

Spin and Isospin in Quaternion Quantum Mechanics

M. D. Maia
Universidade de Brasília, Instituto de Física
Brasília, DF. 70910-900
maia@fis.unb.br

The algebraic consistency of spin and isospin at the level of an unbroken $SU(2)$ gauge theory suggests the existence of an additional angular momentum besides the spin and isospin and also produces a full quaternionic spinor operator. The latter corresponds to a vector boson in space-time, interpreted as a $SU(2)$ gauge field. The existence of quaternionic spinor fields implies in a quaternionic Hilbert space and its necessary mathematical analysis. It is shown how to obtain a unique representation of a quaternion function by a convergent positive power series.

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I. QUATERNION QUANTUM MECHANICS

After six decades, quaternion quantum mechanics is coming out of age. The earliest known reference on a possible generalization of quantum theory with respect to the background field dates back to 1934 with the paper by Jordan et al [1]. The use of quaternions properly was proposed by Birkoff and von Neumann in 1936, later developed by Finkelstein [2], and more recently by Adler and others [3].

The original motivation for quaternion quantum mechanics was formal: The propositional calculus implies that it is possible to represent the pure states of a quantum system by rays on a Hilbert defined on any associative division algebra. This includes the quaternion algebra as the most general case. Contrasting with this, the next division (but non associative) algebra, the octonion algebra, has been always associated with physical arguments, notably in connection with the $SU(3)$ gauge symmetry and strong interactions [4,5,6,7].

In its essence, quaternion quantum mechanics is a modification of the complex quantum theory, in which the wave functions belong to a Hilbert space defined over the quaternion field. As a physical theory, it should prove to be effective at some high energy level, exhibiting experimental evidence which would distinguish it from the complex theory [8].

As in all Clifford algebras, it is possible to represent the quaternion algebra as a tensor product of two independent complex algebras. Therefore, in principle we could write quaternion functions with complex components and use two independent expressions of complex analyticity. However this does not mean that complex concept of analyticity extends trivially to quaternions.

This paper has two purposes. One of them is to show by a simple argument that the combined spin and isospin states leads to quaternion quantum mechanics, and vice

versa, that the algebraic consistency of quaternion quantum mechanics necessarily implies in the existence of an intrinsic angular momentum representing the combined spin and isospin at the level of an unbroken $SU(2)$ gauge theory. The second and more formal topic is a consequence of the first: The emergence of a quaternionic Hilbert space requires the solution of differential equations involving functions of a quaternionic variable, quaternionic Fourier expansions and quaternionic phase transformations. We will see that the traditional complex notion of analyticity characterized by direction independent derivatives does not necessarily apply, so that quaternionic quantum states may exhibit different behavior along different directions in space.

II. ISOSPIN AND QUATERNIONS

The standard textbook explanation on why quantum mechanics should be defined over the complex field is based on the double slit experiment, together with the complex phase difference for the wave functions. However, this can also be explained by a real quantum theory provided a special operator such that $J^2 = 1$ and $J^T = -J$ is introduced [2]. Although most people would agree that this is equivalent to quantum mechanics over the complex field, a more definitive argument for the complex algebra comes from the spin. The existence and classification of the spinor representations of the rotation subgroup of the Lorentz group demands a solution of quadratic algebraic equations for the eigenvalues of the invariant operators. This can be guaranteed only within the complex field. The bottom line is, complex quantum mechanics is a requirement of the spin and its spinor structure [9].

Following a similar argument we may convince ourselves that quaternion quantum mechanics is an algebraic requirement of the spin together with the isospin and the associated spinor structures. In fact, while the spin is associated with the spinor representations of the $SO(3)$ subgroup of the Lorentz group, the isospin is given by a

representation of the gauge group $SU(2)$. Actually the two groups are isomorphic and they have equivalent representations given by the Pauli matrices (they are identical representations, distinguished only by different notations), acting on independent spinor spaces.

