $\mathbf{S}$  on the Fermi surface and the error term is the second sum in Eq. [3.3] which ruins the Kac-Moody algebra because it is not a c-number but rather an operator involving the Fermi fields  $\psi^\dagger$  and  $\psi$ . Note that with the above limit on the size of  $\mathbf{q}$  the magnitude of the quantum anomaly is of order  $\frac{\Lambda^3}{N} \left(\frac{L}{2\pi}\right)^3$ .

Let us estimate now the size of the error term in the commutation relations. It may be estimated by replacing the operator  $(\psi_{\mathbf{k}+\mathbf{q}+\mathbf{p}}^{\dagger\alpha} \psi_{\mathbf{k}\alpha} - \delta_{\mathbf{q}+\mathbf{p},\mathbf{0}}^3 n_{\mathbf{k}})$  by  $1 - \delta_{\mathbf{q}+\mathbf{p},\mathbf{0}}^3$  and computing the volume of the geometrical complement of the intersection of two  $\theta$  functions  $[\theta(\mathbf{S}; \mathbf{k} + \mathbf{q}) - \theta(\mathbf{S}; \mathbf{k} + \mathbf{p})]$  appearing in the second sum. Note that the tops and bottoms of the boxes do not contribute because the matrix elements are assumed to be zero deep inside or outside the Fermi sea. Only the sides of the pill boxes matter. A simple computation then shows that this term is off order  $\text{Max}\{|\mathbf{q}|, |\mathbf{p}|\} \times \Lambda \lambda \left(\frac{L}{2\pi}\right)^3 < \frac{\Lambda^2 \lambda}{N} \left(\frac{L}{2\pi}\right)^3$ . Therefore, the choice of a squat pill box with  $\Lambda = \sqrt{N} \lambda$  makes the error term small (of order  $\frac{1}{\sqrt{N}}$ ) in comparison to the quantum anomaly. This second limit is equivalent to the “ $\omega$ -limit”

of Fermi liquid theory as  $\frac{\omega}{v_f}$  corresponds to  $\lambda$  which is now of order  $\sqrt{N}$  times larger than the momentum  $\mathbf{q}$ . It is satisfying to have this simple geometrical interpretation of the  $\omega$ -limit of Fermi liquid theory.

The current algebra Eq. [3.5a] can be put into a more familiar form with a Fourier transform over the two components of the momentum perpendicular to the Fermi surface normal vector,  $\mathbf{q}_\perp$  and  $\mathbf{p}_\perp$ . Then we obtain:

$$[J(\mathbf{S}; q_\parallel, \mathbf{x}_\perp), J(\mathbf{T}; p_\parallel, \mathbf{y}_\perp)] = 2 \delta_{\mathbf{S}, \mathbf{T}} \frac{q_\parallel L}{2\pi} \delta_{q_\parallel + p_\parallel, 0} L^2 \delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp). \quad (3.5b)$$

Here the current algebra is identical to the usual one-dimensional one, Eq. [3.4], except with additional labels  $\mathbf{S}$ ,  $\mathbf{T}$ ,  $\mathbf{x}_\parallel$  and  $\mathbf{y}_\parallel$  that “come along for the ride.” Thus the well-developed theory of one-dimensional Kac-Moody algebra representations<sup>20</sup> applies equally well to our generalized algebra and we can use this machinery to find the spectrum of states. Of course state counting is simple in the abelian case but null states appear in representations of the non-abelian Kac-Moody algebra.

What choice of parameters  $V_c$  yield the correct spectrum for the charge sector? The non-interacting limit is recovered by making the following choice:  $V_c(\mathbf{S}, \mathbf{T}; \mathbf{q}) = \frac{1}{2} v_f(\mathbf{S}) \Omega^{-1} \delta_{\mathbf{S}, \mathbf{T}}^2$ . Here the factor of  $\Omega \equiv \Lambda^2 (\frac{L}{2\pi})^3$  cancels the factors of volume appearing in the current algebra Eq. [3.5] and the factor of  $\frac{1}{2}$  compensates for the 2 due to up and down spins. With the algebra Eq. [3.5] we then recover the free dispersion relation  $\omega(\mathbf{S}; \mathbf{q}) = v_f(\mathbf{S}) \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} \equiv \mathbf{v}_{\mathbf{S}} \cdot \mathbf{q}$ . To scale the interaction coefficients properly, we appeal to Fermi liquid theory and note that the current evaluated at zero momentum is equal to the occupancy fluctuation operator summed over the interior of the pill box:

$$J(\mathbf{S}; \mathbf{q} = \mathbf{0}) = \sum_{\mathbf{k}} \theta(\mathbf{S}; \mathbf{k}) \delta n_{\mathbf{k}} \quad (3.6)$$

where  $\delta n_{\mathbf{k}} \equiv \psi_{\mathbf{k}}^{\dagger\alpha} \psi_{\alpha\mathbf{k}} - \langle \psi_{\mathbf{k}}^{\dagger\alpha} \psi_{\alpha\mathbf{k}} \rangle$ . Therefore, the Fermi liquid interaction is identical to the zero-momentum piece of our interaction term:

$$\frac{1}{2L^3} \sum_{\mathbf{k}, \mathbf{p}} f(\mathbf{k}, \mathbf{p}) \delta n_{\mathbf{k}} \delta n_{\mathbf{p}} = \frac{1}{2L^3} \sum_{\mathbf{S}, \mathbf{T}} V_c'(\mathbf{S}, \mathbf{T}; \mathbf{q} = \mathbf{0}) J(\mathbf{S}; \mathbf{0}) J(\mathbf{T}; \mathbf{0}) \quad (3.7)$$

if we identify  $f(\mathbf{k}_\mathbf{S}, \mathbf{p}_\mathbf{T}) = V'_c(\mathbf{S}, \mathbf{T}; \mathbf{0})$  by assuming that the Fermi liquid interactions depend only on the momenta  $\mathbf{k}_\mathbf{S}$  and  $\mathbf{p}_\mathbf{T}$  at points  $\mathbf{S}$  and  $\mathbf{T}$  of the Fermi surface, not the component of the momentum perpendicular to the surface. Evidently, a factor of inverse volume ( $\frac{1}{L^3}$ ) should be included in the interaction term,

$$V_c(\mathbf{S}, \mathbf{T}; \mathbf{q}) = \frac{1}{2} v_f(\mathbf{S}) \Omega^{-1} \delta_{\mathbf{S}, \mathbf{T}}^2 + \frac{1}{L^3} V'_c(\mathbf{S}, \mathbf{T}; \mathbf{q}) . \quad (3.8)$$

Note that while our theory contains the same Fermi surface interactions as traditional Landau Fermi liquids, the form of the interaction is more general than Fermi liquid theory as it depends on  $\mathbf{q}$ , the momentum of the collective excitation. In Fermi liquid theory, the parameters  $f(\mathbf{k}, \mathbf{p})$  do not depend on  $\mathbf{q}$  and the momenta  $\mathbf{k}$  and  $\mathbf{p}$  appearing in  $f(\mathbf{k}, \mathbf{p})$  are constrained to lie on the Fermi surface. A *different* extension of Fermi liquid theory which relaxes this constraint on  $\mathbf{k}$  and  $\mathbf{p}$  is described in the next section. Our calculation of non-analytic contributions to the specific heat will highlight the difference between these two types of generalizations.

We see therefore that in higher dimension, as in one dimension, the Bosonization procedure puts the free and interacting components of the Hamiltonian on an equal footing, despite the fact that the free piece is quadratic in the Fermion operators while the interaction is quartic. This simplicity is a result of the current algebra Eq. [3.5] which permits us to express both terms as bilinears in the currents. It is however somewhat deceptive because more general quartic terms, for instance the  $\lambda_5$  interaction in our simplified model of the preceding section, cannot be expressed as bilinears in the current operators. Nevertheless, these interactions have a Bosonic representation, albeit a more complicated one. We show how to Bosonize general interactions below.

