Classification of arithmetic root systems

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Abstract

Arithmetic root systems are invariants of Nichols algebras of diagonal type with a certain finiteness property. They can also be considered as generalizations of ordinary root systems with rich structure and many new examples. On the other hand, Nichols algebras are fundamental objects in the construction of quantized enveloping algebras, in the noncommutative differential geometry of quantum groups, and in the classification of pointed Hopf algebras by the lifting method of Andruskiewitsch and Schneider. In the present paper arithmetic root systems are classified in full generality. As a byproduct many new finite dimensional pointed Hopf algebras are obtained.

Key Words: Hopf algebra, Nichols algebra, Weyl groupoid

MSC2000: 17B37, 16W35

1 Introduction

The theory of Nichols algebras is relatively young, but it is affected by various research areas of mathematics and theoretical physics. It is dominated and motivated by Hopf algebra theory in the following way. Let H be a Hopf algebra with coradical filtration $H_0 \subset H_1 \subset \dots$ such that H_0 is a Hopf subalgebra of H. Let $\operatorname{gr} H$ denote the $\mathbb{N}_0$-graded Hopf algebra $\bigoplus_i H_i/H_{i-1}$. Then H possesses a rich invariant, namely the subalgebra $\mathcal{B}(V) \subset \operatorname{gr} H$ generated by the vector space V of H_0-coinvariants of H_1/H_0. It is called a *Nichols algebra* [4] in commemoration to W. Nichols who started to study these objects systematically [19]. Nichols algebras can be described in many different ways [21], [2], [20]. The importance of such algebras was detected and pointed out in many papers by Andruskiewitsch and Schneider, see for example [4] and [1]. Their structure was enlightened, mainly in the case when H_0 is the group algebra of a finite group, among others in [9], [5], [18], [3]. Nichols algebras were used by Andruskiewitsch and Schneider to start a very promising program [4] to classify pointed Hopf algebras with certain finiteness properties. This so called lifting method was already succesfully performed for finite dimensional pointed Hopf algebras where the nilpotency order of the elements of H_1/H_0 is bigger than 7 [6].

Nichols algebras appear in a natural way also in the construction of quantized Kac–Moody algebras [17, Sect. 8.2.1] [14, Sect. 3.2.9] and their $\mathbb{Z}_2$-graded variants [16]. Using a particular Nichols algebra Yamane [23] described a $\mathbb{Z}/3\mathbb{Z}$-graded quantum group which has a representation theory fitting into the general picture. Nichols algebras are also natural objects in the theory of covariant differential calculus on quantum groups initiated by Woronowicz [22]. Further, Bazlov [8] proved that the cohomology ring of a flag variety can be considered as a subalgebra of a particular Nichols algebra of nonabelian group type. In contrast to these various interesting aspects of the subject one still does not know too much about the structure of Nichols algebras in general.

Color Lie algebras [7] are generalizations of Lie algebras. For the study of their structure methods are developed which are useful also to analyze Nichols algebras. For example, Kharchenko [15] proved that any Hopf algebra generated by skew-primitive and group-like elements has a restricted Poincaré–Birkhoff-Witt basis. Here "restricted" means that the possible powers of the root vectors can be bounded by an integer number. However in contrast to color Lie algebras this bound may be different from 2. Therefore significant new classes of examples can be expected. However note that under some hypotheses all examples are deformations of the upper triangular part of a semisimple Lie algebra [20] [5]. Kharchenko's results apply in particular to Nichols algebras $\mathcal{B}(V)$ of diagonal type, that is when V is a direct sum of 1-dimensional Yetter–Drinfel'd modules over H_0. Motivated by the close relation to Lie theory, to any such Nichols algebra a Weyl groupoid and an arithmetic root system were associated [13]. These constructions were used in [12] and [11] to determine Nichols algebras of rank 2 and 3 with a finite set of PBW generators without dealing with the complicated defining relations of $\mathcal{B}(V)$. In this paper the classification is performed for Nichols algebras of diagonal type and of arbitrary (finite) rank following the ideas in [12] and [11]. The main results are collected in Theorems 16 and 19. Their proof was enabled by the additional development of an efficient technique to recognize that a given triple is an arithmetic root system. Together with the classification results for rank 2 and rank 3 Nichols algebras of diagonal type the first half of Question 5.9 of Andruskiewitsch [1] is answered.

With this classification result a large amount of algebras are detected which did not appear previously in the literature. In particular, infinite series of finite dimensional $\mathbb{Z}/3\mathbb{Z}$-graded algebras are listed. However also many open problems remained unsolved, for example to clarify the relationship of these algebras to super Lie algebras and related structures, and to describe them with help of generators and relations [1, Question 5.9].

In the present paper the notations and conventions in [10] and [11] are followed and several results from these papers will be used.

2 On the finiteness of the Weyl groupoid

Let k be a field of characteristic zero, $d \in \mathbb{N}$, χ a bicharacter on $\mathbb{Z}^d$ with values in $k^* = k \setminus \{0\}$, and $E = \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ a basis of $\mathbb{Z}^d$. Set $q_{ij} := \chi(\mathbf{e}_i, \mathbf{e}_j)$ for all $i, j \in \{1, \dots, d\}$.

In [13] the Weyl groupoid $W_{\chi,E}$ associated to χ and E was defined, see also [10] for the related definition of $W^{\text{ext}}_{\chi,E}$. If $W_{\chi,E}$ (or equivalently $W^{\text{ext}}_{\chi,E}$) is full and finite then one obtains an arithmetic root system $(\mathbf{\Delta}, \chi, E)$ [10], where $\mathbf{\Delta} \subset \mathbb{Z}^d$ is a certain finite subset such that $\mathbf{\Delta} = -\mathbf{\Delta}$. In order to classify arithmetic root systems one has to solve two problems. First, one has to detect Weyl groupoids which are not full or not finite. This can be done effectively using subsystems of arithmetic root systems, see [11]. Second, one has to be able to check the finiteness of full and finite Weyl groupoids. To do the latter in [11] the group $\mathcal{G}_{\chi,E}$ was introduced. However in general it is not easy to determine the structure of this group. As an alternative approach Proposition 5 will be proved. This allows to conclude the finiteness of $W_{\chi,E}$ from the finiteness of standard subgroupoids of $W^{\text{ext}}_{\chi,E}$ and from the existence of one element with a special property.

