The Myth of the Twin Paradox

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Abstract

One of the most discussed peculiarities of Einstein's theory of relativity is the twin paradox, the fact that the time between two events in space-time appears to depend on the path between these events. We show that this time discrepancy results only from faulty assumptions in the transition from one reference system to another. The twin paradox does not exist. But the Lorentz invariance of the theory has strong consequences, if we assume that it is valid not only locally, but also on cosmic scale.

1 Introduction

No other development in physics has changed our view of the world more than the theory of relativity, introduced by Einstein in 1905. Based on the idea that motion of a body cannot be defined absolutely, but only relative with respect to others, he concluded that the laws of physics should look alike in any two reference systems, moving with respect to each other with constant velocity. Together with the experimentally confirmed fact that the speed of light is independent of the reference frame, this led to the conclusion that time cannot be an absolute quantity, but that time and space constitute a 4-dimensional unit. Temporal and spatial distances both must depend on the reference system.

The equivalence of all inertial reference systems requires that linear motion in one system translates into linear motion in another system. Thus the relation between the coordinates (x, y, z, t) in one system and those in another system (x', y', z', t'), moving with velocity v in the direction x, must be linear. These conditions uniquely define the transformation equations (the Lorentz transformation)

$$x' = \gamma(x - \beta t) \qquad y' = y \qquad z' = z \qquad t' = \gamma(-\beta x + t) \tag{1}$$

where the time variable is calibrated to an equivalent length by the speed of light c ($ct \Rightarrow t$). β is the relative velocity as a fraction of c: $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$. The two systems are synchronised by the condition that at $x = t = 0$ we have $x' = t' = 0$.

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The fact that the time scale changes with distance appears somewhat strange to our intuition, but this can be ascribed to the fact that in every days life we are accustomed only to velocities, which are much less than c, so that relativistic effects are negligible. On the other hand the lack of imagination has led us to believe in mathematically derived consequences which can scarcely be proved by experiments.

One of the most discussed consequences of the theory of relativity is the so called twin paradox, which dates back to first decade after the invention, but is discussed in numerous scientific papers still today. Basis of the twin paradox is the dilatation of time, the fact that moving clocks are slowed down, when observed from the rest system. Thus a clock moving with a considerable fraction of c measures a shorter time to reach a distant target than a clock at rest.

The twin paradox is frequently told with the following story: There are twins Alice and Bob. Alice decides to make a journey to a distant star in a spaceship capable of moving at a considerable fraction of c. When she has reached the star, she goes back at the same speed. According to time dilatation a clock moving with her and consequently also Alice herself ages more slowly than her sibling at home, so that, when she comes back, she finds Bob as an old man, while for herself the journey has taken only a few years.

In this paper we will show that this interpretation of relativity is incorrect and that the twin paradox does not exist at all. We will try to show up, where the mathematical flaws come in, and how they can be corrected. Subsequently we will consider some real effects of relativity, especially with respect to the consequences of Lorentz invariance in accelerated systems, as they are discussed in general relativity.

2 Invariants of the Lorentz transformation

The Lorentz transformation describes, how spatial and temporal distances between events change, when observed from different reference systems, which are in relative motion with respect to each other, with the additional condition that the speed of light is independent of the reference system. The equation $x = t$ transforms into $x' = t'$ (time calibrated by c as in the last section), independent of the direction of the relative motion of the systems.

This property can be read immediately from the transformation equation eq.(1), defining the space-time distance between two events by

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2}. \tag{2}$$

or, if the argument of the square root is negative, the proper time

$$\Delta \tau = \sqrt{\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}. \tag{3}$$

Leaving off the y and z coordinates for simplicity, as these are not affected by the transformation, from eq.(1) we get

$$\Delta \tau = \sqrt{\Delta t^2 - \Delta x^2} = \sqrt{\Delta t'^2 - \Delta x'^2}, \tag{4}$$

which reduces to $\Delta\tau = 0$ in the case of light. This is the well known fact that the world lines defined by $\Delta\tau = 0$ constitute the limits of the region, which can be causally connected to some space-time event. Invariance of the condition $\Delta\tau = 0$ means that causality is not affected by any Lorentz transformation.