First we focus on the spin sector. The total Hamiltonian is given by the sum of the charge and spin Hamiltonians. To form the spin Hamiltonian, we define spin currents. In the general SU(n) case we have:

$$J_\beta^\alpha(\mathbf{S}; \mathbf{q}) \equiv \sum_{\mathbf{k}} \theta(\mathbf{S}; \mathbf{k} + \mathbf{q}) \theta(\mathbf{S}; \mathbf{k}) \{ \psi_{\mathbf{k}+\mathbf{q}}^{\dagger\alpha} \psi_{\beta\mathbf{k}} - \frac{1}{n} \delta_\beta^\alpha \psi_{\mathbf{k}+\mathbf{q}}^{\dagger\gamma} \psi_{\gamma\mathbf{k}} \} . \quad (3.9)$$

Like the charge currents, spin currents at different grid points commute, but the non-Abelian Kac-Moody algebra governs currents at the same point in the  $N \rightarrow \infty$  limit:

$$[J_\beta^\alpha(\mathbf{S}; \mathbf{q}), J_\delta^\gamma(\mathbf{S}; \mathbf{p})] = (\delta_\delta^\alpha \delta_\beta^\gamma - \frac{1}{n} \delta_\beta^\alpha \delta_\delta^\gamma) \delta_{\mathbf{q}+\mathbf{p}, \mathbf{0}}^3 \Omega \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} + \delta_\beta^\gamma J_\delta^\alpha(\mathbf{S}; \mathbf{q} + \mathbf{p}) - \delta_\delta^\alpha J_\beta^\gamma(\mathbf{S}; \mathbf{q} + \mathbf{p}). \quad (3.10)$$

The physical SU(2) Kac-Moody algebra can be expressed more succinctly as:

$$[J^a(\mathbf{S}; \mathbf{q}), J^b(\mathbf{S}; \mathbf{p})] = \frac{1}{2} \delta^{ab} \Omega \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} + i\epsilon^{abc} J^c(\mathbf{S}; \mathbf{q} + \mathbf{p}). \quad (3.11)$$

The spin Hamiltonian may then be written:

$$H_s = \frac{1}{2} \sum_{\mathbf{S}, \mathbf{T}} \sum_{\mathbf{q}} V_s(\mathbf{S}, \mathbf{T}; \mathbf{q}) \mathbf{J}(\mathbf{S}; \mathbf{q}) \cdot \mathbf{J}(\mathbf{T}; -\mathbf{q}) \quad (3.12)$$

where  $V_s$  incorporates the Fermi velocity of spin excitations and spin-spin interactions at different points on the Fermi surface:

$$V_s(\mathbf{S}, \mathbf{T}; \mathbf{q}) = \frac{2}{3} v_f(\mathbf{S}) \Omega^{-1} \delta_{\mathbf{S}, \mathbf{T}}^2 + \frac{1}{L^3} V'_s(\mathbf{S}, \mathbf{T}; \mathbf{q}). \quad (3.13)$$

In general it is not possible to exactly diagonalize the Hamiltonian; the non-Abelian nature of the algebra precludes this. We encountered this problem in a simpler form in section (II) where  $\lambda_{1s}$ , the parameter that couples together spin currents on opposing Fermi points, flows by itself (see Eq. [2.9]).

Both the charge and spin currents are invariant under the local U(1) operation which changes the phase of all the Fermions inside a given pill box by the same (time-independent) amount  $\Gamma$ . If  $\mathbf{k}$  lies inside the box centered at grid point  $\mathbf{S}$  then

$$\psi_\alpha(\mathbf{k}) \rightarrow e^{i\Gamma(\mathbf{S})} \psi_\alpha(\mathbf{k}) \text{ [only when } \theta(\mathbf{S}; \mathbf{k}) = 1] \quad (3.14)$$

leaves the currents invariant because the  $\psi^\dagger$  fields, which transform with the opposite phase factor, cancel the overall phase change. Thus the Hamiltonian is automatically invariant under the infinite U(1) symmetry. The physical meaning of the invariance is clear: the Fermi liquid type interactions preserve the Fermion occupancy at each point in momentum space because quasi-particle scattering

is suppressed in the  $N \rightarrow \infty$   $\omega$ -limit. The U(1) symmetry just reflects the local conservation of particle number.

Indeed, it is the existence of an infinite number of conservation laws that makes the charge sector of the problem solvable. On the other hand, the free Fermion system also exhibits local SU(2) symmetry. So it is rather surprising to discover that the spin current interactions in general break the infinite local SU(2) invariances down to a single global SU(2) symmetry. The local invariance is broken because spin currents at different points on the Fermi surface must rotate together to keep the spin Hamiltonian Eq. [3.12] invariant. The special case of purely local current-current coupling,  $V_s(\mathbf{S}, \mathbf{T}; \mathbf{q}) = \frac{2}{3} v_f(\mathbf{S}) \Omega^{-1} \delta_{\mathbf{S}, \mathbf{T}}^2$ , is an exception which restores the full local SU(2) invariance. As expected, the Hamiltonian is now exactly solvable: the Hamiltonian describes free spin excitations propagating at the Fermi velocity. For this special case only the quantum anomaly in Eq. [3.11],  $\frac{1}{2} \delta^{ab} \Omega \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}}$ , not the  $i\epsilon^{abc} J^c(\mathbf{S}; \mathbf{q} + \mathbf{p})$  term, is important because of the symmetry  $\mathbf{q} \rightarrow -\mathbf{q}$ . The factor  $\frac{2}{3}$  compensates for the three spin components. The spectrum of states may now be found either by simply choosing a spin-quantization axis, or in an SU(2) invariant manner with the use of Kac-Moody representation theory.<sup>20</sup>

Actually, we can find the excitation spectrum when the interactions described by  $V'_s$  are non-singular. In this case, we may treat the spin currents as semi-classical objects: the right-hand side of the commutator Eq. [3.11] can be set to zero by rescaling the currents to be of order one. The problem resembles the large-spin limit of a quantum magnet since the currents incorporate a sum over  $\lambda\Omega \gg 1$  points in momentum space. If we rescale  $J^a(\mathbf{S}; \mathbf{q}) \rightarrow (\lambda\Omega)^{-1} J^a(\mathbf{S}; \mathbf{q})$  then the rescaled currents obey:

$$\begin{aligned}
 [J^a(\mathbf{S}; \mathbf{q}), J^b(\mathbf{S}; \mathbf{p})] &= \frac{1}{(\lambda\Omega)^2} \left\{ \frac{1}{2} \delta^{ab} \Omega \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} + i\epsilon^{abc} J^c(\mathbf{S}; \mathbf{q} + \mathbf{p}) \right\} \\
 &\rightarrow 0
 \end{aligned}
 \tag{3.15}$$

as  $L \rightarrow \infty$  with  $\Lambda$  held fixed. The emergence of the classical limit should not be surprising; after all, Landau Fermi Liquid Theory is essentially classical in nature. The free dispersion is still determined by the quantum anomaly; only the interactions are treated classically by replacing the

current operators with their expectation values  $J^a(\mathbf{S}; \mathbf{q}) \rightarrow \langle J^a(\mathbf{S}; \mathbf{q}) \rangle$  evaluated in the excited state of interest. This procedure trivially reproduces the excitation spectrum of Fermi Liquid Theory. Note, however, that the classical limit breaks down in the case of singular interactions. For example, the first term in Eq. [3.13],  $\frac{2}{3} v_f(\mathbf{S}) \Omega^{-1} \delta_{\mathbf{S}, \mathbf{T}}^2$ , is singular because the factor  $\Omega^{-1}$  diverges as the number of mesh points increases. In this case the quantum anomaly cannot be neglected and in fact is needed to reproduce the free dispersion relation. Likewise, any singular spin current interactions that couple different patches on the Fermi surface destroy the classical limit: the small anomaly cannot be neglected because of the large interaction. In fact this is the generic situation in one spatial dimension, where interactions that couple the left and right points are generally of the same order as current-current terms that involve only one point. In other words, there is no sense in which the Fermi-Liquid type interactions can be smooth when there are just two Fermi points. We return to this point in the discussion of section (VI).