In the rest of this section, if not stated otherwise, let $W_{\chi,E}$ be an arbitrary Weyl groupoid. Set $\mathbf{\Delta} = \bigcup\{F \,|\, (\text{id}, F) \in W_{\chi,E}\}$. Note that $\mathbf{\Delta}$ is finite if and only if $W_{\chi,E}$ is finite. For $(\text{id}, F) \in W_{\chi,E}$ let $\mathbf{\Delta}^+_F$ denote the set $\mathbf{\Delta}^+_F := \mathbf{\Delta} \cap \mathbb{N}_0 F$. The proof of [10, Proposition 1] shows the following.

Proposition 1. *Let F be a basis of $\mathbb{Z}^d$ such that* $(\text{id}, F) \in W_{\chi,E}$. *Then* $\mathbf{\Delta} = \mathbf{\Delta}^+_F \cup -\mathbf{\Delta}^+_F$.

Proposition 2. *Let F be a basis of $\mathbb{Z}^d$ such that $(\mathrm{id}, F) \in W_{\chi,E}$ and let $\mathbf{f}_0 \in F$. Then $\boldsymbol{\Delta}^+_{s_{\mathbf{f}_0,F}(F)} = \boldsymbol{\Delta}^+_F \cup \{-\mathbf{f}_0\} \setminus \{\mathbf{f}_0\}$. In particular, $\boldsymbol{\Delta}^+_F \cap -\boldsymbol{\Delta}^+_E$ is finite.*

Proof. By definition of $s_{\mathbf{f}_0,F}$ one has $s_{\mathbf{f}_0,F}(\mathbf{f}) - \mathbf{f} \in \mathbb{Z}\mathbf{f}_0$ for all $\mathbf{f} \in F$. Hence if $f = \sum_{\mathbf{f}\in F} a_\mathbf{f}\mathbf{f}$ for some $f \in \mathbb{Z}^d$, $a_\mathbf{f} \in \mathbb{Z}$, then $f - \sum_{\mathbf{f}\in F} a_\mathbf{f} s_{\mathbf{f}_0,F}(\mathbf{f}) \in \mathbb{Z}\mathbf{f}_0$. By Proposition 1 one has $\boldsymbol{\Delta}^+_{s_{\mathbf{f}_0,F}(F)} \subset \mathbb{N}_0 F \cup -\mathbb{N}_0 F$. Thus if $f \in \boldsymbol{\Delta}^+_{s_{\mathbf{f}_0,F}(F)}$ and $f \notin \mathbb{Z}\mathbf{f}_0$ then $f \in \mathbb{N}_0 F \cap \boldsymbol{\Delta} = \boldsymbol{\Delta}^+_F$. Since F is a basis of $\mathbb{Z}^d$, relation $f \in \mathbb{Z}\mathbf{f}_0$ implies that $f = m\mathbf{f}_0$ with $m^2 = 1$. Therefore equation $s_{\mathbf{f}_0,F}(\mathbf{f}_0) = -\mathbf{f}_0$ gives that $\boldsymbol{\Delta}^+_{s_{\mathbf{f}_0,F}(F)} \subset \boldsymbol{\Delta}^+_F \cup \{-\mathbf{f}_0\} \setminus \{\mathbf{f}_0\}$. Multiplication with -1 yields $-\boldsymbol{\Delta}^+_{s_{\mathbf{f}_0,F}(F)} \subset -\boldsymbol{\Delta}^+_F \cup \{\mathbf{f}_0\} \setminus \{-\mathbf{f}_0\}$, and hence Proposition 1 gives equality in the above relation. The second part of the claim of the proposition follows from the fact that any element of $W_{\chi,E}$ can be written as a finite product of elements of the form $(s_{\mathbf{f},F}, F)$. ∎

Assume that $(\mathrm{id}, F) \in W^{\mathrm{ext}}_{\chi,E}$ and let F' be a subset of F. For any $\mathbf{f} \in \mathbb{Z}^d$ let $[\mathbf{f}]_{F'}$ denote the equivalence class of $\mathbf{f}$ in $\mathbb{Z}^d/\mathbb{Z}F'$. For a subset E' of E set

$$\begin{aligned} W^{\mathrm{ext}}_{\chi,E'\subset E} = \{(T,F) \in W^{\mathrm{ext}}_{\chi,E} \,\big|\, & \text{there exists } (T',E) \in W^{\mathrm{ext}}_{\chi,E} \text{ such that} \\ & T'(E) = F \text{ and } [\mathbf{e}]_{E'} = [T'(\mathbf{e})]_{E'} = [TT'(\mathbf{e})]_{E'} \text{ for all } \mathbf{e} \in E\} \end{aligned} \tag{1}$$

and $W_{\chi,E'\subset E} = W^{\mathrm{ext}}_{\chi,E'\subset E} \cap W_{\chi,E}$. Obviously, $W^{\mathrm{ext}}_{\chi,E'\subset E}$ is a subgroupoid of $W^{\mathrm{ext}}_{\chi,E}$ and $W_{\chi,E'\subset E}$ is a subgroupoid of $W_{\chi,E}$.

Lemma 3. *Assume that $(T,F) \in W^{\mathrm{ext}}_{\chi,E}$ and $F' \subset F$. If $T(F) \cap \mathbb{Z}F' = F'$ and $[T(\mathbf{f})]_{F'} = [\mathbf{f}]_{F'}$ for all $\mathbf{f} \in F \setminus F'$, then $T(F) = F$. Moreover, if additionally $T(\mathbf{f}) = \mathbf{f}$ for all $\mathbf{f} \in F'$ then $T = \mathrm{id}$.*

Proof. By assumption, $T(F \setminus F') \cap \mathbb{Z}F' = \emptyset$ and $T(F) \cap \mathbb{Z}F' = F'$, and hence $T(F') = F'$. In particular, since $T \in \mathrm{Aut}_\mathbb{Z}(\mathbb{Z}^d)$, T induces an automorphism of $\mathbb{Z}^d/\mathbb{Z}F'$. Since $(\mathrm{id}, F), (\mathrm{id}, T(F)) \in W^{\mathrm{ext}}_{\chi,E}$, one has $T(F) \subset \mathbb{N}_0 F \cup -\mathbb{N}_0 F$ and $F \subset \mathbb{N}_0 T(F) \cup -\mathbb{N}_0 T(F)$. Thus equation $[T(\mathbf{f})]_{F'} = [\mathbf{f}]_{F'}$ implies that for all $\mathbf{f} \in F \setminus F'$ relations $T(\mathbf{f}) - \mathbf{f} \in \mathbb{N}_0 F'$ and $\mathbf{f} - T(\mathbf{f}) \in \mathbb{N}_0 T(F') = \mathbb{N}_0 F'$ hold. Therefore $T(\mathbf{f}) = \mathbf{f}$ for all $\mathbf{f} \in F \setminus F'$. ∎

