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Research Article

On Entire and Meromorphic Functions That Share One Small Function with Their Differential Polynomial

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We study the uniqueness of meromorphic functions that share one small function with more general differential polynomial $P[f]$. As corollaries, we obtain results which answer open questions posed by Yu (2003).

1. Introduction and Main Results

In this paper, a meromorphic functions mean meromorphic in the whole complex plane. We use the standard notations of Nevanlinna theory (see [1]). A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r,a) = S(r,f)$, that is, $T(r,a) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of finite linear measure. If $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with same multiplicities (ignoring multiplicities), then we say that $f(z)$ and $g(z)$ share $a(z)$ CM (IM).

For any constant a, we denote by $N_{k)}(r, 1/(f-a))$ the counting function for zeros of $f(z)-a$ with multiplicity no more than k and $\overline{N}_{k)}(r, 1/(f-a))$ the corresponding for which multiplicity is not counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for zeros of $f(z)-a$ with multiplicity at least k and $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding for which the multiplicity is not counted.

Let f and g be two nonconstant meromorphic functions sharing value 1 IM. Let z_0 be common one point of f and g with multiplicity p and q, respectively. We denote by $N_L(r, 1/(f-1))$ ($\overline{N}_L(r, 1/(f-1))$) the counting (reduced) function of those 1 points of f where $p > q$; by $N_E^{1)}(r, 1/(f-1))$ the counting function of those 1-points of f where $p = q = 1$; by $N_E^{(2}(r, 1/(f-1))$ the counting function of those 1-points of f where $p = q \geq 2$. In the same way, we can define $N_L(r, 1/(g-1))$, $N_E^{1)}(r, 1/(g-1))$ and $N_E^{(2}(r, 1/(g-1))$ (see [2]).

In 1996, Brück [3] posed the following conjecture.

Conjecture 1. *Let f be a nonconstant entire function such that the hyper-order $\sigma_2(f)$ of f is not a positive integer and $\sigma_2(f) < \infty$. If f and f' share a finite value a CM, then $(f'-a)/(f-a) = c$, where c is a nonzero constant.*

In [3], under an additional hypothesis, Brück proved that the conjecture holds when $a = 1$.

Theorem A. *Let f be a nonconstant entire function. If f and f' share the value 1 CM and if $N(r, 1/f') = S(r,f)$, then $(f'-1)/(f-1) = c$, for some constant $c \in \mathbb{C} \setminus \{0\}$.*

Many people extended this theorem and obtained many results. In 2003, Yu [4] proved the following theorem.

Theorem B. *Let $k \geq 1$. Let f be a nonconstant meromorphic function and $a(z)$ a meromorphic function such that $a(z) \not\equiv 0, \infty$, f and a do not have any common pole and $T(r,a) = o(T(r,f))$ as $r \to \infty$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and*

$$4\delta(0,f) + 2(8+k)\Theta(\infty,f) > 19 + 2k, \tag{1}$$

then $f \equiv f^{(k)}$.

Theorem C. *Let $k \geq 1$. Let f be a nonconstant entire function and $a(z)$ be a meromorphic function such that $a(z) \not\equiv 0, \infty$ and $T(r,a) = o(T(r,f))$ as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and*

$$\delta(0,f) > \frac{3}{4}, \quad (2)$$

then $f \equiv f^{(k)}$.

In the same paper, the author posed the following questions.

Question 1. Can a CM shared value be replaced by an IM shared value in Theorem C?

Question 2. Is the condition $\delta(0,f) > 3/4$ sharp in Theorem C?

Question 3. Is the condition $4\delta(0,f) + 2(8+k)\Theta(\infty,f) > 19 + 2k$ sharp in Theorem B?

In 2004, Liu and Gu [5] applied different method and obtained the following theorem which answers some questions posed in [4].

Theorem D. *Let $k \geq 1$. Let f be a nonconstant meromorphic function and $a(z)$ a meromorphic function such that $a(z) \not\equiv 0, \infty$ and $T(r,a) = S(r,f)$ as $r \to \infty$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $f^{(k)}$ and $a(z)$ do not have any common poles of same multiplicity and*

$$2\delta(0,f) + 4\Theta(\infty,f) > 5, \quad (3)$$

then $f \equiv f^{(k)}$.