By introducing Boson fields conjugate to the currents, the Fermi fields and interaction terms can be Bosonized<sup>18,3</sup>. We proceed by analogy to our construction in one-dimension [section (I)] and concentrate on spinless Fermions; it is straightforward to include spin via either Abelian or non-Abelian Bosonization. We introduce the coarse-grained Boson field  $\phi(\mathbf{S}; \mathbf{x})$  and the associated Boson current in the direction normal to the Fermi surface:

$$J(\mathbf{S}; \mathbf{x}) = \sqrt{4\pi} \hat{\mathbf{n}}_{\mathbf{S}} \cdot \nabla \phi(\mathbf{S}; \mathbf{x}) . \quad (3.16)$$

The Boson field is related to the microscopic fields  $\phi(\mathbf{p})$  by coarse graining over the pill box:

$$\phi(\mathbf{S}; \mathbf{x}) = \frac{\sqrt{\Omega}}{2\pi} \sum_{\mathbf{p}, \mathbf{p} \cdot \hat{\mathbf{n}}_{\mathbf{S}} > 0} \frac{\theta(\mathbf{S}; \mathbf{p})}{2\sqrt{|\mathbf{p} \cdot \hat{\mathbf{n}}_{\mathbf{S}}|}} \{e^{i\mathbf{p} \cdot \mathbf{x}} \phi(\mathbf{p}) + e^{-i\mathbf{p} \cdot \mathbf{x}} \phi(-\mathbf{p})\} . \quad (3.17)$$

The microscopic Boson fields satisfy equal-time commutation relations:

$$[\phi(\mathbf{p}) , \phi(\mathbf{q})] = \delta_{\mathbf{p}+\mathbf{q}, \mathbf{0}}^3 . \quad (3.18a)$$

Note that the reality of the microscopic fields  $\phi(\mathbf{x})$  means that  $\phi(-\mathbf{k}) = \phi^\dagger(\mathbf{k})$  and with this in mind the commutation relations Eq. [3.18a] take on the more familiar form:

$$[\phi(\mathbf{p}) , \phi^\dagger(\mathbf{q})] = \delta_{\mathbf{p}, \mathbf{q}}^3 . \quad (3.18b)$$

Consequently, the coarse-grained fields obey a natural three-dimensional generalization of the one-dimensional equal-time commutation relations Eq. [1.8]:

$$[\phi(\mathbf{S}; \mathbf{x}) , \phi(\mathbf{T}; \mathbf{y})] = i \frac{\Omega}{4} \frac{L^3}{2\pi} \delta_{\mathbf{S},\mathbf{T}}^2 \epsilon(\hat{\mathbf{n}}_{\mathbf{S}} \cdot [\mathbf{x} - \mathbf{y}]) \delta^2(\mathbf{x}_{\perp} - \mathbf{y}_{\perp}) \quad (3.19a)$$

where again  $\epsilon(x) = 1$  for  $x > 0$ ; otherwise it equals  $-1$ . Here  $\mathbf{x}_{\perp}$  denotes the two components of  $\mathbf{x}$  that are perpendicular to the surface normal  $\hat{\mathbf{n}}_{\mathbf{S}}$ . Note that  $\delta^2(0) = (\frac{\Lambda}{2\pi})^2$  which is the area of the base of the pill box. Thus, when  $\mathbf{x}_{\perp} = \mathbf{y}_{\perp}$  we have:

$$[\phi(\mathbf{S}; \mathbf{x}) , \phi(\mathbf{T}; \mathbf{y})] = i \frac{\Omega^2}{4} \delta_{\mathbf{S},\mathbf{T}}^2 \epsilon(\hat{\mathbf{n}}_{\mathbf{S}} \cdot [\mathbf{x} - \mathbf{y}]) ; \quad (3.19b)$$

otherwise the  $\phi(\mathbf{S}; \mathbf{x})$  fields commute. Furthermore, the Boson currents Eq. [3.16] satisfy the same U(1) Kac-Moody algebra Eq. [3.5] as the Fermion charge currents Eq. [3.2] (with half the anomaly because we have removed the spin index):

$$[J(\mathbf{S}; \mathbf{q}) , J(\mathbf{T}; \mathbf{p})] = \Omega \delta_{\mathbf{S},\mathbf{T}} \delta_{\mathbf{q}+\mathbf{p},0}^3 \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} \quad (3.20a)$$

or in real space,

$$[J(\mathbf{S}; \mathbf{x}) , J(\mathbf{T}; \mathbf{y})] = -i \Omega \delta_{\mathbf{S},\mathbf{T}} L^3 \hat{\mathbf{n}}_{\mathbf{S}} \cdot \nabla_{\mathbf{x}} \delta^3(\mathbf{x} - \mathbf{y}) . \quad (3.20b)$$

Here the Fourier transform of the currents is given by:

$$J(\mathbf{S}; \mathbf{x}) \equiv \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} J(\mathbf{S}; \mathbf{q}) . \quad (3.21)$$

The Hamiltonian Eq. [3.1] then becomes (for spinless Fermions):

$$H = 2\pi \int d^3x d^3y \sum_{\mathbf{S},\mathbf{T}} V_c(\mathbf{S}, \mathbf{T}; \mathbf{x} - \mathbf{y}) [\hat{\mathbf{n}}_{\mathbf{S}} \cdot \nabla \phi(\mathbf{S}; \mathbf{x})] [\hat{\mathbf{n}}_{\mathbf{T}} \cdot \nabla \phi(\mathbf{T}; \mathbf{y})] \quad (3.22)$$

and Fermi fields are expressed in terms of the Boson fields as:

$$\psi(\mathbf{S}; \mathbf{x}) = \frac{\Omega^{\frac{1}{2}}}{\sqrt{2\pi a}} e^{i\mathbf{k}_{\mathbf{S}}\cdot\mathbf{x}} \exp\{i \frac{\sqrt{4\pi}}{\Omega} \phi(\mathbf{S}; \mathbf{x})\} \quad (3.23)$$

where  $\mathbf{k}_{\mathbf{S}}$  is the Fermi momentum at grid point  $\mathbf{S}$ .