In the case of the unbroken $SU(2)$ gauge theory, spin and isospin are present in a combined symmetry scheme, so that the total spinor space is a direct sum of the spinor space and the isospinor spaces: $K = I \oplus J$. The spinor space is represented as a complex plane (a Gauss plane) generated by the basis

$$I=(\text{spinor space}) : \left\{ 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

and the isospinor space represented by another complex plane generated by

$$J=(\text{isopinor space}) : \left\{ 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, j = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Thus, I and J can be taken as two independent Gauss planes sharing the same real unit 1, but with different and independent imaginary units i and j respectively. As long as the isospin symmetry and the spinor representation of the Lorentz group remain combined, the corresponding angular momenta add up to generate a total angular momentum represented in the direct sum of the two spinor spaces. On the other hand, it has been argued that when this combined symmetry is broken the $SU(2)$ degree of freedom reappears as a spin degree of freedom [10]. Following these ideas a fermionic state is derived from a magnetic monopole model in four dimensions associated with an $SU(2)$ soliton [11,12,13,14].

The question we address here concerns with the algebraic consistency of the combined spinor-isospinor symmetry as it remains unbroken. In this case, we end up with the total spinor space generated by the hypercomplex basis $\{1, i, j\}$. According to Hamilton this direct sum is algebraically consistent only if a third imaginary unit k such that $k = ij$ is introduced. That is, we can only close the algebra if a third complex plane

$$K=(\text{new spinor space}) : \left\{ 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, k = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is introduced. We conclude that an additional spin-half field should also be present. This new spinor may be the generator of the Jackiw-Rebbi spin degree of freedom after the symmetry is broken. However, this is not all. When the combined symmetry remains unbroken the three spinors produce a full quaternion algebra with basis $\{1, i, j, k\}$, such that its group of automorphisms carry the combined symmetry. If we add to the space of these quaternion wave functions a Hilbert product compatible with the quaternion algebra we obtain a quaternionic Hilbert space. This space should reduce to the usual Hilbert space of complex quantum mechanics with separate spin, isospin plus one extra spinor degree of freedom at the level of the combined symmetry breaking.

In conclusion, quaternion quantum mechanics appears as consistent condition of the combined spin and isospin symmetries. A possible relation between quaternions and the isotopic spin was suggested by C. N. Yang [2] and by E. J. Schrepf [15]. However, their basic arguments are distinct from the ones based on the combination of symmetries.

When the two spinor spaces are taken together, they give way to a quaternionic spin operator given by a 2×2 matrix representation of the quaternion algebra given by the Pauli matrices

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

This has a one to one correspondence with the quaternion algebra* whose elements are the quaternions

$$\Psi = \sum_{\alpha=0}^3 \Psi_{\alpha} \sigma^{\alpha} = \begin{pmatrix} \Psi_0 + \Psi_3 & \Psi_1 - i\Psi_2 \\ \Psi_1 + i\Psi_2 & \Psi_0 - \Psi_3 \end{pmatrix}, \quad (1)$$

As it is well known, the above matrix corresponds to a vector field in space-time, associated to a pair of two component spinors $\Psi^{\alpha} = \sigma_{AB}^{\alpha} \xi^A \bar{\zeta}^B$. A possible interpretation for this pair of spinors is given by the spin and isospin states in a $SU(2)$ model [11,12,13,14].

Since the inner automorphism of the quaternion algebra correspond to the isometries of space-time, the existence of a combined spin-isospin structure also have implications on the classical notion of derivative of a function. In ordinary complex analysis the derivative of a function does not depend on the direction in the complex plane along which the limit is taken. This has to be so because the complex plane is generated by only one real and one imaginary direction, leading to the Cauchy-Riemann conditions for analyticity. On the other hand, in the quaternionic case there are three imaginary directions generating a space that is isomorphic to \mathbb{R}^3 , where in principle there is no reason for the derivatives to be all equal. This means that the properties of differentiable equations involving quaternion functions of a quaternionic variable does not necessarily coincide with those of ordinary complex quantum theory. Fortunately

*Greek indices are space-time indices and they run from 0 to 3. Small case Latin indices run from 1 to 3. Capital Latin indices are spinor indices running from 1 to 2. The quaternion multiplication table is taken to be

$$e_i e_j = -\delta_{ij} + \sum \epsilon_{ijk} e_k, \quad e_i e_0 = e_0 e_i = e_i$$

The conjugate of a quaternion X is \bar{X} , with $\bar{e}_i = -e_i$, $\bar{e}_0 = e_0$. The quaternion norm is $|X|^2 = \sum (X^{\alpha})^2$ and the inverse of X is $X^{-1} = \bar{X}/|X|$.

this can be examined through the methods of classical analysis.