Theorem E. *Let $k \geq 1$. Let f be a nonconstant entire function and $a(z)$ a meromorphic function such that $a(z) \not\equiv 0, \infty$ and $T(r,a) = S(r,f)$ as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and*

$$\delta(0,f) > \frac{1}{2}, \quad (4)$$

then $f \equiv f^{(k)}$.

Recently, Zhang and Lü [6] considered the problem of meromorphic functions sharing one small function with its kth derivative and proved the following theorem.

Theorem F. *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f a nonconstant meromorphic function. Also let $a(z) \not\equiv 0, \infty$ be a small meromorphic function with respect to f. If f^n and $f^{(k)}$ share the value $a(z)$ IM and*

$$(2k+6)\Theta(\infty,f) + 4\Theta(0,f) + 2\delta_{k+2}(0,f) > 2k + 12 - n \quad (5)$$

or f^n and $f^{(k)}$ share the value $a(z)$ CM and

$$(k+3)\Theta(\infty,f) + 2\Theta(0,f) + \delta_{k+2}(0,f) > k + 6 - n, \quad (6)$$

then $f \equiv f^{(k)}$.

Regarding these results, a natural question is what can be said when a nonconstant meromorphic function f shares one nonzero small meromorphic function $a(z)$ with $P[f]$, where $P[f]$ is a differential polynomial in f.

Definition 2. Any expression of the type

$$P[f] = \sum_{i=1}^{n} \alpha_i(z) f^{n_{i_0}} (f')^{n_{i_1}} (f'')^{n_{i_2}} \cdots (f^{(m)})^{n_{i_m}} \quad (7)$$

is called differential polynomial in f of degree $\overline{d}(P)$, lower degree $\underline{d}(P)$, and weight Γ_P, where $n_{i_0}, n_{i_1}, \ldots, n_{i_m}$ are nonnegative integers, $\alpha_i = \alpha_i(z)$ are meromorphic functions satisfying $T(r,\alpha_i) = S(r,f)$ and

$$\overline{d}(P) = \max\left\{\sum_{j=0}^{m} n_{ij} : 1 \leq i \leq n\right\},$$

$$\underline{d}(P) = \min\left\{\sum_{j=0}^{m} n_{ij} : 1 \leq i \leq n\right\}, \quad (8)$$

$$\Gamma_P = \max\left\{\sum_{j=0}^{m} (j+1) n_{ij} : 1 \leq i \leq n\right\}.$$

Further, if $\overline{d}(P) = \underline{d}(P) = n$ (say), then the differential polynomial $P[f]$ is called a homogeneous differential polynomial in f of degree n.

Correspond to the above question, we obtain the following results, which extend and improve Theorems A–F and give answers to the questions possed by Yu [4] for more general differential polynomial.

Theorem 3. *Let f be a nonconstant meromorphic function and $a(z)$ be a small meromorphic function such that $a(z) \not\equiv 0, \infty$. P[f] be a nonconstant differential polynomial in f as defined in (7). If f and $P[f]$ share the value a IM and*

$$(2Q+6)\Theta(\infty,f) + (2 + 3\underline{d}(P))\delta(0,f) > 2Q + 2\underline{d}(P) + \overline{d}(P) + 7, \quad (9)$$

then $f \equiv P[f]$.

Remark 4. Taking $P[f] = f^{(k)}$, that is, $Q = k$, $\overline{d}(P) = \underline{d}(P) = 1$ in (9), we get $(2k+6)\Theta(\infty,f) + 5\delta(0,f) > 2k + 10$, which improves (5) and extends the theorem to more general differential polynomial $P[f]$ as defined in (7).

Theorem 5. *Let f be a nonconstant meromorphic function and $a(z)$ be a small meromorphic function such that $a(z) \not\equiv 0, \infty$. Let $P[f]$ a nonconstant differential polynomial in f as defined in (7). If f and $P[f]$ share the value a CM and*

$$3\Theta(\infty,f) + (\underline{d}(P) + 1)\delta(0,f) > 4, \quad (10)$$

then $f \equiv P[f]$.

Remark 6. Taking $P[f] = f^{(k)}$, that is, $Q = k$, $\overline{d}(P) = \underline{d}(P) = 1$ in (10), we get $3\Theta(\infty, f) + 2\delta(0, f) > 4$, which improves (6) and extends the theorem to more general differential polynomial $P[f]$ as defined in (7).