The N-point Fermion correlation functions are reproduced with the use of the Bosonization formula Eq. [3.23]. If, for example, we use the operator identity:

$$e^A e^B =: e^{A+B} : \exp\langle AB + \frac{1}{2}(A^2 + B^2)\rangle \quad (3.24)$$

then we find that the Fermion two-point function is given by:

$$\langle \psi^\dagger(\mathbf{S}; \mathbf{x}) \psi(\mathbf{T}; \mathbf{0}) \rangle = \frac{1}{2\pi a} \delta_{\mathbf{S}, \mathbf{T}}^2 e^{i\mathbf{k}_S \cdot \mathbf{x}} \exp\left\{\frac{4\pi}{\Omega^2} \langle \phi(\mathbf{S}; \mathbf{x})\phi(\mathbf{S}; \mathbf{0}) - \phi^2(\mathbf{S}; \mathbf{0}) \rangle\right\}. \quad (3.25)$$

The Boson correlation function can be computed using the relation Eq. [3.17] and the result is:

$$\begin{aligned} G_c(\mathbf{S}; \mathbf{z}) &\equiv \langle \phi(\mathbf{S}; \mathbf{x}) \phi(\mathbf{S}; \mathbf{0}) - \phi^2(\mathbf{S}; \mathbf{0}) \rangle \\ &= -\frac{\Omega^2}{4\pi} \ln\left(\frac{\hat{\mathbf{n}}_S \cdot \mathbf{x} + ia}{ia}\right); \quad |\mathbf{x}_\perp \Lambda| \ll 1 \\ &\rightarrow -\infty; \quad |\mathbf{x}_\perp \Lambda| \gg 1. \end{aligned} \quad (3.26)$$

Consequently we obtain the correct Fermion correlation function, coarse-grained over the pill box:

$$\langle \psi^\dagger(\mathbf{S}; \mathbf{x}) \psi(\mathbf{T}; \mathbf{0}) \rangle = \frac{i\Omega}{2\pi} \delta_{\mathbf{S}, \mathbf{T}}^2 \frac{e^{i\mathbf{k}_S \cdot \mathbf{x}}}{\mathbf{k}_S \cdot \mathbf{x} + ia} \left(\frac{2\pi}{\Lambda}\right)^2 \delta^2(\mathbf{x}_\perp). \quad (3.27)$$

It should be emphasized that it is the average over the pill box that results in the  $\delta^2(\mathbf{x}_\perp)$  term.

To close the circle (Bosons  $\rightarrow$  Fermions  $\rightarrow$  Bosons) we form the Fermion current Eq. [3.2]. In real space we utilize the point-splitting procedure:

$$\begin{aligned} J(\mathbf{S}; \mathbf{x}) &=: \psi^\dagger(\mathbf{S}; \mathbf{x})\psi(\mathbf{S}; \mathbf{x}) : \\ &= \frac{\Omega}{2\pi a} \lim_{\epsilon \rightarrow 0} : \exp[-i\frac{\sqrt{4\pi}}{\Omega} \phi(\mathbf{S}; \mathbf{x} + \hat{\mathbf{n}}_S \epsilon)] \exp[i\frac{\sqrt{4\pi}}{\Omega} \phi(\mathbf{S}; \mathbf{x})] : \end{aligned} \quad (3.28)$$

then using the operator identity Eq. [3.24] again we obtain:

$$\begin{aligned} J(\mathbf{S}; \mathbf{x}) &= \frac{\Omega}{2\pi a} \lim_{\epsilon \rightarrow 0} \exp[-i\frac{\sqrt{4\pi}}{\Omega} \{\phi(\mathbf{S}; \mathbf{x} + \hat{\mathbf{n}}_S \epsilon) - \phi(\mathbf{S}; \mathbf{x})\}] \exp[\frac{4\pi}{\Omega^2} G_c(\mathbf{S}; \mathbf{x})] \\ &= \frac{1}{\sqrt{\pi}} \hat{\mathbf{n}}_S \cdot \nabla \phi(\mathbf{S}; \mathbf{x}) \end{aligned} \quad (3.29)$$

which is identical to Eq. [3.16]. A similar calculation shows that the free Fermion Hamiltonian is of the same form as the Boson Hamiltonian Eq. [3.22]:

$$\begin{aligned} H^0 &\equiv \sum_{\mathbf{S}} v_f(\mathbf{S}) \int d^3x \psi^\dagger(\mathbf{S}; \mathbf{x})(\hat{\mathbf{n}}_S \cdot \nabla)\psi(\mathbf{S}; \mathbf{x}) \\ &= 2\pi \sum_{\mathbf{S}} \frac{v_f(\mathbf{S})}{\Omega} \int d^3x \{(\hat{\mathbf{n}}_S \cdot \nabla)\phi(\mathbf{S}; \mathbf{x})\}^2. \end{aligned} \quad (3.30)$$

As it stands,  $\psi$  fields located in the same patch and at the same perpendicular coordinates anti-commute. For example, the Fermion two-point function Eq. [3.27] is odd under the transformation  $\mathbf{x} \rightarrow -\mathbf{x}$  followed by complex conjugation which is equivalent to interchanging the creation and annihilation operators in a translationally invariant system. However, fields in different patches, and fields in the same patch with  $\mathbf{x}_\perp \neq \mathbf{y}_\perp$ , commute:

$$\begin{aligned} \{\psi(\mathbf{S}; \mathbf{x}), \psi(\mathbf{S}; \mathbf{y})\} &= 0; \quad \mathbf{x}_\perp = \mathbf{y}_\perp \\ [\psi(\mathbf{S}; \mathbf{x}), \psi(\mathbf{S}; \mathbf{y})] &= 0; \quad \mathbf{x}_\perp \neq \mathbf{y}_\perp \\ [\psi(\mathbf{S}; \mathbf{x}), \psi(\mathbf{T}; \mathbf{y})] &= 0; \quad \mathbf{S} \neq \mathbf{T}. \end{aligned} \tag{3.31}$$

The commutation relations can be transformed into the correct anticommutation relations by introducing an ordering operator analogous to a Jordan-Wigner transformation<sup>18</sup>. Let  $O(\mathbf{S})$  be the ordering operator defined by:

$$O(\mathbf{S}) \equiv \exp\left\{i \frac{\pi}{2} \sum_{\mathbf{T}=1}^{\mathbf{S}-1} J(\mathbf{T}; \mathbf{q} = \mathbf{0})\right\} \tag{3.32}$$

where the mesh points  $\mathbf{T}$  have been arranged in consecutive order. To be definite, we could follow Luther's prescription and choose the mesh points to begin at some point (the "north pole") on the Fermi surface, spiral outwards, and converge at the antipode ("south pole"). It is straightforward to check that the combination  $\psi(\mathbf{S}; \mathbf{x})O(\mathbf{S})$  anticommutes with  $\psi(\mathbf{T}; \mathbf{y})O(\mathbf{T})$  when  $\mathbf{S} \neq \mathbf{T}$ . Commuting statistics are still obeyed when the fields are in the same pill box, but this discrepancy can be neglected in the continuum limit  $\Lambda \rightarrow 0$ . Alternatively, a second ordering operator may be introduced to implement anticommuting statistics within the pill box.

Thus we see that charge sector of the semi-classical Landau theory has been replaced by a quantum mechanical theory. The Fermi liquid should be thought of as a zero-temperature quantum critical Gaussian fixed point with infinite  $U(1)$  symmetry and parameters  $V_c(\mathbf{S}, \mathbf{T}; \mathbf{q})$ . No longer do semi-classical entities like  $\delta n_k$  appear: these have been replaced by charge current operators that are quantized with the Kac-Moody algebras. On the other hand, we have to resort to a semi-classical description of the spin sector because the quantum version appears to be intractable. A geometrical meaning has been given to the  $\omega$ -limit and a direct connection between the quasiparticle operators

and the Boson fields is made via the Bosonization formulas. To exercise the new framework, we rederive some well known results in the next two sections. We concentrate on the charge sector to illustrate how the quantum theory reproduces these results.

#### IV. $T^3 \ln(T)$ CONTRIBUTION TO THE SPECIFIC HEAT

As a concrete application of the proceeding formalism, we calculate the specific heat of an interacting Fermi liquid in three spatial dimensions. We obtain a non-analytic  $T^3 \ln(T)$  contribution to the specific heat. The existence of such a term is consistent with careful measurements<sup>21</sup> of the specific heat in Helium-3.