Proposition 4. *Suppose that $W^{\mathrm{ext}}_{\chi,E}$ is full. Let $E' \neq \emptyset$ be a subset of E and d' the number of elements of E'. Let χ' denote the bicharacter on $\mathbb{Z}^{d'} = \mathbb{Z}E' \subset \mathbb{Z}^d$ such that $\chi'(\mathbf{e}, \mathbf{e}') = \chi(\mathbf{e}, \mathbf{e}')$ for all $\mathbf{e}, \mathbf{e}' \in E'$. Then the map $\Phi^{\mathrm{ext}} : W^{\mathrm{ext}}_{\chi,E'\subset E} \to W^{\mathrm{ext}}_{\chi',E'}$ defined by $(T,F) \mapsto (T\restriction_{\mathbb{Z}E'}, F \cap \mathbb{Z}E')$ is an isomorphism.*

Proof. Suppose that $(T,F)\in W^{\rm ext}_{\chi,E'\subset E}$. By definition of $W^{\rm ext}_{\chi,E'\subset E}$ the set $F\cap\mathbb{Z}E'$ is a basis of $\mathbb{Z}E'$. In the same way one obtains that $T(F\cap\mathbb{Z}E')=T(F)\cap\mathbb{Z}E'$ is a basis of $\mathbb{Z}E'$. Thus $T\restriction_{\mathbb{Z}E'}\in\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}E')$, and hence the map $\widetilde{\Phi^{\rm ext}}:W^{\rm ext}_{\chi,E'\subset E}\to\widetilde{W_{d'}}$, where

$$\widetilde{W_{d'}}=\{(T',F')\,|\,T'\in\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}E'),F'\text{ is a basis of }\mathbb{Z}E'\},$$

given by $(T,F)\mapsto(T\restriction_{\mathbb{Z}E'},F\cap\mathbb{Z}E')$, is a well-defined map of groupoids.

First it will be proved that the map $\widetilde{\Phi^{\rm ext}}$ is injective. If $(T_1,F_1),(T_2,F_2)\in W^{\rm ext}_{\chi,E'\subset E}$ then there exist $(T,F_1),\in W^{\rm ext}_{\chi,E}$ such that $T(F_1)=F_2$ and $[\mathbf{f}]_{E'}=[T(\mathbf{f})]_{E'}$ for all $\mathbf{f}\in F_1$. Assume now that $\widetilde{\Phi^{\rm ext}}((T_1,F_1))=\widetilde{\Phi^{\rm ext}}((T_2,F_2))$. Then $F_1\cap\mathbb{Z}E'=F_2\cap\mathbb{Z}E'$ and hence Lemma 3 implies that $F_1=F_2$. Moreover, $(T_2^{-1}T_1,F_1)$ with $F'=F_1\cap\mathbb{Z}E'$ satisfies the assumptions of the second part of Lemma 3 and therefore one has $T_1=T_2$.

It remains to show that the image of $\widetilde{\Phi^{\rm ext}}$ is equal to $W^{\rm ext}_{\chi',E'}$. First of all $W^{\rm ext}_{\chi',E'}$ is contained in $\widetilde{\Phi^{\rm ext}}(W^{\rm ext}_{\chi,E'\subset E})$. Indeed, if $(T',E)\in W^{\rm ext}_{\chi,E}$, $[T'(\mathbf{e})]_{E'}=[\mathbf{e}]_{E'}$ for all $\mathbf{e}\in E$, and $\mathbf{e}'\in E'$, then $(s_{T'(\mathbf{e}'),T'(E)},T'(E))\in W^{\rm ext}_{\chi,E'\subset E}$ since $W^{\rm ext}_{\chi,E}$ is full. Therefore $\widetilde{\Phi^{\rm ext}}((s_{T'(\mathbf{e}'),T'(E)},T'(E)))=(s_{T'(\mathbf{e}'),T'(E')},T'(E'))$.

Take now $(T,F)\in W^{\rm ext}_{\chi,E'\subset E}$ and let (T',E) be in $W^{\rm ext}_{\chi,E}$ satisfying the conditions in Equation (1). Then equation $(T,F)=(TT',E)\circ(T',E)^{-1}$ holds, and $(TT',E),(T',E)\in W^{\rm ext}_{\chi,E'\subset E}$. Therefore it is sufficient to show that

$$(*)\qquad \widetilde{\Phi^{\rm ext}}((T,E))\in W^{\rm ext}_{\chi',E'}\text{ whenever }(T,E)\in W^{\rm ext}_{\chi,E'\subset E}.$$

By Proposition 2 the set $\boldsymbol{\Delta}^+_{T(E)}\cap-\boldsymbol{\Delta}^+_E$ is finite. In particular, if $T(E')\cap-\boldsymbol{\Delta}^+_E\neq\emptyset$ then (*) holds for (T,E) if and only if it holds for $(s_{\mathbf{f},T(E)}T,E)$, where $\mathbf{f}\in T(E')$. Hence without loss of generality one can assume that $T(E')\cap-\boldsymbol{\Delta}^+_E=\emptyset$, that is $T(E')\subset\boldsymbol{\Delta}^+_E\cap\mathbb{N}_0E'$. Moreover, Equation (1) gives that $T(\mathbf{e})-\mathbf{e}\in\mathbb{Z}E'$ for all $\mathbf{e}\in E$, and hence $T(E)\subset\boldsymbol{\Delta}^+_E$. This and Proposition 1 yield that $E\subset\mathbb{N}_0T(E)$ and hence $E=T(E)$. ∎