Remark 6 gives answer to Question 3 of [4].

Theorem 7. *Let f be a nonconstant entire function and $a(z)$ a small meromorphic function such that $a(z) \not\equiv 0, \infty$. Let $P[f]$ be a nonconstant differential polynomial in f as defined in* (7) *If f and $P[f]$ share the value a IM and*

$$\left(3\underline{d}(P) + 2\right)\delta(0, f) > 2\overline{d}(P) + 2, \tag{11}$$

then $f \equiv P[f]$.

Remark 8. Taking $P[f] = f^{(k)}$, that is, $Q = k$, $\overline{d}(P) = \underline{d}(P) = 1$ in (11), we get

$$\delta(0, f) > \frac{4}{5}. \tag{12}$$

Remark 8 gives answer to Question 1 of Yu [4].

Theorem 9. *Let f be a nonconstant entire function and $a(z)$ be a small meromorphic function such that $a(z) \not\equiv 0, \infty$. $P[f]$ be a nonconstant differential polynomial in f as defined in* (7). *If f and $P[f]$ share the value a CM and*

$$\left(\underline{d}(P) + 1\right)\delta(0, f) > 1, \tag{13}$$

then $f \equiv P[f]$.

Remark 10. Taking $P[f] = f^{(k)}$, that is, $Q = k$, $\overline{d}(P) = \underline{d}(P) = 1$ in (13), we get $\delta(0, f) > 1/2$, which improves Theorem C and extends the theorem to more general differential polynomial $P[f]$ as defined in (7).

Remark 10 gives answer to Question 2 of Yu [4].

Remark 11. By proving Remarks 6, 8, and 10 we have answered Questions 3, 1, and 2 (of [4]), respectively, for the case $f^{(k)}$. Theorems 3–9 improve and generalize Theorems A–F for more general differential polynomial $P[f]$.

2. Lemma

Lemma 12 (see [7]). *Let f ba a meromorphic function and $P[f]$ be a differential polynomial in f. Then*

$$\begin{aligned}
& m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \\
&\quad \leq \left(\overline{d}(P) - \underline{d}(P)\right) m\left(r, \frac{1}{f}\right) + S(r, f), \\
& m\left(r, \frac{P[f]}{f^{\underline{d}(P)}}\right) \\
&\quad \leq \left(\overline{d}(P) - \underline{d}(P)\right) m(r, f) + S(r, f), \\
& N\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq \left(\overline{d}(P) - \underline{d}(P)\right) N\left(r, \frac{1}{f}\right) \\
&\qquad + Q\left[\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + S(r, f), \\
& N(r, P[f] \leq \overline{d}(P) N(r, f) + Q\overline{N}(r, f) + S(r, f), \\
& T(r, P(f)) \leq Q\overline{N}(r, f) + \overline{d}(P) T(r, f) + S(r, f),
\end{aligned} \tag{14}$$

where $Q = \max\{n_{i_1} + 2n_{i_2} + 3n_{i_3} + \cdots + mn_{i_m}; 1 \leq i \leq n\}$.

Lemma 13 (see [8]). *Let f be a nonconstant meromorphic function, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f) \tag{15}$$

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \tag{16}$$

Lemma 14 (see [9]). *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right), \tag{17}$$

where F and G are two nonconstant meromorphic functions. If F and G share 1 IM and $H \not\equiv 0$, then

$$N_E^{1)}\left(r, \frac{1}{F - 1}\right) \leq N(r, H) + S(r, F) + S(r, G). \tag{18}$$

Lemma 15. *Let f be a transcendental meromorphic function. Let $P[f]$ be defined as in (7). If $P[f] \not\equiv 0$, we have*

$$\begin{aligned} N\left(r, \frac{1}{P[f]}\right) &\le T(r, P[f]) - T\left(r, f^{\overline{d}(P)}\right) \\ &\quad + \left(\overline{d}(P) - \underline{d}(P)\right) m\left(r, \frac{1}{f}\right) \\ &\quad + N\left(r, \frac{1}{f^{\overline{d}(P)}}\right) + S(r, f), \end{aligned} \tag{19}$$

$$\begin{aligned} N\left(r, \frac{1}{P[f]}\right) &\le Q\overline{N}(r, f) + \left(\overline{d}(P) - \underline{d}(P)\right) m\left(r, \frac{1}{f}\right) \\ &\quad + N\left(r, \frac{1}{f^{\overline{d}(P)}}\right) + S(r, f). \end{aligned} \tag{20}$$