But with respect to the twin paradox it is more important that the quantity $\Delta\tau$ is generally invariant under Lorentz transformations, not only in the case $\Delta\tau = 0$. The space-time distance between two events is a uniquely defined quantity, independent of the reference system. If two events take place at the same physical location, we must assume that their spatial distance is zero in every reference system. As their space-time distance is uniquely defined, too, the necessary consequence according to eq(4) is that the temporal distance is also uniquely defined. It must be independent of the choice of the reference system.

For Alice and Bob there are two events, where they are physically at the same spatial position: The first one, when Alice leaves the earth with her spaceship, and the second one, when she returns from her journey. The space-time distance between these events is uniquely defined and thus also the time interval between the events. There is no room for any discrepancy in the aging of Alice and Bob.

Switching from one reference system to another may change the local time scale during her journey, but changes of the reference system cannot change the underlying physics. In the next section we will try to find out, where the pitfalls are, which lead to positive results for the twin paradox.

3   False solutions

False solutions of the twin paradox date back to the very beginning of the theory of relativity. Even Einstein himself has mentioned time discrepancies as a peculiar result of his theory [1] and the story of the traveling twins was introduced by Langevin as early as 1911 [2]. Numerous papers on the topic have been published since then, but most of them give more or less sophisticated explanations, why the twins age differently, but the existence of the effect is scarcely disputed.

The most simple argument in favour of a different aging runs as follows. The clock in the moving spaceship is slowed down with respect to a clock on earth by a factor $1/\gamma = \sqrt{1 - \beta^2}$. Thus the time to reach the point of return is reduced just by this factor. As the return journey is symmetric with respect to time, the total time is also reduced by this factor compared to the elapsed time on earth.

This reasoning contains several errors, however. The first one is that it regards the spatial geometry, which is fixed in the rest frame of the earth, as fixed also in the comoving frame and only considers the change of the time scale. The second is that at the return point there is a further change of the reference system. While the Lorentz transformation between the earth and the spaceship has been synchronised at the starting point, there is no such synchronisation at the return point. That means that the zero point of the time scale is altered. Closely related to the synchronisation problem is the fact that the Lorentz

transformation does not conserve simultaneity. Simultaneous events at different space points do not remain simultaneous with changes to a moving coordinate system.

It should be stressed here that a proper time interval is not the change of some scalar property between two events, but it is the equivalent in a pseudo-Euclidean metric to the vector length in Euclidean geometry. Thus adding proper time intervals, measured in different reference frames, must be done by the rules of vector algebra and not like the addition of scalar data. Change to a moving reference system is analog to rotation in Euclidean space.

The problem can easily be visualised in Euclidean geometry (see fig.1). We consider a set of transformations, consisting of a rotation of the (x, y) coordinate system by some angle ϑ about the zero-point to (x', y'), subsequent shifting the coordinate system in x'-direction to (x'', y''), rotating the system back by $-\vartheta$ to (x''', y''') and finally shift it back in direction of x''' to (x'''', y''''), so that $x'''' = x$. At the end y'''' will not be equal to y. But this operation does of course not change any distance between points in the (x, y) plane.
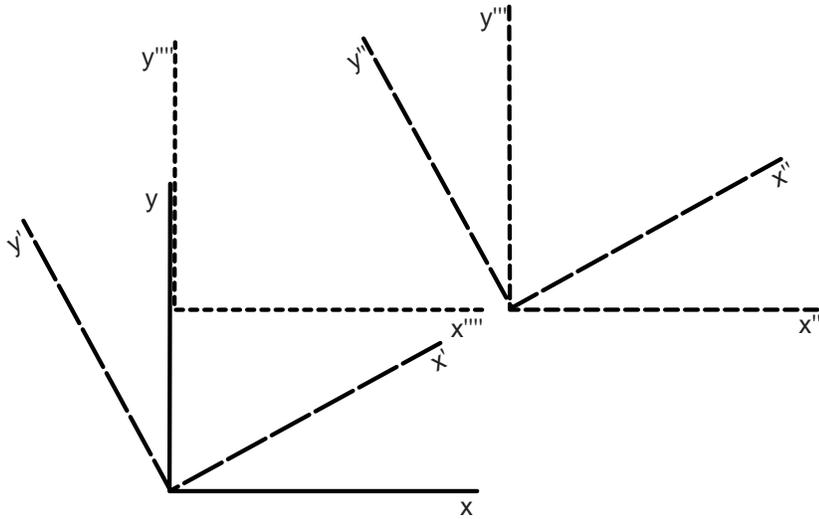