Proposition 5. *Assume that $W^{\rm ext}_{\chi,E}$ is full and that there exist $(T,E)\in W^{\rm ext}_{\chi,E}$, $\mathbf{e}\in E$ and $\mathbf{f}\in\boldsymbol{\Delta}^+$ such that $T(E)=E\setminus\{\mathbf{e}\}\cup\{-\mathbf{f}\}$. Set $E':=E\setminus\{\mathbf{e}\}$. Let χ' denote the bicharacter on $\mathbb{Z}^{d-1}=\mathbb{Z}E'\subset\mathbb{Z}^d$ such that $\chi'(\mathbf{e}',\mathbf{e}'')=\chi(\mathbf{e}',\mathbf{e}'')$ for all $\mathbf{e}',\mathbf{e}''\in E'$, and assume that $W^{\rm ext}_{\chi',E'}$ is finite. Then $W^{\rm ext}_{\chi,E}$ is finite.*

Proof. It suffices to show that the set $\mathbf{\Delta}$ is finite. By assumption one has $(\mathrm{id}, T(E)) \in W_{\chi,E}$ and hence Proposition 2 gives that $\mathbf{\Delta}^+_{T(E)} \cap -\mathbf{\Delta}^+_E$ is finite. Moreover, Proposition 1 implies that $\mathbf{\Delta} = \mathbf{\Delta}^+_{T(E)} \cup -\mathbf{\Delta}^+_{T(E)} = \mathbf{\Delta}^+_E \cup -\mathbf{\Delta}^+_E$ and hence it remains to show that the set

$$\mathbf{\Delta}^+_{T(E)} \cap \mathbf{\Delta}^+_E = \mathbf{\Delta} \cap (-\mathbb{N}_0\mathbf{f} + \mathbb{N}_0 E') \cap (\mathbb{N}_0\mathbf{e} + \mathbb{N}_0 E') = \mathbf{\Delta} \cap \mathbb{N}_0 E'$$

is finite. By assumption $W^{\mathrm{ext}}_{\chi',E'}$ is finite. Hence [10, Proposition 2] implies that there exists $(\mathrm{id}, E'') \in W^{\mathrm{ext}}_{\chi',E'}$ such that $E'' \subset -\mathbb{N}_0 E'$. Thus by Proposition 4 there exists $(T', E) \in W^{\mathrm{ext}}_{\chi,E}$ such that $T'(E') = E''$ and $T'(\mathbf{e}) \in (\mathbf{e} + \mathbb{Z}E') \cap \mathbf{\Delta}^+_E$. Since also $(\mathrm{id}, T'(E)) \in W^{\mathrm{ext}}_{\chi,E}$, Proposition 2 yields that $\mathbf{\Delta}^+_{T'(E)} \cap -\mathbf{\Delta}^+_E$ is a finite set. Thus the set $\mathbf{\Delta} \cap \mathbb{N}_0 E'$ is finite because of the relations $\mathbf{\Delta} \cap -\mathbb{N}_0 E' = \mathbf{\Delta} \cap \mathbb{N}_0 E'' \cap -\mathbb{N}_0 E' \subset \mathbf{\Delta} \cap \mathbf{\Delta}^+_{T'(E)} \cap -\mathbf{\Delta}^+_E$. ■

3 Connected arithmetic root systems of rank two and three

Recall the notation in Section 2. In [10] and [11] arithmetic root systems of rank 2 and rank 3 were classified. For the considerations in rank 4 and higher the following facts will be needed.

Lemma 6. *Let $(\mathbf{\Delta}, \chi, E)$ be a connected rank 2 arithmetic root system. If $q_{11}q_{12}q_{21}q_{22} = -1$ then one of the following two systems of equations holds:*

$$q_{11} + 1 = q_{12}q_{21}q_{22} - 1 = 0, \qquad q_{22} + 1 = q_{11}q_{12}q_{21} - 1 = 0.$$

Proof. Since $(\mathbf{\Delta}, \chi, E)$ is connected, one has $q_{12}q_{21} \neq 1$. Thus the claim follows from [11, Proposition 9(i)]. ■

According to Table 2 in [11] one obtains the following lemma.

Lemma 7. *Let $(\mathbf{\Delta}, \chi, E)$ be a connected rank 3 arithmetic root system. Then the following assertions hold.*

(i) If $q_{13}q_{31} = 1$ and $q_{11}, q_{22}, q_{33} \neq -1$ then either (χ, E) is of Cartan type or relations $q_{ii} \in R_3$, $q_{22}, q_{jj} \in R_6 \cup R_9$, $q_{jj}q_{j2}q_{2j} = q_{22}q_{2i}q_{i2} = 1$, $(q_{j2}q_{2j}q_{22} - 1)(q_{j2}q_{2j}q_{22}^2 - 1) = 0$ hold for an $i \in \{1, 3\}$ and $j = 4 - i$.

(ii) If $q_{ij}q_{ji} \neq 1$ for all $i \neq j$ then $\prod_{i<j} q_{ij}q_{ji} = 1$ and $\prod_{i=1}^{3}(q_{ii} + 1) = 0$. Moreover, if also relations $q_{11} = -1$ and $q_{22}, q_{33} \neq -1$ hold then $q_{12}^2 q_{21}^2 = q_{13}^2 q_{31}^2 \in R_3$ and $q_{12}q_{21}q_{22} = q_{13}q_{31}q_{33} = 1$.

(iii) If $q_{13}q_{31}=1$ *and* $q_{22}=-1$, $q_{11}q_{12}q_{21}=1$, $q_{11},q_{33}\neq -1$, *then either* $q_{23}q_{32}q_{33}=1$ *or* $q_{23}q_{32}q_{33}^2=1$, $(q_{11}q_{33}^2-1)(q_{11}q_{33}^3+1)=0$, *or* $q_{33}=-q_{11}\in R_3$, $q_{23}q_{32}\in\{-1,-q_{33}\}$.

(iv) If $q_{13}q_{31}=1$, $q_{33}=-1$, $q_{11},q_{22}\neq -1$, *and* $q_{11}q_{12}q_{21}=q_{12}q_{21}q_{22}=1$ *then either equation* $q_{22}q_{23}q_{32}=1$ *or relations* $q_{22}^2q_{23}q_{32}=1$, $q_{11}\in R_3\cup R_4\cup R_6$ *hold.*

(v) If $q_{13}q_{31}=1$, $q_{33}=-1$, *and* $q_{12}q_{21}q_{22}=q_{22}q_{23}q_{32}=1$ *then either* $(q_{11}-q_{22})(q_{11}^2-q_{22})=0$ *or* $q_{11}=-1$ *or* $q_{11}\in R_3$, $q_{11}q_{22}=-1$.