Proof. By the first fundamental theorem, we have

$$N\left(r, \frac{1}{P[f]}\right) = T(r, P[f]) - m\left(r, \frac{1}{P[f]}\right) + O(1). \tag{21}$$

We have

$$\begin{aligned} m\left(r, \frac{1}{f^{\overline{d}(P)}}\right) &\le m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + m\left(r, \frac{1}{P[f]}\right) \\ m\left(r, \frac{1}{f^{\overline{d}(P)}}\right) - m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) &\le m\left(r, \frac{1}{P[f]}\right) \\ -m\left(r, \frac{1}{f^{\overline{d}(P)}}\right) + m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) &\ge -m\left(r, \frac{1}{P[f]}\right) \end{aligned} \tag{22}$$

or

$$-m\left(r, \frac{1}{P[f]}\right) \le -m\left(r, \frac{1}{f^{\overline{d}(P)}}\right) + m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right). \tag{23}$$

By (21), (23) and Lemma 12, we obtain (19).

Since

$$\begin{aligned} T(r, P[f]) &= m(r, P[f]) + N(r, P[f]) \\ &\le m\left(r, \frac{P[f]}{f^{\underline{d}(P)}}\right) + m\left(r, f^{\underline{d}(P)}\right) + N(r, P[f]) \\ &\le \left(\overline{d}(P) - \underline{d}(P)\right) m(r, f) + \underline{d}(P)\, m(r, f) \\ &\quad + \overline{d}(P)\, N(r, f) + Q\overline{N}(r, f) + S(r, f) \\ &\le \overline{d}(P)\, m(r, f) + \overline{d}(P)\, N(r, f) \\ &\quad + Q\overline{N}(r, f) + S(r, f), \end{aligned} \tag{24}$$

we get

$$T(r, P[f]) \le \overline{d}(P)\, T(r, f) + Q\overline{N}(r, f) + S(r, f). \tag{25}$$

Substituting (25) in (19), we obtain (20). □

Lemma 16 (see [10]). *Let f be a transcendental meromorphic function, $P[f]$ a differential polynomial in f of degree $\overline{d}(P)$ and weight Γ_P. Then $T(r, P) = O(T(r, f))$, $S(r, P) = S(r, f)$.*

3. Proof of Theorems

Proof of Theorem 3. Let

$$F = \frac{P[f]}{a}, \qquad G = \frac{f}{a}. \tag{26}$$

From the conditions of Theorem 3, we know that F and G share 1 IM. From (26), we have

$$\begin{aligned} T(r, F) &= O(T(r, f)) + S(r, f), T(r, G) \\ &\le T(r, f) + S(r, f), \end{aligned} \tag{27}$$

$$\overline{N}(r, F) = \overline{N}(r, G) + S(r, f), \tag{28}$$

$$\begin{aligned} \overline{N}(r, F) &= \overline{N}(r, f) + S(r, f), \overline{N}(r, G) \\ &= \overline{N}(r, f) + S(r, f), \end{aligned} \tag{29}$$

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) = N_E^{1)}\left(r, \frac{1}{G-1}\right) + S(r, f), \tag{30}$$

$$\overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right) = \overline{N}_E^{(2}\left(r, \frac{1}{G-1}\right) + S(r, f), \tag{31}$$

$$\overline{N}_L\left(r, \frac{1}{F-1}\right) \le \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, F) + S(r, f), \tag{32}$$

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F-1}\right) &= \overline{N}\left(r, \frac{1}{G-1}\right) + S(r, f) \\ &\le N_E^{1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right) \\ &\quad + N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + S(r, f). \end{aligned} \tag{33}$$

Let H be defined by (17). Suppose that $H \not\equiv 0$. By Lemma 14, (18) holds.