We turn off the spin-spin interactions in the following and for simplicity eliminate the spin index. As the nonanalytic behavior arises from small momentum processes, it is permissible to treat the Fermi surface in a locally flat approximation. Let the surface normal point in the  $\hat{z}$  direction. Then the U(1) Kac-Moody algebra can be written as:  $[J(\mathbf{S}; \mathbf{q}), J(\mathbf{T}; \mathbf{p})] = 2\Omega q_z \delta_{\mathbf{S},\mathbf{T}}^2 \delta_{\mathbf{p}+\mathbf{q},\mathbf{0}}^3$ . These commutation relations are equivalent to those obeyed by Bosonic harmonic oscillator creation and annihilation operators once we rescale by a factor of the square root of the momentum perpendicular to the surface:

$$\begin{aligned} J(\mathbf{S}; \mathbf{q}) &= \sqrt{-2\Omega q_z} a^\dagger(\mathbf{S}; -\mathbf{q}) ; \quad q_z \leq 0 \\ &= \sqrt{2\Omega q_z} a(\mathbf{S}; \mathbf{q}) ; \quad q_z > 0 \end{aligned} \quad (4.1)$$

where  $[a(\mathbf{S}; \mathbf{q}), a^\dagger(\mathbf{T}; \mathbf{p})] = \delta_{\mathbf{S},\mathbf{T}}^2 \delta_{\mathbf{p},\mathbf{q}}^3$ . Thus we can find the spectrum by direct diagonalization of the Bosonic harmonic oscillator Hamiltonian:

$$H_c = \sum_{\mathbf{S},\mathbf{T}} \sum_{\mathbf{q}, q_z > 0} V_c(\mathbf{S}, \mathbf{T}; \mathbf{q}) (2\Omega) q_z a^\dagger(\mathbf{S}; \mathbf{q}) a(\mathbf{T}; \mathbf{q}) . \quad (4.2)$$

We again place the system in a box of dimensions  $L^3$  and use periodic boundary conditions so the momenta are quantized as  $\mathbf{q}_m = \frac{2\pi}{L} \mathbf{m}$ . The Fermi velocity is given in terms of the Fermi energy  $\epsilon_f$  by  $v_f = \sqrt{2\epsilon_f/m}$  and the number of states at the Fermi surface,  $A$ , is given by  $A \equiv \sum_{\mathbf{S}} \sum_{\mathbf{q}} \theta(\mathbf{S}; \mathbf{q}) \delta_{\hat{\mathbf{n}}_{\mathbf{S}}, \mathbf{q}, \mathbf{0}} = \frac{2m\epsilon_f}{\pi} L^2 = \frac{v_f^2}{\pi} L^2$ . Because the pill boxes completely tile the surface we have the sum rule:

$$\sum_{\mathbf{S}} \sum_{\mathbf{q}} \theta(\mathbf{S}; \mathbf{q}) = A \frac{\lambda L}{2\pi} . \quad (4.3)$$

The specific heat is computed by using the standard formula for *Bosons*:

$$C_V = \frac{1}{4k_B T^2} \sum_{\mathbf{S}} \sum_{\mathbf{q}, q_z > 0} \theta(\mathbf{S}; \mathbf{q}) \frac{\epsilon^2(\mathbf{S}; \mathbf{q})}{\sinh^2\left(\frac{\epsilon(\mathbf{S}; \mathbf{q})}{2k_B T}\right)} \quad (4.4)$$

where  $\epsilon(\mathbf{S}; \mathbf{q})$  is an eigenvalue of the Hamiltonian Eq. [4.2] which depends on the momentum  $\mathbf{q}$  as well as the index  $\mathbf{S}$  that labels the vector space of the patches covering the Fermi surface. Let us first consider the case of non-interacting Fermions. The eigenenergies of the Bosonized Hamiltonian are simply:

$$\epsilon(\mathbf{S}; \mathbf{q}) = v_f \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} . \quad (4.5)$$

The sum over the patch index  $\mathbf{S}$  and the components of  $\mathbf{q}$  parallel to the surface just yields the number of states at the Fermi surface,  $A$ . The sum over the component of  $\mathbf{q}$  perpendicular to the surface can be converted to an integral. Assuming that the temperature is small (so that the thermally excited particle-hole pairs lie within the pill box, in other words  $k_B T \ll v_f \lambda$ ) we then find:

$$\begin{aligned} C_V &= \frac{A}{4k_B T^2} \frac{L}{2\pi} \int_0^\infty \frac{v_f^2 q_z^2}{\sinh^2(\frac{v_f q_z}{2k_B T})} dq_z \\ &= \frac{A}{4k_B T^2} \frac{(2k_B T)^3}{v_f} \frac{L}{2\pi} (\pi^2/6) \\ &= k_B^2 T \left( \frac{m p_f}{2\pi^2} \right) \frac{\pi^2}{3} L^3 . \end{aligned} \quad (4.6)$$

This result is the correct answer for spinless Fermions and of course it should be multiplied by a factor of two to account for the spin. It is remarkable that the Boson formula, Eq. [4.4], yields the full specific heat. We take it as further evidence that even for spatial dimensions greater than one Bosonization reproduces the entire Fermion Hilbert space.

We now follow Pethick and Carneiro<sup>22</sup> and focus on quasi-particles separated only by a small momentum  $\mathbf{W} \equiv \mathbf{k}_{\mathbf{S}} - \mathbf{k}_{\mathbf{T}}$  (ie.  $|\mathbf{W}| \ll k_f$ ) since a consideration of these processes is sufficient to demonstrate the existence of non-analytic contributions to the specific heat. Define two vectors  $\mathbf{u} \equiv \mathbf{k}_{\mathbf{S}} + \mathbf{q}$  and  $\mathbf{u} + \mathbf{p} \equiv \mathbf{k}_{\mathbf{T}} - \mathbf{q}$ . The quantity  $\hat{\mathbf{u}} \cdot \hat{\mathbf{p}}$  then functions as a small, rotationally-invariant, dimensionless expansion parameter. Here the normalized momenta are defined by  $\hat{\mathbf{p}} \equiv \mathbf{p}/p$  where  $p \equiv |\mathbf{p}|$ . Figure [7] exhibits the geometry of the interaction. The interaction coefficient  $V'_c$  may be expanded in our cylindrical coordinate system. Note that odd powers of  $\hat{\mathbf{u}} \cdot \hat{\mathbf{p}}$  do not appear in the

expansion because the sum over grid points and momentum eliminates terms odd in  $\mathbf{p}$ .

$$\begin{aligned} V'_c(\mathbf{S}, \mathbf{T}; \mathbf{q}) &= a + b (\hat{\mathbf{u}} \cdot \hat{\mathbf{p}})^2 + \dots \\ &= a + b \frac{4q_z^2}{4q_z^2 + \mathbf{W}^2} + \dots \end{aligned} \quad (4.7)$$

The expansion parameter is controlled in the low-temperature limit which keeps  $q_z$ , the particle-hole momentum perpendicular to the Fermi surface, small. (Recall that squat pill-boxes force  $|\mathbf{W}| \gg |\mathbf{q}| \geq |q_z|$  in the  $N \rightarrow \infty$  limit.)

Actually, the interaction differs from the one that Pethick and Carneiro studied: it couples particle-hole pairs at points  $\mathbf{S}$  and  $\mathbf{T}$  whereas the Pethick-Carneiro interaction couples the occupancies  $n_{\mathbf{u}}$  and  $n_{\mathbf{u}+\mathbf{p}}$ . To be precise, the Pethick-Carneiro interaction has the form:

$$\sum_{\mathbf{u}, \mathbf{p}} b (\hat{\mathbf{u}} \cdot \hat{\mathbf{p}})^2 \delta n_{\mathbf{u}} \delta n_{\mathbf{u}+\mathbf{p}} . \quad (4.8)$$

This interaction cannot be directly expressed in terms of the currents since it involves products of distinct occupancies above and below the Fermi surface whereas the current operator evaluated at zero momentum,  $J(\mathbf{S}; \mathbf{0})$ , averages the occupancy operator over the interior of the pill box. Therefore a direct connection with the earlier calculation cannot be made. Nevertheless, our purpose here is to show how non-analytic contributions to the specific heat arise in the new framework. Other terms may make non-analytic contributions; the interaction Eq. [4.7] is the simplest such term within our framework.