Figure 1: Coordinate transformations in Euclidean space: rotation, shift along x' axis, back rotation, back shift along x''' axis

It is the fact that we have used rotations with different centers, which leads to a shift of the y coordinate. In just the similar way the age shift of the twin paradox results from the fact that the reference frame is changed twice, first at the starting point, and then a second time at the point of return, but now with a different center of 'rotation'. It is this change of velocity at the return point, which causes the supposed age shift. To reach the velocity of the new comoving reference frame, the spaceship has to be accelerated. During the acceleration phase the proper time changes continuously. An instantaneous switch to the

new comoving system requires infinite acceleration and results in a jump of proper time.

But we can try to consider the space trip of Alice without the change of the reference system at the return point. We compare the situations as seen from the earth and from the reference system of the spaceship, synchronised at the starting point.

In the rest frame of the earth the target star of the journey is at a fixed distance x_S, while in the reference system of the spaceship according to eq.(1) at the state of synchronisation the local time at x_S is $t'_S = -\gamma\beta x_S$, the position is $x'_S = \gamma x_S$ and the star is moving towards the spaceship with velocity $-\beta$. The star passes the spaceship at

$$t'_1 = \frac{x'_S}{\beta} + t'_S = \frac{\gamma x_S}{\beta} - \gamma\beta x_S = \frac{\gamma x_S}{\beta}(1 - \beta^2) = \frac{x_S}{\beta\gamma} \tag{5}$$

We could, of course, derive the same relation immediately from eq.(1) by setting $t = \beta x_S$ and $x = x_S$. The detour was only to demonstrate the importance of taking into account missing simultaneity of distant events. In the comoving system the proper time interval is equal to t'_1. In the earth-bound system we have

$$\Delta\tau = \sqrt{-x_S^2 + (x_S/\beta)^2} = \frac{x_S}{\beta\gamma}, \tag{6}$$

the same value as t'_1, as must be expected. The coordinate times are different, as times at different locations are compared, the proper time is the same in both systems.

The back journey is a little bit more complicated. Alice in her spaceship changes the velocity at the turning point. But to keep things simple, we consider this change only in the system (x', t'), as we know that a further change of the reference system at the turning point would bring in problems of acceleration and thus of synchronisation. The situation is different now from the first part of the journey, however. Now for Alice the earth is no longer at a fixed location, but moving apart at velocity $-\beta$. She is hunting for a moving target. Thus in her reference system the speed must be higher than that of the earth. In the non-relativistic case, if the time back to earth should be equal to t'_1, her speed must be -2β. But if $\beta > 0.5$, she comes into trouble, as her spaceship is limited to the speed of light. The back journey will take more time than the first part.