From (17) and (28), we have

$$\begin{aligned} N(r, H) &\le N_{(2}\left(r, \frac{1}{F}\right) + N_{(2}\left(r, \frac{1}{G}\right) + \overline{N}(r, G) \\ &\quad + N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) \\ &\quad + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right), \end{aligned} \tag{34}$$

where $N_0(r, 1/F')$ denotes the counting function corresponding to the zeros of F' which are not the zeros of F and $F - 1$. Similarly, $N_0(r, 1/G')$ is defined.

From the second fundamental theorem, we have

$$\begin{aligned} &T(r,F)+T(r,G)\\ &\quad\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)\\ &\quad+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\ &\quad-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{35}$$

Since F and G share 1 IM, we get from (33):

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\ &\quad=2N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{F-1}\right)\\ &\quad+2N_L\left(r,\frac{1}{G-1}\right)+2\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right). \end{aligned} \tag{36}$$

From this, (18), and (34), we have

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\ &\quad\le N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+N_0\left(r,\frac{1}{F'}\right)\\ &\quad+3N_L\left(r,\frac{1}{G-1}\right)+3N_L\left(r,\frac{1}{F-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)\\ &\quad+2\overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{37}$$

It is clear that

$$\begin{aligned} &N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)\\ &\quad+2\overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)\\ &\quad\le N\left(r,\frac{1}{G-1}\right)\\ &\quad\le T(r,G)+O(1). \end{aligned} \tag{38}$$

Combining (37), and (38), we obtain

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\ &\quad\le N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)\\ &\quad+2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)+T(r,G)\\ &\quad+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{39}$$

Substituting (39) in (35) and using (28), we obtain

$$\begin{aligned} T(r,F)&\le 3\overline{N}(r,G)+N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)\\ &\quad+2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)+S(r,f). \end{aligned} \tag{40}$$

Using (26) and (19), we get

$$\begin{aligned} &\overline{d}(P)\,T(r,f)\\ &\quad\le 3\overline{N}(r,G)+\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)\\ &\quad+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{\overline{d}(P)}}\right)\\ &\quad+2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)+S(r,f). \end{aligned} \tag{41}$$

From (16), (20), and (26) we have

$$\begin{aligned} &2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)\\ &\quad\le 2N\left(r,\frac{1}{F'}\right)+N\left(r,\frac{1}{G'}\right)\\ &\quad\le 2\left[N\left(r,\frac{1}{F}\right)+\overline{N}(r,F)\right]+N\left(r,\frac{1}{f}\right)\\ &\quad+\overline{N}(r,f)+S(r,f)\\ &\quad\le 2N\left(r,\frac{1}{P[f]}\right)+3\overline{N}(r,f)\\ &\quad+N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\quad\le 2Q\overline{N}(r,f)+2N\left(r,\frac{1}{f^{\overline{d}(P)}}\right)\\ &\quad+2\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)\\ &\quad+3\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+S(r,f)\\ &\quad\le (2Q+3)\,\overline{N}(r,f)+\left(2\overline{d}(P)+1\right)N\left(r,\frac{1}{f}\right)\\ &\quad+2\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned} \tag{42}$$

From (41) and (42), we get

$$\begin{aligned}
&\overline{d}(P)T(r,f)\\
&\quad\le (2Q+6)\overline{N}(r,f)+\left(2+3\overline{d}(P)\right)N\left(r,\frac{1}{f}\right)\\
&\qquad+3\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)+S(r,f)\\
&\quad\le (2Q+6)\overline{N}(r,f)+(2+3\underline{d}(P))N\left(r,\frac{1}{f}\right)\\
&\qquad+3\left(\overline{d}(P)-\underline{d}(P)\right)T\left(r,\frac{1}{f}\right)+S(r,f)\\
&\left(3\underline{d}(P)-2\overline{d}(P)\right)T(r,f)\\
&\quad\le (2Q+6)\overline{N}(r,f)+(2+3\underline{d}(P))N\left(r,\frac{1}{f}\right)+S(r,f)\\
&\quad\le \{(2Q+6)(1-\Theta(\infty,f))+(2+3\underline{d}(P))(1-\delta(0,f))\}\\
&\qquad\times T(r,f)+S(r,f)\\
&\quad\le \{(2Q+3\underline{d}(P)+8)-[(2Q+6)\Theta(\infty,f)\\
&\qquad\quad+(2+3\underline{d}(P))\delta(0,f)]\}T(r,f)+S(R,f).
\end{aligned}\tag{43}$$

Therefore, we have

$$\begin{aligned}
&\Big\{(2Q+6)\Theta(\infty,f)+(2+3\underline{d}(P))\delta(0,f)\\
&\quad-\left(2Q+2\overline{d}(P)+8\right)\Big\}T(r,f)\le S(r,f),
\end{aligned}\tag{44}$$

which is a contradiction to our hypothesis (9).