III. ANALYSIS OF QUATERNION FIELDS

The earliest known study on the analysis of quaternion functions using the same concepts of complex analysis was made by Fueter in 1932, finding very restrictive generalizations of the Cauchy-Riemann conditions [16]. Some alternative criteria for defining quaternion analyticity have been suggested [17,18,19,20,21,22], but to date there is not a consensus on what is meant by quaternion analyticity. To understand the nature of the difficulties we need to start from the basic principles. Denoting a generic quaternion function by $f(X) = \sum U_\alpha(X)e^\alpha$, and $\Delta f = [f(X + \Delta X) - f(X)]$ we may define its left derivative as

$$f'(X) = \lim_{\Delta X \rightarrow 0} \delta f(X)(\Delta X)^{-1},$$

and the right derivative as

$$'f(X) = \lim_{\Delta X \rightarrow 0} (\Delta X)^{-1} \Delta f(X)$$

where the limits are taken with $|\Delta X| \rightarrow 0$ along the direction of the four-vector ΔX which depends on the 3-dimensional vector \vec{X} .

To compare with the complex case, we may use the exponential form of a quaternion: A vector of \mathbb{R}^3 , $\xi = \sum X_i e^i$ associate a quaternion ξ with norm $|\xi|^2 = \xi \bar{\xi} = \sum X_i^2$ and square $\xi^2 = -\xi \xi = -\sum X_i^2$. Defining the unit quaternion (iota) $\iota = \xi / \sqrt{\xi \bar{\xi}}$ such that $\iota^2 = -1$, we may construct a Gauss plane generated by ι and the quaternion unit e^0 . In this plane a quaternion $X = X_0 e^0 + \sum X_i e^i$ can be expressed as $X = |X|(\cos \gamma + \iota \sin \gamma)$, with $\tan \gamma = X_0 / \sqrt{\sum X_i^2}$. Therefore, if we define the quaternion exponential as

$$\exp(\iota \gamma) = e^{\iota \gamma} = \cos \gamma + \iota \sin \gamma,$$

then we may express $\Delta X = |\Delta X|e^{\iota \gamma}$ and $\Delta X^{-1} = e^{-\iota \gamma} / |\Delta X|$, where the three dimensional direction is included in the definition of ι .

Contrasting with the complex case, we cannot neglect the phase factor $e^{\iota \gamma}$ during the limiting process because we have a functions depending on three variables. The independence of the limit with the phase is a privilege (or rather a limitation) of complex theory. Furthermore, here we have the added complication that the left and right derivatives do not necessarily coincide. To see the consequences of this, consider the derivatives of a quaternion function $f(X)$ along a fixed direction $\Delta X = \Delta X_\beta e^\beta$ (no sum), indicated by the index within parenthesis

$$f'(X)_{(\beta)} = \frac{\partial U_0}{\partial X_\beta} e^0 (e^\beta)^{-1} + \sum_i \frac{\partial U_i}{\partial X_\beta} e^i (e^\beta)^{-1} \quad (2)$$

$$'f(X)_{(\beta)} = \frac{\partial U_0}{\partial X_\beta} (e^\beta)^{-1} e^0 + \sum_i \frac{\partial U_i}{\partial X_\beta} (e^\beta)^{-1} e^i \quad (3)$$

From direct calculation we find that

$$\begin{aligned} f'(X)_{(0)} &= 'f(X)_{(0)} \\ f'(X)_{(j)} &= 'f(X)_{(j)} - 2 \sum_{i,k} \epsilon^{ijk} \frac{\partial U_i}{\partial X_j} e^k \end{aligned}$$

Imposing that these derivatives are selectively equal, four basic classes of complex-like analytic functions are obtained

Class A) Right analytic functions

$$\left. \begin{aligned} f'(X)_{(0)} &= f'(X)_{(i)} \\ f'(X)_{(i)} &= f'(X)_{(j)} \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} \frac{\partial U_\alpha}{\partial X_\alpha} &= \frac{\partial U_\beta}{\partial X_\beta} \\ \frac{\partial U_i}{\partial X_0} &= -\frac{\partial U_0}{\partial X_i} \\ \frac{\partial U_i}{\partial X_j} &= \sum \epsilon^{ijk} \frac{\partial U_k}{\partial X_0} \end{aligned} \right. \quad (4)$$