(vi) If one has $q_{13}q_{31}=1$, $q_{11}=q_{22}=-1$, *and* $q_{33}\neq -1$, *then either relations* $q_{12}q_{21}=-1$, $q_{33}\in R_3$, $q_{23}^2q_{32}^2q_{33}=1$, *or equations* $q_{23}q_{32}q_{33}=1$, $(q_{12}q_{21}+1)(q_{12}q_{21}q_{23}q_{32}+1)(q_{12}^2q_{21}^2q_{23}q_{32}-1)(q_{12}^3q_{21}^3q_{23}q_{32}-1)=0$, *or equations* $q_{12}q_{21}q_{23}q_{32}=1$, $(q_{23}q_{32}q_{33}-1)(q_{23}q_{32}q_{33}^2-1)(q_{23}q_{32}+q_{33})=0$ *hold.*

(vii) If $q_{13}q_{31}=1$ *and* $-q_{22},q_{12}q_{21}q_{22},q_{22}q_{23}q_{32}\neq 1$, *then* $q_{12}q_{21}q_{22}q_{23}q_{32}=-1$ *and one has* $q_{22}\in R_3\cup R_6$, $q_{ii}=-1$, $q_{22}^2q_{2i}q_{i2}=1$, $q_{2j}q_{j2}=-q_{22}$, $q_{jj}\in\{-1,-q_{22}^{-1}\}$ *for some* $i\in\{1,3\}$ *and* $j=4-i$.

(viii) If $q_{13}q_{31}=1$, $q_{11}=q_{33}=-1$, $q_{12}q_{21}q_{22}=1$, *and* $q_{22}\neq -1$, *then either* $q_{22}q_{23}q_{32}=1$ *or* $q_{23}q_{32}=-1$, $q_{22}\in R_3\cup R_4\cup R_6$, *or* $q_{23}^2q_{32}^2=q_{22}^2\in R_3$.

(ix) If $q_{ij}q_{ji}\neq 1$ *for all* $i\neq j$ *then for all* i *there exists* $j\neq i$ *such that* $(q_{ii}+1)(q_{ii}q_{ij}q_{ji}-1)=0$.

4 Connected arithmetic root systems of rank four

Let E be a basis of $\mathbb{Z}^d$, where $d\geq 2$, and let χ be a bicharacter on $\mathbb{Z}^d$. The following proposition is one of the key tools in the classification of arithmetic root systems. In most of the cases it will be used without referring to it.

Proposition 8. *[11, Prop. 4,Lemma 5] Let* $r\in\mathbb{N}$ *with* $r<d$. *Assume that* $F=\{\mathbf{f}_1,\dots,\mathbf{f}_r\}\subset\mathbf{\Delta}_E^+$ *is a set of linearly independent elements of* $\mathbb{Z}^d$ *such that for all* $j\leq r$ *and all* $m_1,\dots,m_{j-1}\in\mathbb{N}_0$ *one has*

$$\mathbf{f}_j-\sum_{i=1}^{j-1}m_i\mathbf{f}_i\notin\mathbf{\Delta}\setminus\{\mathbf{f}_j\}. \tag{2}$$

Then $\mathbf{\Delta}(\chi;\mathbf{f}_1,\dots\mathbf{f}_r)=(\mathbf{\Delta}\cap\mathbb{Z}F,\chi\restriction_{\mathbb{Z}F\times\mathbb{Z}F},F)$ *is an arithmetic root system.*

In the following also the terminology "$\mathbf{\Delta}(\chi;\mathbf{f}_1,\dots\mathbf{f}_r)$ is finite" will be used in order to emphasize the finiteness of the set $\bigcup\{F'\mid(\mathrm{id},F')\in W_{\chi',F}\}=$

$\boldsymbol{\Delta} \cap \mathbb{Z}F$, where $\chi' = \chi\restriction_{\mathbb{Z}F\times\mathbb{Z}F}$.

Note that relation (2) holds in particular if

$$\mathbf{f}_j - \sum_{i=1}^{j-1} m_i \mathbf{f}_i \notin (\mathbb{N}_0 E \setminus \{\mathbf{f}_j\}) \cup -\mathbb{N}_0 E. \tag{3}$$

In the remaining part of this section let $(\boldsymbol{\Delta}, \chi, E)$ be a connected rank 4 arithmetic root system.

Lemma 9. *One has* $\prod_{i\neq j}(q_{ij}q_{ji} - 1) = 0$.

Proof. Assume that $q_{ij}q_{ji} \neq 1$ for all $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Since $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4)$ is finite by Proposition 8, one obtains the equation $\prod_{i=2}^4 q_{1i}q_{i1} \prod_{i=3}^4 q_{2i}q_{i2} = 1$ from Lemma 7(ii). Again by Lemma 7(ii) and the finiteness of $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4)$ this yields $1 = q_{14}q_{41}q_{24}q_{42} = (q_{12}q_{21})^{-1}$ which is a contradiction. ∎

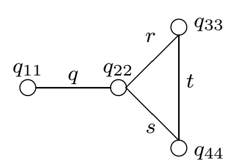
Lemma 10. *For given $a, b \in \{1, 2, 3, 4\}$ with $a \neq b$ one has the equation* $\prod_{i\neq a}(q_{ai}q_{ia} - 1) \prod_{i\neq b}(q_{bi}q_{ib} - 1) = 0$.

Proof. Without loss of generality assume that $a = 1$ and $b = 2$ and that the claim of the lemma is false. By the previous lemma one has $q_{34}q_{43} = 1$. Consider $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$. Then one gets $\left(\prod_{i=2}^4 q_{1i}q_{i1}\right) q_{24}q_{42} = 1$ which is a contradiction to $q_{13}q_{31} \neq 1$ and the finiteness of $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4)$. ∎