Thus $H\equiv 0$. By integration, we get from (17) that

$$\frac{1}{G-1}=\frac{A}{F-1}+B,\tag{45}$$

where $(A\neq 0)$ and B are constants. Thus

$$\begin{aligned}
G&=\frac{(B+1)F+(A-B-1)}{BF+(A-B)},\\
F&=\frac{(B-A)G+(A-B-1)}{BG-(B+1)}.
\end{aligned}\tag{46}$$

We discuss the following three cases.

Case 1. Suppose that $B\neq 0,-1$. From (46), we have

$$\overline{N}\left(r,\frac{1}{G-(B+1)/B}\right)=\overline{N}(r,F).\tag{47}$$

From this and second fundamental theorem, we have

$$\begin{aligned}
T(r,f)&\le T(r,G)+S(r,f)\\
&\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)\\
&\quad+\overline{N}\left(r,\frac{1}{G-(B+1)/B}\right)+S(r,f)\\
&\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+S(r,f)\\
&\le 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\
&\le (2Q+6)\overline{N}(r,f)+(2+3\underline{d}(P)N\left(r,\frac{1}{f}\right)\\
&\quad+S(r,f)\\
&\le \{(2Q+3\underline{d}(P)+8)-[(2Q+6)\Theta(\infty,f)\\
&\quad+(2+3\underline{d}(P))\delta(0,f)]\}T(r,f)+S(R,f).
\end{aligned}\tag{48}$$

Therefore, we have

$$\begin{aligned}
&\{(2Q+6)\Theta(\infty,f)+(2+3\underline{d}(P))\delta(0,f)\\
&\quad-(2Q+3\underline{d}(P)+7)\}T(r,f)\le S(r,f),
\end{aligned}\tag{49}$$

which is a contradiction to our hypothesis (9).

Case 2. Suppose that $B=0$, From (46), we get

$$G=\frac{F+(A-1)}{A},\qquad F=AG-(A-1),\tag{50}$$

we claim $A=1$.

If $A\neq 1$ from (50), we obtain

$$N\left(r,\frac{1}{G-(A-1)/A}\right)=N\left(r,\frac{1}{F}\right).\tag{51}$$

From this, second fundamental theorem, and (20), we have

$$\begin{aligned} T(r,f) &\le T(r,G)+S(r,f)\\ &\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{G-(A-1)/A}\right)+S(r,f)\\ &\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{F}\right)+S(r,f)\\ &\le \overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+Q\overline{N}(r,f)\\ &\quad+\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)\\ &\quad+N\left(r,\frac{1}{f^{\overline{d}(P)}}\right)+S(r,f)\\ &\le (Q+1)\overline{N}(r,f)+\left(1+\underline{d}(P)\right)N\left(r,\frac{1}{f}\right)\\ &\quad+\left(\overline{d}(P)-\underline{d}(P)\right)T\left(r,\frac{1}{f}\right)+S(r,f)\\ &\left(1-\overline{d}(P)+\underline{d}(P)\right)T(r,f)\\ &\quad\le(2Q+6)\overline{N}(r,f)+\left(2+3\underline{d}(P)\right)N\left(r,\frac{1}{f}\right)\\ &\quad+S(r,f)\\ &\quad\le\{(2Q+3\underline{d}(P)+8)-[(2Q+6)\Theta(\infty,f)\\ &\qquad+(2+3\underline{d}(P))\delta(0,f)]\}T(r,f)+S(R,f). \end{aligned} \tag{52}$$

Hence, we have

$$\begin{aligned} &\left\{(2Q+6)\Theta(\infty,f)+(2+3\underline{d}(P))\delta(0,f)\right.\\ &\quad\left.-\left(2Q+2\underline{d}(P)+\overline{d}(P)+7\right)\right\}T(r,f)\le S(r,f), \end{aligned} \tag{53}$$

which is a contradiction to our hypothesis (9)

Thus, $A=1$.