Class B) Left analytic functions

$$\left. \begin{aligned} 'f(X)_{(0)} &= 'f(X)_{(i)} \\ 'f(X)_{(i)} &= 'f(X)_{(j)} \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} \frac{\partial U_\alpha}{\partial X_\alpha} &= \frac{\partial U_\beta}{\partial X_\beta} \\ \frac{\partial U_i}{\partial X_0} &= -\frac{\partial U_0}{\partial X_i} \\ \frac{\partial U_i}{\partial X_j} &= -\sum \epsilon^{ijk} \frac{\partial U_k}{\partial X_0} \end{aligned} \right. \quad (5)$$

Class C) Left-right analytic functions

$$f'(X)_{(\alpha)} = 'f(X)_{(\alpha)} \Rightarrow \left\{ \begin{aligned} \frac{\partial U_\alpha}{\partial X_\alpha} &= \frac{\partial U_\beta}{\partial X_\beta} \\ \frac{\partial U_i}{\partial X_j} &= -\frac{\partial U_j}{\partial X_i} \\ \frac{\partial U_i}{\partial X_0} &= -\frac{\partial U_0}{\partial X_i} \end{aligned} \right. \quad (6)$$

Class D) The total analytic functions

$$\left. \begin{aligned} f'(X)_{(\alpha)} &= f'(X)_{(\beta)} \\ 'f(X)_{(\beta)} &= 'f(X)_{(\alpha)} \\ 'f(X)_{(\alpha)} &= f'(X)_{(\alpha)} \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} \frac{\partial U_\alpha}{\partial X_\alpha} &= \frac{\partial U_\beta}{\partial X_\beta} \\ \frac{\partial U_i}{\partial X_\beta} &= 0 \quad \alpha \neq \beta \end{aligned} \right. \quad (7)$$

As we can see, these conditions are very restrictive, specially when we consider the applications to quantum mechanics, suggesting the adoption of different criteria for analyticity.

IV. HARMONICITY

Consider the operator $\partial = \sum e^\alpha \partial_\alpha = \sum e^\alpha \partial / \partial X_\alpha$ acting on the right and on the left of a function $f(X)$

$$\begin{aligned} \partial f(X) &= \left(\frac{\partial U_0}{\partial X_0} + \sum \frac{\partial U_i}{\partial X_0} e^i \right) \\ &+ \sum \left[\frac{\partial U_0}{\partial X_i} e^i - \sum \frac{\partial U_i}{\partial X_j} (\delta^{ij} - \epsilon^{ijk} e^k) \right] \\ &= 'f(X)_{(0)} + \sum_j 'f(X)_{(j)} \end{aligned} \quad (8)$$

$$\begin{aligned}
f(x)\bar{\vartheta} &= \left(\frac{\partial U_0}{\partial X_0} + \sum \frac{\partial U_i}{\partial X_0} e^i \right) \\
&+ \sum \left[\frac{\partial U_0}{\partial X_i} e_i - \sum \frac{\partial U_i}{\partial X_j} (\delta^{ij} + \epsilon^{ijk} e^k) \right] \\
&= f'(X)_{(0)} - \sum_j f'(X)_{(j)}
\end{aligned} \tag{9}$$

Using these results four new classes of quaternion functions can be defined:

Class E) The functions such that

$$\bar{\vartheta}f(X) = f(X)\bar{\vartheta} \Rightarrow \left\{ \frac{\partial U_i}{\partial X_j} = \frac{\partial U_j}{\partial X_i} \right. \tag{10}$$

Class F) The left harmonic functions

$$\bar{\vartheta}f(X) = 0 \Rightarrow \left\{ \begin{aligned} \frac{\partial U_0}{\partial X_0} &= \sum_i \frac{\partial U_i}{\partial X_i} \\ \frac{\partial U_0}{\partial X_k} + \frac{\partial U_0}{\partial X_k} &= \sum_{ij} \epsilon^{ijk} \frac{\partial U_i}{\partial X_j} \end{aligned} \right. \tag{11}$$