To proceed we diagonalize the Hamiltonian with the aid of a Fourier transform from Fermi surface patch index  $\mathbf{S}$  space to  $\mathbf{X}$ -space. Let

$$a^\dagger(\mathbf{S}; \mathbf{q}) = \frac{\Lambda}{\sqrt{4\pi p_f^2}} \int d^2 X e^{-i\mathbf{X} \cdot \mathbf{S}} a^\dagger(\mathbf{X}; \mathbf{q}) \quad (4.9)$$

where  $\int d^2 X 1 = \frac{4\pi p_f^2}{\Lambda^2} =$  (number of patches) then

$$H = L^3 \int d^2 X \int_{-\Lambda/2}^{\Lambda/2} \frac{dq_x}{2\pi} \int_{-\Lambda/2}^{\Lambda/2} \frac{dq_y}{2\pi} \int_0^\lambda \frac{dq_z}{2\pi} \epsilon(\mathbf{X}; \mathbf{q}) a^\dagger(\mathbf{X}; \mathbf{q}) a(\mathbf{X}; \mathbf{q}) . \quad (4.10)$$

Using Eq. [3.8] and Eq. [4.1] we then obtain the eigenenergies:

$$\epsilon(\mathbf{X}; \mathbf{q}) = v_f q_z + \frac{8b\Lambda^2 q_z^3}{(2\pi)^3} \sum_{\mathbf{W}} \frac{e^{-i\mathbf{X} \cdot \mathbf{W}/\Lambda}}{4q_z^2 + \mathbf{W}^2} . \quad (4.11)$$

The sum can be converted to a Riemann integral with the substitution  $\Lambda^2 \sum_{\mathbf{w}} \rightarrow \int d^2W$  and we find:

$$\epsilon(\mathbf{X}; \mathbf{q}) = v_f q_z - \frac{8bq_z^3}{(2\pi)^3} \pi \ln(4q_z^2 \mathbf{X}^2) . \quad (4.12)$$

In this equation we discard uninteresting terms proportional to  $b$  that make additional analytic contributions to the specific heat and keep only the logarithmic piece. We treat this term as a perturbation and calculate the specific heat to  $O(b)$ ; then the change in the specific heat  $\delta C_V$  due to the perturbation is:

$$\begin{aligned} \delta C_V &\approx -b \frac{A}{4k_B T^2} \frac{32v_f \pi}{(2\pi)^3} \frac{L}{2\pi} \int_0^\infty dq_z \frac{q_z^4 \ln(q_z)}{\sinh^2\left(\frac{v_f q_z}{2k_B T}\right)} \\ &\approx -b \frac{16ALk_B^4}{\pi^3 v_f^4} T^3 \ln(T) \int_0^\infty \frac{x^4}{\sinh^2(x)} dx . \end{aligned} \quad (4.13)$$

In the second line we retain only the term containing the  $T^3 \ln(T)$  temperature dependence; analytic contributions also appear but again these are not interesting. The integral in the second line equals  $\pi^4/30$  so the final result is:  $\frac{\delta C_V}{L^3} \approx -\frac{8}{15} b \frac{k_B^4 m^4}{p_f^2} T^3 \ln(T)$ . Not surprisingly, this result has the same form as that found by Pethick and Carneiro<sup>22</sup> as dimensional analysis guarantees this. A direct comparison of the coefficient is meaningless however since our interaction is not the same.

## V. COLLECTIVE MODES

The curvature of the Fermi surface did not play an important role in the calculation of the specific heat. In fact we took the Fermi surface to be flat; consequently the Hamiltonian could be rewritten as the sum of products of a single creation and a single annihilation operator (see Eq. [4.2]). Collective excitations of the Fermi surface, on the other hand, arise from the curvature. It is therefore interesting to derive the spectrum of collective modes within the new framework. For a curved Fermi surface the Hamiltonian can contain products, for example, of two creation operators, so the more general Bogoliubov transformation is required to diagonalize it.

Again we concentrate on the charged excitations to illustrate the quantum theory. We diagonalize the Hamiltonian Eq. [3.1] by first taking the matrix square root of  $V_c$

$$V_c(\mathbf{S}, \mathbf{T}; \mathbf{q}) = \sum_{\mathbf{U}} V_c^{\frac{1}{2}}(\mathbf{S}, \mathbf{U}; \mathbf{q}) V_c^{\frac{1}{2}}(\mathbf{U}, \mathbf{T}; \mathbf{q}) , \quad (5.1)$$

then we rewrite the Hamiltonian as:

$$\begin{aligned} H &= \frac{1}{2} \sum_{\mathbf{q}} \sum_{\mathbf{U}} \left[ \sum_{\mathbf{S}} V_c^{\frac{1}{2}}(\mathbf{U}, \mathbf{S}; \mathbf{q}) J(\mathbf{S}; \mathbf{q}) \right] \left[ \sum_{\mathbf{T}} V_c^{\frac{1}{2}}(\mathbf{U}, \mathbf{T}; \mathbf{q}) J(\mathbf{T}; -\mathbf{q}) \right] \\ &= \frac{1}{2} \sum_{\mathbf{U}} \sum_{\mathbf{q}} \tilde{J}(\mathbf{U}; \mathbf{q}) \tilde{J}(\mathbf{U}; -\mathbf{q}) . \end{aligned} \quad (5.2)$$

Here we have introduced new charge currents:

$$\tilde{J}(\mathbf{U}; \mathbf{q}) \equiv \sum_{\mathbf{S}} V_c^{\frac{1}{2}}(\mathbf{U}, \mathbf{S}; \mathbf{q}) J(\mathbf{S}; \mathbf{q}) \quad (5.3)$$

and also use the fact the  $V_c$  is a real symmetric matrix [ie.  $V_c(\mathbf{S}, \mathbf{T}; \mathbf{q}) = V_c(\mathbf{T}, \mathbf{S}; \mathbf{q})$ ] so therefore  $V_c^{\frac{1}{2}}$  is also symmetric. These new currents obey modified Kac-Moody commutation relations:

$$[\tilde{J}(\mathbf{S}; \mathbf{q}) , \tilde{J}(\mathbf{T}; \mathbf{p})] = 2\Omega \delta_{\mathbf{p}, \mathbf{q}}^3 [V_c^{\frac{1}{2}} D V_c^{\frac{1}{2}}](\mathbf{S}, \mathbf{T}; \mathbf{q}) \quad (5.4)$$

where the diagonal matrix

$$D(\mathbf{S}, \mathbf{T}; \mathbf{q}) \equiv \delta_{\mathbf{S}, \mathbf{T}}^2 \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} \quad (5.5)$$

appears naturally in the implicit matrix product on the right hand side of the equation. (The sum over the internal indices in Eq. [5.4] has been suppressed for clarity.) We obtain the spectrum by



diagonalizing this modified anomaly. Let the eigenvectors  $u^{\mathbf{A}}(\mathbf{S}; \mathbf{q})$  and eigenvalues  $\omega^{\mathbf{A}}(\mathbf{q})$  of the spectrum carry the label  $\mathbf{A}$ . Suppressing again internal matrix indices and the momentum  $\mathbf{q}$  we have:

$$2\Omega \sum_{\mathbf{T}} [V_c^{\frac{1}{2}} D V_c^{\frac{1}{2}}](\mathbf{S}, \mathbf{T}) u^{\mathbf{A}}(\mathbf{T}) = \omega^{\mathbf{A}} u^{\mathbf{A}}(\mathbf{S}) . \quad (5.6)$$