In the limiting case $\beta = 1$ the time would even be infinite. No light pulse can catch up with another pulse, emitted at an earlier time by the same source. In the relativistic case we must change our problem a little bit. Now we put the question, when and where will Alice catch up with the moving earth, when her spaceship now moves with speed $-\beta'_2$, measured in the reference frame (x', t'). The earth is receding at $-\beta$ since $t' = 0$, she herself is moving with $-\beta'_2$ since $t' = t'_1$ Thus she will meet her twin on earth at t'_2, given by

$$-\beta t'_2 = -\beta'_2(t'_2 - t'_1), \tag{7}$$

5

leading to

$$t_2' = \frac{\beta_2'}{\beta_2' - \beta}\, t_1', \qquad x_2' = -\frac{\beta \beta_2'}{\beta_2' - \beta}\, t_1' \tag{8}$$

and

$$\Delta\tau_2' = \sqrt{1 - \beta^2}\frac{\beta_2'}{\beta_2' - \beta}\, t_1' = \frac{\beta_2'}{\beta}\frac{1 - \beta^2}{\beta_2' - \beta}\, x_S. \tag{9}$$

If we insert the values of eq.(8) into eq.(1) and solve for the time interval in rest frame of the earth, we find $\Delta\tau_2 \equiv t_2 = \Delta\tau_2'$. There is no change of the proper time between Alice and Bob.

This simple example shows that it is only the mixing up time intervals, measured in different reference frames, which leads to contradictions. If perceived time intervals depend on distance, it is no longer meaningful to add time intervals measured in different reference systems, which are in relative motion. The only quantity, which is of physical relevance, is the space-time distance of events or the proper time interval, which must not confounded with the perceived time interval measured in some reference frame. This topic has already been discussed by Kracklauer [3] in 2001, but still there are numerous newer papers, which ignore the invariance of proper time and insist on the existence of the time discrepancy.

4   Real effects of Lorentz invariance

Though there exist no local time discrepancies which depend on the course of the world line between two events, there are several real effects, which are explained by the Lorentz invariance of the basic laws of physics. But all these effects are related to our local observations of physical processes, generated in systems, which are moving with respect to the local rest frame.

One well known example is the apparent increase of life time of unstable particles like muons, when they approach the earth at velocities close to the speed of light.

Another is the red shift of light, emitted from moving sources. Though the light always reaches us with the velocity c, the wavelength is shifted. There is no change of the relative velocity, as we know it from Doppler shift, but it is the different time scale at emission, which leads to the wavelength shift at observation.

The difference between Doppler shift and Lorentz shift clearly shows up in cosmological observations. The observed red shift of light from distant objects can be explained by a continuous expansion of space, which is equivalent to a continuous local acceleration or a recession velocity proportional to distance. This leads to a change of time scale proportional to distance between emission and observation. This change does not only affect the frequency of light, but influences all time dependent processes. One well observed effect is the dilatation of the time scale of distant supernovas.

But there is a strong caveat in this interpretation of cosmological red shift and time dilatation. Contrary to Doppler shift, which occurs only with motions

in the direction of observation, the Lorentz shift of time scale is independent of the direction of acceleration. A continuous acceleration perpendicular to the direction of observation and acceleration in this direction will result in exactly the same time shift. If space is curved, every geodesic motion must be regarded as accelerated. Thus from red shift or time dilatation measurements we cannot decide, if space is expanding or if space is curved.

Only independent measurements, which are affected only by the spatial component of space-time, like the size distribution of distant galaxies or clusters, can help to decide if the cause of red shift is expansion or curvature of space. There is one consequence which remains in both cases. If Lorentz invariance is valid throughout the entire universe, there is no sensible definition of a global time scale. Time varies with distance. As a reasonable definition of time to describe distant events in the universe, one can only use the running time of light with respect to our local reference frame.

The persistent discussions of the twin paradox demonstrate that we have not yet fully understood all the consequences of Lorentz invariance of interactions as well in the regime of mechanics as in gravity. But in the last decades a huge amount of measurements and observations has been accumulated, so that we should be able to decide, if Lorentz invariance is really the governing principle, not only of electromagnetism, but also of gravity.

References

[1] Einstein, A., Ann.Phys. 17,891 (1905)

[2] Langevin, P., Scientia 10,31 (1911)

[3] Kracklauer, A.F. and Kracklauer, P.T., arXiv:physics/0012041 (2001)