Lemma 11. *The graph $\mathcal{D}_{\chi,E}$ is not a labeled cycle graph.*

Proof. Assume that $(q_{12}q_{21} - 1)(q_{23}q_{32} - 1)(q_{34}q_{43} - 1)(q_{14}q_{41} - 1) \neq 0$ and $q_{ij}q_{ji} = 1$ if $|i - j| \geq 2$. Then the finiteness of $\boldsymbol{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ implies that $q_{23}q_{32}q_{34}q_{43}q_{14}q_{41} = 1$. By symmetry this gives that $q_{ij}q_{ji} = q_{12}q_{21}q_{23}q_{32}q_{34}q_{43}q_{14}q_{41}$ whenever $|i-j| = 1$, and $q_{12}q_{21} \in R_3$. Moreover, from Lemma 7(ii) one obtains that $(q_{11}q_{12}q_{21}q_{22} + 1)(q_{33} + 1)(q_{44} + 1) = 0$. By Lemma 6 one gets that $\prod_{i=1}^4(q_{ii} + 1) = 0$. Without loss of generality assume that $q_{11} = -1$. Then one can apply $s_{\mathbf{e}_1,E}$, and hence Lemma 10 implies the claim. ∎

Lemma 12. *If $\mathcal{D}_{\chi,E}$ is*

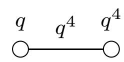


(4)

then one of the two systems of equations $q_{22}+1=(qr-1)(qs-1)=0$, $q_{22}q-1=(q-r)(q-s)=0$ *is valid.*

Proof. The finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_2+\mathbf{e}_4)$ implies that equation $(q_{22}^2-1)(q_{22}^2q^2-1)=0$ holds. Similarly, considering $\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_4)$ one obtains that $(q_{22}^2qr-1)(q_{22}^2qs-1)=0$. Thus one has two possibilities. In the first case one has $q_{22}=-1$ and $(qr-1)(qs-1)=0$, and in the second the equations $q_{22}^2q^2=1$ and $(q-r)(q-s)=0$ hold. Suppose that $q_{22}q=-1$ and $q_{22}^2qr=1$. Then $q_{22}\neq -1$, and hence Lemma 7(vii) applied to $\mathbf{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ gives that $q_{22}qr=-1$. This is a contradiction to $r\neq 1$. ■

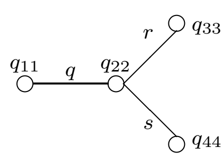
Lemma 13. *If* $\mathcal{D}_{\chi,E}$ *is as in Figure (4) then* $(q_{33}+1)(q_{44}+1)=0$*. In particular,* $(\mathbf{\Delta},\chi,E)$ *is Weyl equivalent to an arithmetic root system* $(\mathbf{\Delta}',\chi',E)$ *such that* $\mathcal{D}_{\chi',E}$ *is a labeled path graph.*

Proof. Suppose that $(q_{33}+1)(q_{44}+1)\neq 0$. By the finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4)$ one gets $q_{22}=-1$, $q_{33}r=1$, $q_{44}s=1$, $r^2=s^2$, and $r\in R_3\cup R_6$. Further, the finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4)$ gives that $q_{11}q=1$. By Lemma 12 one can assume that $qr=1$. Since $\chi\chi^{\mathrm{op}}(\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)=s\neq 1$, one has $\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4\in\mathbf{\Delta}$. Then one obtains a contradiction to the finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4,\mathbf{e}_3+\mathbf{e}_4)$ 

and relation $q\in R_3\cup R_6$.

To obtain the last assertion of the lemma assume that $q_{33}=-1$ and apply $s_{\mathbf{e}_3,E}$. ■

Lemma 14. *If* $\mathcal{D}_{\chi,E}$ *is*



(5)

then one has either $q_{22}=-1$*, and at least two of* qr,qs *and* rs *are equal to 1, or at least two of* $q_{22}q,q_{22}r$ *and* $q_{22}s$ *are equal to 1.*

Proof. Assume first that $q_{22}=-1$. Then one can apply $s_{\mathbf{e}_2,E}$ and hence Lemma 10 gives the claim.

Consider now the case $q_{22}\neq -1$. Assume first that $q_{11}qq_{22}=-1$. By Lemma 6 this implies that $q_{11}=-1$ and $q_{22}q=1$. Then one can apply $s_{\mathbf{e}_1,E}$, and the previous case implies that $(q^{-1}r-1)(q^{-1}s-1)=0$.

Finally, by twist equivalence one can assume that $q_{11}qq_{22}, q_{22}rq_{33}$ and $q_{22}sq_{44}$ are different from -1. Then the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_2+\mathbf{e}_4)$ and Lemma 7(ii) imply that at least one of q_{22}^2qr, q_{22}^2qs and q_{22}^2rs is equal to 1. Without loss of generality suppose that $q_{22}^2qr=1$. Then relation $q_{22}\neq -1$ gives that $q_{22}qr\neq -1$, and hence Lemma 7(vii) applied to $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ implies that $q_{22}q=q_{22}r=1$. ∎

Proposition 15. *If $(\boldsymbol{\Delta},\chi,E)$ is not of Cartan type then it is Weyl equivalent to an arithmetic root system $(\boldsymbol{\Delta}',\chi',E)$ such that $\mathcal{D}_{\chi',E}$ is a labeled path graph.*

Proof. By Lemmata 9, 10, 11 and 13 it suffices to show the claim under the assumption that $\mathcal{D}_{\chi,E}$ is of the form (5).

If $q_{22}=-1$ then by Lemma 14 one can assume that $qr=1$ and $qs=1$. If $q=-1$ then by [11, Lemma 16] one can suppose additionally that $q_{33}=q_{44}=-1$. Then the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ implies that (χ,E) is of Cartan type which is a contradiction. On the other hand, if $q\neq -1$ then one can apply $s_{\mathbf{e}_2,E}$ and Lemma 13 gives the claim.

Assume now that $q_{22}\neq -1$. Further, by Lemma 14 one can suppose additionally that $q_{22}r=q_{22}s=1$. Then Lemma 7(ii) and the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_2+\mathbf{e}_4)$ give that equation

$$(q_{22}q-1)(q_{22}-q)=0 \tag{6}$$

holds. Moreover, since (χ,E) is not of Cartan type, [11, Cor. 13] implies that one of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2)$, $\boldsymbol{\Delta}(\chi;\mathbf{e}_2,\mathbf{e}_3)$ and $\boldsymbol{\Delta}(\chi;\mathbf{e}_2,\mathbf{e}_4)$ has a generalized Dynkin diagram appearing in rows 3, 5, 6, or 7 of [11, Table 1]. One has the following possibilities.

(i) $q_{33}=-1$. Apply $s_{\mathbf{e}_3,E}$. The claim follows from the second paragraph of the proof of this lemma.