Upon matrix multiplying both sides of this equation by  $V_c^{\frac{1}{2}}$  and defining new eigenvectors  $\tilde{u}^{\mathbf{A}}(\mathbf{S}) \equiv \sum_{\mathbf{T}} V_c^{\frac{1}{2}}(\mathbf{S}, \mathbf{T}) u^{\mathbf{A}}(\mathbf{T})$  we arrive at the collective mode equation (with an implicit sum over repeated indices):

$$2\Omega V_c(\mathbf{U}, \mathbf{S}; \mathbf{q}) D(\mathbf{S}, \mathbf{T}; \mathbf{q}) \tilde{u}^{\mathbf{A}}(\mathbf{T}; \mathbf{q}) = \omega^{\mathbf{A}}(\mathbf{q}) \tilde{u}^{\mathbf{A}}(\mathbf{U}; \mathbf{q}) . \quad (5.7)$$

This equation can be rephrased in a more familiar form by writing  $V_c$  explicitly as  $\frac{1}{2} v_f \Omega^{-1} \delta_{\mathbf{S}, \mathbf{T}}^2 + \frac{1}{L^3} V'_c(\mathbf{S}, \mathbf{T}; \mathbf{q})$  and taking the interaction  $V'_c$  to be independent of  $\mathbf{q}$ . Dropping the label  $\mathbf{A}$  and the tilde we find the dispersion relation:

$$(\mathbf{q} \cdot \mathbf{v}_{\mathbf{S}} - \omega) u(\mathbf{S}) + \frac{2\Omega}{L^3} \sum_{\mathbf{T}} V'_c(\mathbf{S}, \mathbf{T}) (\mathbf{q} \cdot \mathbf{v}_{\mathbf{T}}) u(\mathbf{T}) = 0 . \quad (5.8)$$

(Recall that  $\mathbf{v}_{\mathbf{S}} \equiv v_f \hat{\mathbf{n}}_{\mathbf{S}}$  is the Fermi velocity at grid point  $\mathbf{S}$ .) Now we multiply each term in Eq. [5.8] by  $\mathbf{q} \cdot \mathbf{v}_{\mathbf{S}}$  and make another change of variable by redefining  $u(\mathbf{T}) \rightarrow \mathbf{q} \cdot \mathbf{v}_{\mathbf{T}} u(\mathbf{T})$  (with no sum over  $\mathbf{T}$ ). The result is:

$$(\mathbf{q} \cdot \mathbf{v}_{\mathbf{S}} - \omega) u(\mathbf{S}) + \mathbf{q} \cdot \mathbf{v}_{\mathbf{S}} \frac{2\Omega}{L^3} \sum_{\mathbf{T}} V'_c(\mathbf{S}, \mathbf{T}) u(\mathbf{T}) = 0 . \quad (5.9)$$

Recognizing that the sum is just a coarse-grained version of the sum over momenta  $\mathbf{k}$  lying on the Fermi surface (FS):

$$\Omega \sum_{\mathbf{T}} \approx \frac{L}{2\pi} \sum_{\mathbf{k} \in \text{FS}} \quad (5.10)$$

we see that we have arrived at the collective mode equation.

Since zero sound excitations involve global distortions of the Fermi surface that slosh Fermions back and forth, the curvature of the Fermi surface plays an important role. For example, solving this equation for a perfectly spherical Fermi surface with  $V'_c(\mathbf{S}, \mathbf{T})$  assumed to be a constant independent

of the angle between  $\mathbf{S}$  and  $\mathbf{T}$  we find the zero-sound mode:

$$u(\theta, \phi) \propto \frac{\cos(\theta)}{s - \cos(\theta)} \quad (5.11)$$

where  $(\theta, \phi)$  are polar coordinates with the polar axis in the  $\hat{\mathbf{q}}$  direction and  $s \equiv \frac{\omega}{v_f |\mathbf{q}|}$ . Also implicit in Eq. [5.9] is the renormalization of the Fermion mass. Again assuming a spherical Fermi surface, we may use Galilean invariance<sup>19</sup> to find the well-known result:

$$\frac{1}{m} = \frac{1}{m^*} + \frac{k_f}{3\pi^2} f_1^c \quad (5.12)$$

if we identify  $V_c'(\theta) = \sum_{l=0}^{\infty} f_l^c P_l(\cos\theta)$ . Finally, collective excitations in the spin sector of our theory are given by the corresponding Fermi liquid formula. Apparently our new formulation of the Fermion liquid reproduces well known Fermi liquid theory results.

## VI. DISCUSSION

In the preceding sections we showed that a simple model of interacting Fermions in two spatial dimensions can lead to a fixed point with local U(1) symmetry despite the fact that the bare Hamiltonian is only invariant under global U(1) transformations. We also presented a framework for the Bosonization of Fermion liquids in higher dimension. We enlarge upon the connection between the simplified model of section (II) and the general problem of Bosonization here. First it is clear that the model is pathological in the sense that the residual fixed point current-current interactions that couple the four Fermi points are singular; for example,  $\lambda_{1c}$  (which couples currents at opposite points) is typically of the same order as  $v_c$  (which couples currents at the same point). In the physical case of a continuous Fermi surface in spatial dimensions of two or higher, interactions of this type would be equivalent to a current-current coupling of the form:

$$V_c(\mathbf{S}, \mathbf{T}; \mathbf{q}) = \frac{1}{2}v_c\Omega^{-1} \delta_{\mathbf{S},\mathbf{T}}^2 + \lambda_{1c} \Omega^{-1}\delta_{\mathbf{S},-\mathbf{T}}^2 \quad (6.1)$$

where  $-\mathbf{T}$  denotes a mesh point directly opposite point  $\mathbf{T}$ . In the  $\Lambda \rightarrow 0$  limit of a finer and finer mesh,  $\Omega^{-1} \propto \Lambda^{-2} \rightarrow \infty$ . Thus the second term in Eq. [6.1] amounts to a singular interaction that might be expected to destroy Fermi-liquid type behavior<sup>23</sup>.

For singular interactions, however, the connection between the multidimensional Bosonization and one-dimensional behavior begins to break down for at least two reasons. First, as we noted in section (I), Luttinger liquids in one spatial dimension are characterized by the elimination of the discontinuity in the Fermion occupancy at the Fermi surface. Consequently, the Fermion distribution is smeared out over some energy scale (set by the energy cut-off in the interaction). As long as this cut-off is small compared to lattice energy scales, the continuum analysis of section (I) holds. In higher dimensions, however, a second energy scale  $v_f\lambda$  has to be introduced since the Kac-Moody algebra is obtained in our construction only in the  $\omega$ -limit of  $\Lambda \gg \lambda \gg |\mathbf{q}|$  where  $\Lambda \rightarrow 0$ . This limit apparently precludes the incorporation of interactions which eliminate the Fermi discontinuity.

We alluded to a second problem with singular spin-spin interactions earlier: the semi-classical limit breaks down because the terms on the right-hand side of the spin current commutation relations Eq. [3.11] cannot be neglected when interactions diverge in the  $\Lambda \rightarrow 0$  limit. A return to the original Luther Bosonization prescription using narrow tubes instead of squat pill boxes appears to offer a way out of both difficulties. In this case the energy scale  $v_f \lambda$  need not be introduced; we can think of the higher-dimensional problem as a collection of purely one-dimensional theories. However, now *only* singular “tomographic<sup>24</sup>” type interactions are permitted: the current in any given tube can couple only to itself or to the current in a tube emerging from a point directly opposite on the Fermi surface. Actually, the simplified model of section (II) illustrates this problem. Interactions  $\lambda_{2c}$  and  $\lambda_{2s}$  only couple the zero-momentum components of the currents and therefore cannot change the excitation spectrum. Nevertheless the special case of tomographic Bosonization may exhibit features of interest.