Class G) The right harmonic functions

$$f(X)\bar{\vartheta} = 0 \Rightarrow \left\{ \begin{aligned} \frac{\partial U_0}{\partial X_0} &= \sum_i \frac{\partial U_i}{\partial X_i} \\ \frac{\partial U_0}{\partial X_k} + \frac{\partial U_0}{\partial X_k} &= -\sum_{ij} \epsilon^{ijk} \frac{\partial U_i}{\partial X_j} \end{aligned} \right. \tag{12}$$

Class H) The left and right harmonic functions

$$\bar{\vartheta}f(X) = 0 \text{ and } f(X)\bar{\vartheta} = 0 \Rightarrow \left\{ \begin{aligned} \frac{\partial U_0}{\partial X_0} &= \sum_i \frac{\partial U_i}{\partial X_i} \\ \frac{\partial U_i}{\partial X_0} &= \frac{\partial U_0}{\partial X_i} \\ \frac{\partial U_i}{\partial X_j} &= \frac{\partial U_j}{\partial X_i} \end{aligned} \right. \tag{13}$$

Notice that for the classes F, G and H we have

$$\sum \delta^{ij} \frac{\partial^2 U_0}{\partial X_i \partial X_j} + \frac{\partial^2 U_0}{\partial X_0^2} = \square^2 U_0 = 0$$

where $\square^2 = \bar{\vartheta}\bar{\vartheta}$. Similarly, $\square^2 U_k = 0$, so that those classes describe harmonic functions in the sense that $\square^2 f(X) = 0$.

A non trivial example of class H quaternion function is given by a instantons field expressed in terms of quaternions [23]. The connection of an anti self dual $SU(2)$ gauge field is given by the form

$$\omega = \sum_{\alpha} A_{\alpha(X)} dx^{\alpha} \tag{14}$$

where $A_0 = \sum U_k e^k$ and $A_k = U_0 e^k - \epsilon_{ijk} U_i e^j$ and where

$$U_0 = \frac{\frac{1}{2} X_0}{1 + |X|^2}, \quad U_i = \frac{-\frac{1}{2} X_i}{1 + |X|^2}$$

are the components of the quaternion function $f(X) = U_{\alpha} e^{\alpha}$. We can easily see that $f(X)$ satisfy the conditions (13) in the region of space-time defined by $\sum X_i^2 = -2X_0$. In fact, this is a particular case of a wider class of quaternion functions with components $U_{\alpha} = g_{\alpha}(X)/(1 + |X|^2)$, where $g_{\alpha}(X)$ are some real functions. The case of instantons correspond to the choice $g_0 = \frac{-1}{2} \sum X_i^2$ and $g_i = \frac{\partial g_0}{\partial X_i}$. It is also interesting to notice that the anti instantons do not belong to the same class of analyticity as the instantons.

V. INTEGRAL THEOREMS

Given a quaternion function $f(X)$ defined on a orientable 3-dimensional hypersurface S with, unit normal vector η we may define two integrals

$$\int_S f(X) dS_{\eta}, \text{ and } \int_S dS_{\eta} f(X)$$

where $dS_{\eta} = \sum dS_i e^i$ denotes the quaternion hypersurface element with components

$$\begin{aligned}
dS_0 &= dX_1 dX_2 dX_3, \quad dS_1 = dX_0 dX_2 dX_3, \\
dS_2 &= dX_0 dX_1 dX_3, \quad dS_3 = dX_0 dX_1 dX_2.
\end{aligned}$$

On the other hand, denoting by $dv = dX_0 dX_1 dX_2 dX_3$ the 4-dimensional volume element in a region Ω bounded by S , we obtain after integrating in one of the variables we obtain

$$\begin{aligned}
\int_{\Omega} \bar{\vartheta}f(X) dv &= \int_{\Omega} e^{\alpha} \partial_{\alpha} e^{\beta} U_{\beta} dv = \\
\int_{\Omega} [(\partial_0 U_0 - \sum_i \partial_i U_i) + \sum_i (\partial_0 U_i + \partial_i U_0) e^i + \epsilon^{ijk} \partial_i U_j e^k] dv
\end{aligned}$$

and noting that

$$\begin{aligned}
\int_{\Omega} \partial_0 U_0 dv &= \int_S U_0 dS_0, \quad \int_{\Omega} \partial_0 U_i dv = \int_S U_i dS_0, \\
\int_{\Omega} \partial_i U_0 dv &= \int_S U_0 dS_i, \quad \int_{\Omega} \partial_i U_j dv = \int_S U_j dS_i
\end{aligned}$$

it follows that

$$\begin{aligned}
\int_{\Omega} \bar{\vartheta}f(X) dv &= \int_S [(U_0 dS_0 - \sum \delta^{ij} U_i dS_j) e^0 \\
&+ \sum (U_i dS_0 + U_0 dS_i) e^i - \sum \epsilon^{ijk} U_i dS_j e^k]
\end{aligned}$$