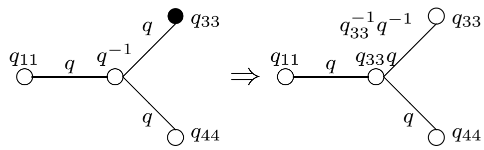
(ii) $q_{11}=-1$, $q=q_{22}^{-2}$. Apply $s_{\mathbf{e}_1,E}$ and use Lemma 14 to obtain that $-q_{22}^{-1}r=1$, that is $q_{22}\in R_4$. This is a contradiction to Equation (6).

(iii) $q_{22}\in R_3$, $q=-q_{22}$, $q_{11}=-1$. This is again a contradiction to (6).

(iv) $q_{22}\in R_3$, $q_{11}q=1$, $q^3\neq 1$. Recall again (6).

(v) $q_{33}\in R_3$, $q_{22}^3\neq 1$. Since $q_{33}\neq -1$ and $q_{33}r\neq 1$, Lemma 7(ix) and the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_2+\mathbf{e}_4)$ imply that $q_{22}q=1$, that is one has $q=r=s=q_{22}^{-1}$. Thus Lemma 7(vii) applied to $\boldsymbol{\Delta}(\chi;\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_2+\mathbf{e}_4)$

gives that equation $q_{33}q^2 = -1$ is fulfilled. Apply now the transformation

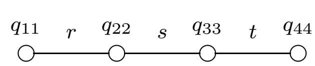


If $q_{33}q = -1$ then the claim follows from the second paragraph of the proof of this lemma. Otherwise Lemma 14 gives that $q_{33}q^2 = 1$ which is a contradiction to equation $q_{33}q^2 = -1$. ∎

Theorem 16. *Let k be a field of characteristic 0. Then twist equivalence classes of connected arithmetic root systems of rank 4 are in one-to-one correspondence to generalized Dynkin diagrams appearing in Table B. Moreover, two such arithmetic root systems are Weyl equivalent if and only if their generalized Dynkin diagrams appear in the same row of Table B and can be presented with the same set of fixed parameters.*

Proof. One can check case by case that each row of Table B contains the generalized Dynkin diagrams of the representants of a unique Weyl equivalence class (χ, E), where χ is a bicharacter on $\mathbb{Z}^4$ and E is a fixed basis of $\mathbb{Z}^4$. In order to prove that these diagrams correspond to arithmetic root systems one has to show that $W^{\mathrm{ext}}_{\chi,E}$ is finite. If (χ, E) is of Cartan type then this follows from [13, Theorem 1]. In all other cases Proposition 5 will be applied. One checks first that $W^{\mathrm{ext}}_{\chi,E}$ is full and using [11, Theorem 12] one recognizes that $W^{\mathrm{ext}}_{\chi,E'\subset E}$ is finite for all subsets $E' \subset E$ of 3 elements. It remains to find an element (T, E) with respect to one fixed representant of the Weyl equivalence class which has the property $T(E) \subset E \setminus \{\mathbf{e}\} \cup \{-\mathbf{f}\}$ for some $\mathbf{e} \in E$ and $\mathbf{f} \in \boldsymbol{\Delta}^+_E$. These elements (T, E) are listed in Appendix A.

It remains to prove that all arithmetic root systems have a generalized Dynkin diagram appearing in Table B. By [13, Theorem 1] and Proposition 15 it suffices to consider pairs (χ, E) such that $\mathcal{D}_{\chi,E}$ is a labeled path graph



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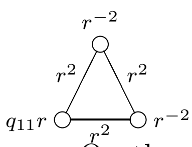
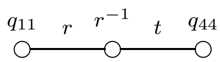
In what follows several cases will be considered according to the set $I = \{i \in \{1, 2, 3, 4\} \,|\, q_{ii} = -1\}$.

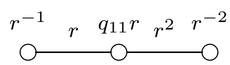
Step 1. $q_{ii}^2 \neq 1$ for all $i \in \{1, 2, 3, 4\}$. By Lemma 7(i) for $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $\boldsymbol{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ one obtains that there exist $m_1, m_2, m_3, m_4 \in \{1, 2\}$ such that $q_{22}^{m_1}r = q_{22}^{m_2}s = q_{33}^{m_3}s = q_{33}^{m_4}t = 1$. More precisely, either (χ, E) is of Cartan type, in which case $\mathcal{D}_{\chi,E}$ appears in rows 1–5 of Table B, or one has without loss of generality $q_{11} \in R_3$, $q_{22}r = 1$, $q_{33}s = 1$, $(q_{22}s-1)(q_{22}^2s-1) = 0$, and $q_{22}^3, q_{33}^3 \neq 1$. If $q_{22}^2s = 1$ then Lemma 7(i) for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ gives that

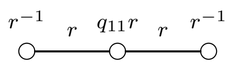
$q_{33}t = q_{44}t = 1$. In this case the finiteness of $\mathbf{\Delta}(\chi; 2\mathbf{e}_1+\mathbf{e}_2, 2\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_4)$



, relation $q_{33}^2 = r^{-6} \neq 1$ and Lemma 7(ii) give a contradiction.

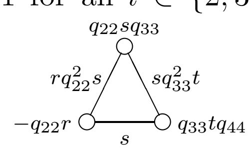
On the other hand, if $q_{22}s = 1$ then Lemma 7(i) for $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ 

implies that $q_{44}t = 1$, and either $r = t$, $r^3 \neq 1$, or $r^2 = t$, $r^6 \neq 1$. Consider first the case $r^2 = t$. Applying Lemma 7(i) to $\mathbf{\Delta}(\chi; \mathbf{e}_2 + \mathbf{e}_3, 2\mathbf{e}_1+\mathbf{e}_2, 2\mathbf{e}_3+\mathbf{e}_4)$ 

one obtains that $q_{11}r = -1$ or $q_{11}r^2 = 1$ or $q_{11}r^3 - 1 = q_{11}^2r^3 - 1 = 0$ which is a contradiction to $q_{11} \in R_3$ and $r^6 \neq 1$. In the last case one has $r = t$. Apply now Lemma 7(i) to $\mathbf{\Delta}(\chi; \mathbf{e}_2 + \mathbf{e}_3, 2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4)$ 

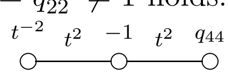
and conclude that $(q_{11}r + 1)(q_{11}r^2 - 1) = 0$. If $q_{11}r = -1$ then $\mathcal{D}_{\chi,E}$ appears in row 17 of Table B. Otherwise relation $r^3 \neq 1$ implies that $r = -q_{11}$, and hence $\mathcal{D}_{\chi,E}$ appears in row 15 of Table B.