From (50) we have $F\equiv G$.

Therefore, we have $f\equiv P[f]$.

*Case 3.*Suppose that $B=-1$, from (46) we have

$$G=\frac{A}{-F+A+1},\qquad F=\frac{(1+A)G-A}{G}. \tag{54}$$

If $A\neq-1$, we obtain from (54) that

$$N\left(r,\frac{1}{G-A/(A+1)}\right)=N\left(r,\frac{1}{F}\right). \tag{55}$$

By the same argument as in Case 2, we obtain a contradiction. Hence, $A=-1$.

From (54), we get

$$FG\equiv 1, \tag{56}$$

that is,

$$f\cdot P[f]\equiv a^2. \tag{57}$$

From (57), we have

$$N(r,f)+N\left(r,\frac{1}{f}\right)=S(r,f). \tag{58}$$

Using (54), (57), Lemma 12, and first fundamental theorem, we get

$$\begin{aligned} &\left(\overline{d}(P)+1\right)T(r,f)\\ &\quad=T\left(r,\frac{1}{f^{\overline{d}(P)+1}}\right)\\ &\quad=T\left(r,\frac{1}{f^{\overline{d}(P)}f}\right)\\ &\quad=T\left(r,\frac{P[f]}{f^{\overline{d}(P)}a^2}\right)+S(r,f)\\ &\quad=m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+N\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+S(r,f)\\ &\quad\le\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)+\left(\overline{d}(P)-\underline{d}(P)\right)N\left(r,\frac{1}{f}\right)\\ &\qquad+Q\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,f)\\ &\quad\le\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)+S(r,f)\\ &\quad\le\left(\overline{d}(P)-\underline{d}(P)\right)T\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned} \tag{59}$$

From this, we have

$$(\underline{d}(P))+1)T(r,f)\le S(r,f), \tag{60}$$

which is a contradiction. This completes the proof of Theorem 3. □

Proof of Theorem 5. Let F and G be given by (26). From the assumption of Theorem 5, we know that F and G share 1 CM:

$$\overline{N_L}\left(r,\frac{1}{F-1}\right)=\overline{N_L}\left(r,\frac{1}{G-1}\right)=0. \tag{61}$$

Proceeding as in Theorem 3, we obtain (41).

Using (61) in (41), we get

$$\begin{aligned}
&\overline{d}(P)\,T(r,f)\\
&\quad\le 3\overline{N}(r,G)+\left(\overline{d}(P)-\underline{d}(P)\right)m\left(r,\frac{1}{f}\right)\\
&\qquad+N\left(r,\frac{1}{f^{\overline{d}(P)}}\right)+N\left(r,\frac{1}{f}\right)+S(r,f)\\
&\quad\le 3\overline{N}(r,f)+\left(\overline{d}(P)-\underline{d}(P)\right)\left(T(r,f)-N\left(r,\frac{1}{f}\right)\right)\\
&\quad+\left(\overline{d}(P)+1\right)N\left(r,\frac{1}{f}\right)+S(r,f),\\
&\underline{d}(P)\,T(r,f)\\
&\le 3\overline{N}(r,f)+\left(\underline{d}(P)+1\right)N\left(r,\frac{1}{f}\right)+S(r,f)\\
&\le\{(\underline{d}(P)+4)\\
&\quad-[3\Theta(\infty,f)+(\underline{d}(P)+1)\,\delta(0,f)]\}\,T(r,f)+S(r,f).
\end{aligned} \tag{62}$$

We have

$$\{3\Theta(\infty,f)+(\underline{d}(P+1))\,\delta(0,f)-4\}\,T(r,f)\le S(r,f), \tag{63}$$

which contradicts (10).

Thus, $H\equiv 0$. Proceeding as in Theorem 3, we prove Theorem 5. □

Proof of Theorem 7. f is a nonconstant entire function. Taking $N(r,f)=0$ in proof of Theorem 3, we obtain Theorem 7. □

Proof of Theorem 9. f is a nonconstant entire function. Taking $N(r,f)=0$ in proof of Theorem 5, we obtain Theorem 9. □

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