A separate, but related, problem of interest arises when the velocity of charge excitations differs from that of spin excitations. Fermi liquid theory breaks down in this case because the Fermion propagator no longer exhibits a simple pole; instead there is a branch cut. Thus the quasi-particle weight  $Z = 0$  even though, as mentioned in section (I), a discontinuity in the Fermion occupancy remains at the Fermi surface. Since the two velocities are just parameters appearing in  $V_c(\mathbf{S}, \mathbf{T}; \mathbf{q})$  and  $V_s(\mathbf{S}, \mathbf{T}; \mathbf{q})$  of our theory, we need not restrict ourselves to setting both velocities equal to a Fermi velocity  $v_f$  as we did in section (III). Anderson has argued that a difference in velocities between the two sectors, rather than singular interactions at the antipode, might account for the anomalous normal state properties of the copper oxide superconductors<sup>24</sup>. It might be interesting to explore the consequences of spin-charge separation within the Bosonization framework presented here. Other open problems include the incorporation of van Hove singularities and energy gaps within the Bosonization framework. It may also be possible to include fluctuations in the Fermi surface shape or topology within a renormalization group approach.

Finally, the Bosonization procedure outlined in this paper may permit the application of semi-

classical approximations to the interacting Fermion problem. Semiclassical approximations cannot be directly applied to Fermions because the Pauli exclusion principle guarantees that occupation numbers are of order one and thus far from the classical limit. Bosonization bypasses this problem by replacing the Fermion variables with Bosonic ones. Indeed, semiclassical approximations to Bosonized versions of certain one-dimensional problems have been remarkably useful in the past. For example, an analysis of quantum spin chains that begins with the weak-coupling Hubbard model describing interacting electrons ends up mapping the low-energy theory onto the non-linear WZW sigma model. The behavior of this model in the semi-classical limit explains many of the known properties of quantum antiferromagnets<sup>25</sup>.

#### Acknowledgements

We thank Ian Affleck, Sasha Finkel'stein, Matthew Fisher, Duncan Haldane, Mike Kosterlitz, Hyok-Jon Kwon, Patrick Lee, Andreas Ludwig, M. Randeria, N. Read, and R. Shankar for helpful discussions. J.B.M. thanks the Institute for Theoretical Physics (NSF Grant No. PHY89-04035) for its hospitality during his stay there. A. H. was supported by the National Science Foundation through grant DMR9008239.

## REFERENCES

- [1] R. Shankar, *Physica A* **177**, 530 (1991).
- [2] For a review, see V. J. Emery in *Highly Conducting One-dimensional Solids*, eds. J. T. Devreese, R. P. Evrard, and V. E. Van Doren (Plenum 1979).
- [3] F. D. M. Haldane, seminar at Brown University, November 1991.
- [4] For a good review of Abelian Bosonization, see R. Shankar, “Bosonization: how to make it work for you in condensed matter,” lectures given at the BCSPIN school in Katmandu, Nepal, May 1991.
- [5] E. Lieb and F. Wu, *Phys. Rev. Lett.* **20**, 1445 (1968).
- [6] Ian Affleck and J. Brad Marston, *J. Phys. C* **21**, 2511 (1988).
- [7] We thank I. Affleck for pointing this fact out to us.
- [8] F. D. M. Haldane, *J. Phys. C* **14**, 2585 (1981).
- [9] H. J. Schulz, *Europhysics Lett.* **4**, 609 (1987).
- [10] I. E. Dzyaloshinskii, *Sov. Phys. JETP* **66**, 848 (1987); I. E. Dzyaloshinskii and V. M. Yakovenko, *Sov. Phys. JETP* **67**, 844 (1988).
- [11] P. W. Anderson, *Phys. Rev. Lett.* **67**, 3844 (1991).
- [12] A. Finkel’stein, unpublished.
- [13] C. Castellani, C. Di Castro, and W. Metzner, *Phys. Rev. Lett* **69**, 1703 (1992).
- [14] J. C. Campuzano *et al.*, *Phys. Rev. Lett.* **64**, 2308 (1990).
- [15] R. Shankar, private communication.

- [16] P. W. Anderson, *Science* **235**, 1196 (1987).
- [17] E. Witten, *Commun. Math. Phys.* **92**, 455 (1984); A. B. Zamolodchikov and V. A. Fateev, *Sov. J. Nucl. Phys.* **43**, 657 (1986); D. Gepner and E. Witten, *Nucl. Phys.* **B278**, 493 (1986); A. W. W. Ludwig and Ian Affleck, *Nucl. Phys. B.* **360**, 641 (1991).
- [18] A. Luther, *Phys. Rev. B* **19**, 320 (1979).
- [19] G. Baym and C. Pethick, *Landau Fermi-Liquid theory: concepts and applications* (John Wiley & Sons, New York, 1991).
- [20] Peter Goddard and David Olive, *Int. J. of Mod. Phys. A* **1**, 303 (1986).
- [21] Dennis S. Greywall, *Phys. Rev. B* **27**, 2747 (1983).
- [22] C. J. Pethick and G. M. Carneiro, *Physical Review A* **7**, 304 (1973).
- [23] P. C. E. Stamp, *Phys. Rev. Lett.* **68**, 2180 (1992).
- [24] P. W. Anderson, *Phys. Rev. Lett.* **67**, 2092 (1991).
- [25] Ian Affleck and F. D. M. Haldane, *Phys. Rev. B* **36**, 5291 (1987).

## FIGURE CAPTIONS

- (1) Typical band structure and the left and right Fermi points of the one-dimensional problem. The dashed lines denote filled states.
- (2) A one-loop diagram that gives a second order contribution to the renormalization of  $\lambda_s$ . All diagrams that contribute at this order have a left and a right moving propagator which connect the point  $(x, t)$  with the point  $(0, 0)$ .
- (3) The four Fermi points kept in the model with linear dispersion along the lines. The dotted line delineates the Fermi surface of the half-filled nearest-neighbor tight binding model. The inset shows the  $(u, v)$  coordinate system.
- (4) The nine types of marginal interactions: (a) The four current-current interactions that respect spin-charge separation. (b) The one non-Umklapp mixed process. (c) The two Umklapp processes that transport charge-2 spin singlets and therefore respect spin-charge separation. (d) The two other “mixed” Umklapp processes that transport both spin and charge and thus break spin-charge separation. (The Umklapp processes only occur at half-filling.) Note that these diagrams only depict representative process – the missing diagrams are generated by performing the various symmetry operations on the square lattice (reflections that exchange points 1 and -1 or 2 and -2 and rotations through 90 degrees).
- (5) Renormalization group flow of the model away from half-filling. The initial coupling is a repulsive Hubbard interaction with  $U/t = 1$ . The couplings flow toward the fixed line  $\lambda_{1s} = \lambda_5 = 0$  and  $\lambda_{1c} > 0$ .
- (6) Currents at each grid point  $\mathbf{S}$  on the Fermi surface are constructed with the use of squat pill boxes that tile the surface. The box has dimensions  $\Lambda \times \Lambda$  along the surface, height  $\lambda = \frac{\Lambda}{\sqrt{N}}$  (where  $N \rightarrow \infty$ ) and is bisected through the mid-plane by the Fermi Surface. The function  $\theta(\mathbf{S}; \mathbf{k}) = 1$  inside the box; otherwise it is zero. The momentum  $\mathbf{q}$  must be small:  $|\mathbf{q}| \leq \frac{\Lambda}{N}$ . Thus,  $|\mathbf{q}| \ll \lambda \ll \Lambda$ .



- (7) Geometry of the Fermi liquid interactions that lead to non-analytic contributions to the specific heat. Two squat boxes lie on the locally flat Fermi surface at grid points **S** and **T** (see text).