It is a simple matter to see that this is exactly the same expression of the surface integral

$$\int_S dS_{\eta} f(X) = \sum \int_S U_{\alpha} dS_{\beta} e^{\beta} e^{\alpha}$$

Therefore, we obtain the result

$$\int_{\Omega} \not\partial f(X) dv = \int_S dS_{\eta} f(X) \quad (15)$$

and similarly we obtain for the left hypersurface integral

$$\int_{\Omega} f(X) \not\partial dv = \int_S f(X) dS_{\eta} \quad (16)$$

The above integrals hold for any of the previously defined classes of functions and they difference is

$$\sum \epsilon^{ijk} e^k \int_S (U_i dS_j - U_j dS_i) = - \sum \epsilon^{ijk} e^k \int_{\Omega} \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} \right) dv$$

which vanish on account of Green's theorem in the (i, j) plane. The following result extends the first Cauchy's Theorem for quaternion functions:

If $f(X)$ is of H class in the interior of a region Ω bounded by a hypersurface S then

$$\int_S f(X) dS_{\eta} = \int_S dS_{\eta} f(X) = 0 \quad (17)$$

This follows immediately from eqns. (15), (16) and the condition for a class H function (13) where $\not\partial f(X) = 0$ and $f(X) \not\partial = 0$.

The second Cauchy's theorem is also true only for class H functions

If $f(X)$ satisfy the conditions of class H , in a region bounded by a simple closed 3-dimensional hypersurface S , then for $P \in S$,

$$f(P) = \frac{1}{\pi^2} \int_S f(X) (X - P)^{-3} dS_{\eta} \quad (18)$$

In fact, the integrand does not satisfy the class H conditions in Ω as it is not defined at P and consequently the previous theorem does not apply. However this point may be isolated by a sphere with surface S_0 with center at P and radius ϵ such that it is completely inside Ω . Applying the previous theorem in the region bounded by S and S_0 we obtain

$$\int_S f(X) (X - P)^{-3} dS_{\eta} + \int_{S_0} f(X) (X - P)^{-3} dS_{\eta} = 0$$

Now, the primary condition for a function belonging to class E through H is that its components are regular so that we may calculate their Taylor series around P : $U_{\alpha}(X) = U_{\alpha}(P) + \epsilon^{\beta} \frac{\partial U_{\alpha}}{\partial x^{\beta}} \Big|_P + \dots$. Using this expansion in the integral over S_0 and taking the limit $\epsilon \rightarrow 0$, it follows that

$$f(P) = \left(\int_S f(X) (X - P)^{-3} dS_{\eta} \right) \left(\int_{S_0} (X - P)^{-3} dS_{\eta} \right)^{-1} \quad (19)$$

In order to calculate the integral over the sphere it is convenient to use four dimensional spherical coordinates $(r, \theta, \phi, \gamma)$, such that $X_0 = r \sin \gamma$, $X_1 = r \cos \gamma \sin \theta \cos \phi$,

$X_2 = r \cos \gamma \sin \theta \sin \phi$ and $X_3 = r \cos \gamma \cos \theta$ where $\theta \in (0, \pi)$, $\phi \in (0, 2\pi)$, $\gamma \in (-\pi/2, \pi/2)$. With this, the coordinates X_0, X_1, X_2, X_3 correspond to the coordinates of a space-time point with quaternion norm $|X|^2 = r^2$, while γ span values from the past to the future. Then the volume element is $dv = J dr d\theta d\phi d\gamma$ where $J = -\epsilon^3 \cos^2 \gamma \sin \theta$ is the Jacobian determinant. Using the polar form, the unit normal to the sphere centered at P can be written as $\eta = e^{\iota \gamma}$ and $X - P = \epsilon e^{\iota \gamma} = \epsilon \eta$, so that

$$\int_{S_0} (X - P)^{-3} dS_{\eta} = \int_{S_0} e^{-2\iota \gamma} \sin^2 \gamma \sin \theta d\theta d\phi d\gamma = \pi^2$$

After replacing in (19) we obtain the result (18).

Notice that the power (-3) in (18) is not accidental as it is the right power required to cancel the Jacobian determinant as $\epsilon \rightarrow 0$.

VI. POWER SERIES

To conclude, consider the particular function $f(X) = (1 - X)^{-3}$, with $|X| < 1$. It is a simple matter to see that it can be expanded as

$$(1 - X)^{-3} = \sum_1^{\infty} \frac{n(n+1)}{2} X^{n-1} = \sum_{m=0}^{\infty} \frac{(m+1)(m+2)}{2} X^m \quad (20)$$

Using this particular case we may prove the following general result for quaternion functions:

Let $f(X)$ be of class H inside a region Ω bounded by a surface S . Then for all X inside Ω there are coefficients a_n such that

$$f(X) = \sum_0^{\infty} a_n (X - Q)^n \quad (21)$$

The proof is a straightforward adaptation from the similar complex theorem. If S_0 is the largest sphere in Ω centered at Q , the integral (18) for a point $P = X$ inside Ω gives

$$f(X) = \frac{1}{\pi^2} \int_S f(X') (X' - Q)^{-3} [1 - (X' - Q)^{-1} (X - Q)]^{-3} dS'_{\eta}$$

Assuming that $|X - Q| < |X' - Q|$ and using (20), the integrand is equivalent to

$$\begin{aligned} & [1 - (X' - Q)^{-1} (X - Q)]^{-3} \\ &= \sum_0^{m=\infty} \frac{(m+1)(m+2)}{2} (X' - Q)^{-m} (X - Q)^m \end{aligned}$$

so that

$$f(X) = \frac{1}{\pi^2} \sum_{m=0}^{\infty} \frac{(m+1)(m+2)}{2} \times \int_{S_0} f(X')(X' - Q)^{-3-m} (X - Q)^m dS'_\eta \quad (22)$$

Now we may write $(X - Q)^m = \epsilon^m e^{m\iota\gamma}$ and $dS'_\eta = e^{\iota\gamma} dS'$, and it follows that

$$(X - Q)^m dS'_\eta - dS'_\eta (X - Q)^m = \epsilon dS' (e^{m\iota\gamma} e^{\iota\gamma} - e^{\iota\gamma} e^{m\iota\gamma}) = 0$$

Therefore (22) is equivalent to

$$f(X) = \frac{1}{\pi^2} \sum_{m=0}^{\infty} \frac{(m+1)(m+2)}{2} \times \int_{S_0} f(X')(X' - Q)^{-3-m} dS_\eta (X - Q)^m$$

or, after defining the coefficients

$$a_m = \frac{1}{\pi^2} \frac{(m+1)(m+2)}{2} \int_{S_0} f(X')(X' - Q)^{-3-m} dS'_\eta \quad (23)$$

we obtain (21).

This important result shows that class H functions can be expressed as a convergent positive power series. Therefore the class H or the latter property could be taken to represent a class of analyticity for quaternion functions, in the same sense of the real and complex analyticity. However, unlike the complex case from (13) we see that their derivatives depend on the direction in which the limit is taken.

VII. DISCUSSION

At the level of an unbroken $SU(2)$ gauge theory the algebraic properties of the spinors predicts a combined spin-isospin angular momentum here called the k-spin (after i-spin for complex and j-spin for isospin). The three resulting spinors give way to a full quaternionic spinor operator obtained from the linear combination of the Pauli matrices in a specific representation. In this way, we conclude that quaternion quantum mechanics may be effective at the level of the combined spinor symmetry. The quaternionic spinor operator naturally associates a vector in space-time whose physical interpretation depends on the $SU(2)$ model of gauge field considered. We have suggested the 't Hooft-Poliakov monopole as a possible interpretation of that vector field.

The emergence of quaternionic spinor fields requires a proper analysis of quaternion functions of a quaternionic variable. We have shown that there is a class of analytic quaternion functions which can be represented by a positive power series, a property which is shared with the other two associative division algebras (the real and complex functions). Outside the conditions for class H

the the power expansions would also have negative powers and the associated poles as points in space-time and their corresponding residues [19].

The harmonic property implicit in class H implies in the possibility that the quaternion quantum fields and states can be represented in terms of quaternionic Fourier expansions, something that is required to represent the quaternion wave packets. As it has been noted, quaternion analyticity does not imply that the derivatives are independent of direction in space and in this respect complex analysis and the corresponding quantum theory may be considered to be somewhat limited as compared with quaternion analysis. The direction dependent property should be detectable at the level of the combined symmetry

It is conceivable that the characterization of analyticity either by class H or more generally by positive power series expansions will not hold at higher energy levels, where the wave functions are subjected to fast variation at a sort time. In this case, the best we may hope that these functions remain differentiable and any appeal to analyticity in the sense of a converging power series may be regarded as an unduly luxury. In this respect, the above results may hold for quaternions quantum mechanics at an intermediate energy theory, where the combined symmetry includes the $SU(2)$ group. For higher energies we would expect the emergence of the $SU(3)$ group and the octonion algebra.

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- [1] F. Jordan, J. von Neumann & E. P. Wigner, Ann. Maths. **35**, 29 (1934)
 - [2] D. Finkelstein, *Notes on Quaternion Quantum Mechanics*, Preprint CERN-Th 59-9 (1959), D. Finkelstein et al, Jour. Math. Phys. **3**, 207 (1962), ibid **4**, 788, (1963)
 - [3] S. L. Adler, *Quaternion Quantum Mechanics and Quantum Fields*, Oxford U. P. (1995). Updates can be found in the Internet site <http://www.sns.ias.edu/~adler/Htmlquaternionic.html>
 - [4] A. Pais, Phys. Rev. Lett. **7**, 291, (1961)
 - [5] J. Tiomno, Theor. Phys. Intl. (published by the Atomic Energy Agency, Vienna) **pag. 251**, (1963)
 - [6] C. G. Oliveira & M. D. Maia, Jour. Math. Phys. **20**, 923, (1979)
 - [7] S. P. Brumby et al *Implications of quaternionic dark matter*. University of Melbourne UM-P-96/90 (1996)
 - [8] A. Peres, Phys. Rev. Lett. **42**, 683 (1979)
 - [9] C. Chevalley *The Construction and Study of Certain Important Algebras*, The Mathematical Soc. of Japan (1955)
 - [10] R. Jackiw & C. Rebbi Phys. Rev. Lett. **36**, 1116, (1976)
 - [11] P. Hazenfrantz & G. 't Hooft, Phys. Rev. Lett. **36**, 1119 (1976)
 - [12] T. Vachaspati *Spin from isospin in purely bosonic...* Preprint, Western reserve University CWU-P11-97 (1997), hep-th 9709119.

- [13] D. Singleton, Int. Jour. Theor. Phys. **34**, 2453 (1995).
- [14] G. G. Emch & A. Z. Jadczyk, *On Quaternions and Monopoles* Florida University (1998), quant -ph 9803002.
- [15] E. J. Schremp, NRL Quaterly on Nuclear Science & Technology **October, 7**, (1967)
- [16] R. Fueter, Comment. Math. Helv. **4**, 9, (1932), ibid **9**, 320, (1936) **10**, 306, (1937)
- [17] P. N. Ketchum, Am. Math. Soc. Trans. **30**, 641, (1928)
- [18] C. A. Ferraro, Proc. Roy. Irish. Acad. **44**, 101, (1938).
- [19] M. D. Maia Bol. Soc. Bras. Mat. **3**, 57, (1972). *errata available by request*
- [20] C. Nash & C. G. Joshi, Jour. Math Phys. **28**, 463 (1987)
- [21] M. Evans, F. Gursev & V. Ogievetsky, *From 2D conformal to 4D self-dual theories: quaternionic analyticity* CERN-TH.6533/92 hep-th 9207089 (1992).
- [22] Khaled Abdel-Khalek *Quaternion Analysis*, University of Lecce (1996), gr-qc 9607152.
- [23] M. F. Atiah, *Geometry of the Yang-Mills Fields*. Academia Nazionale dei Licei, Scuola Normale Superiore, Pisa (1979).