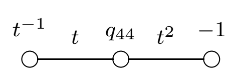
Step 2. $q_{11} = -1$, $q_{ii} \neq -1$ for all $i \in \{2, 3, 4\}$. By Lemma 7(ii) for

$\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$ 

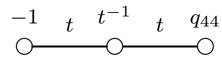
and Lemma 6 one obtains that either equation $(rq_{22}^2s - 1)(sq_{33}^2t - 1) = 0$ is fulfilled, or relations $q_{22}sq_{33}, q_{33}tq_{44} \neq -1$, $q_{22}r = 1$ and $(rq_{22}^2s - s)(rq_{22}^2s + s) = 0$ hold. The last case is a contradiction to $q_{22}^2 \neq 1$.

Step 2.1. Assume first that $rq_{22}^2s = 1$. Since $q_{22} \neq -1$, Lemma 7(vii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ gives that $q_{22}r = q_{22}s = 1$. Further, Lemma 7(i) for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ implies that $(q_{33}s - 1)(q_{33}^2s - 1) = 0$.

If $q_{22}r = q_{22}s = q_{33}^2s = 1$ then again by Lemma 7(i) for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ one obtains that $q_{33}t = 1$, and either $q_{44} \in R_3$, $t^6 \neq 1$, or $q_{44}t = 1$. Note that also relation $t^4 = q_{22}^{-2} \neq 1$ holds. Therefore Lemma 7(iii) for $\mathbf{\Delta}(\chi; \mathbf{e}_2 + 2\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4)$ 

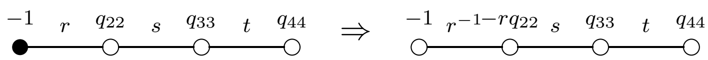
implies that $q_{44} = t^{-1} \in R_{10}$. This is a contradiction to the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_1+\mathbf{e}_2)$ 

and Lemma 7(iv).

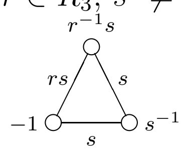
On the other hand, if $q_{22}r = q_{22}s = q_{33}s = 1$ then by Lemma 7(i) for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ one obtains that either equations $q_{33}^2 t = 1$, $q_{44}t = 1$ hold, or equation $q_{33}t = 1$ is valid. In the first case $\mathcal{D}_{\chi,E}$ appears in row 8 of Table B. In the second case the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4)$ 

and Lemma 7(v) imply that $q_{44}t = 1$ or $q_{44}^2 t = 1$ or $q_{44} = -t \in R_3$. Then $\mathcal{D}_{\chi,E}$ appears in rows 6, 7, and 16 of Table B, respectively.

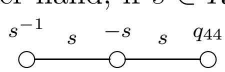
Step 2.2. Now suppose that $rq_{22}^2 s \neq 1$ and $sq_{33}^2 t = 1$. Since relation $q_{33} \neq -1$ holds, Lemma 7(vii) for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ gives $q_{33}s = q_{33}t = 1$. Further, Lemma 7(vii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ yields $(q_{22}r - 1)(q_{22}s - 1)(q_{22}rs + 1) = 0$. If $(q_{22}r - 1)(q_{22}s - 1) \neq 0$ then one can change the representant of the Weyl equivalence class via the transformation



and therefore one can assume that $(q_{22}r - 1)(q_{22}s - 1) = 0$.

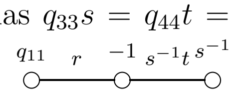
In the first case one has $q_{22}r = 1$, and hence $r \neq s$. Lemma 7(i) for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ implies that $q_{33} = q_{44}$ and either $q_{22} \in R_3$ or $q_{22}^2 s = 1$. If $q_{22}^2 s = 1$ then $\mathcal{D}_{\chi,E}$ appears in row 9 or 18 of Table B. Otherwise one has the relations $r \in R_3$, $s^3 \neq 1$, and hence Lemma 7(ii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, 2\mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$ 

implies that $rs^3 = 1$, and either $r^{-1}s = -1$ or $r^2 s^2 = s^2$. This is impossible since $r \in R_3$.

In the second case equation $q_{22}s = 1$ holds, and hence relation $rq_{22}^2 s \neq 1$ implies that $r \neq s$. Therefore Lemma 7(iv) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ tells that $r = s^2$ and $s \in R_3 \cup R_4 \cup R_6$. If $s \in R_4$ then Lemma 7(i) for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ implies that (χ, E) is of Cartan type and hence $\mathcal{D}_{\chi,E}$ appears in row 3 of Table B. On the other hand, if $s \in R_3 \cup R_6$ then Lemma 7(i) for $\mathbf{\Delta}(\chi; \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4)$ 

gives a contradiction.

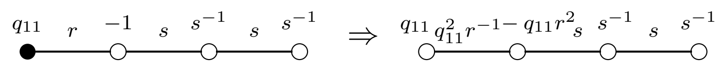
Step 3. $q_{22} = -1$, $q_{ii} \neq -1$ for all $i \in \{1, 3, 4\}$.

Step 3.1. Assume first that $q_{11}r \neq 1$. Then by [11, Lemma 16] for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$ one has $q_{33}s = q_{44}t = 1$. Further, [11, Lemma 16] for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$ 

implies that $(s - t)(s^2 - t) = 0$.

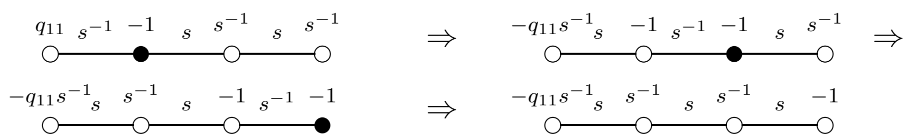
If $s^2 = t$ then relation $s^4 \neq 1$ and Lemma 7(iii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$ imply that $q_{11}^2 r = 1$ and $(s^{-1}q_{11}^2 - 1)(s^{-1}q_{11}^3 + 1) = (s^{-2}q_{11}^2 - 1)(s^{-2}q_{11}^3 + 1) = 0$. The last equations contradict to the assumption $s^4 \neq 1$ and $q_{11}^2 \neq 1$.

If $s = t$ then consider first two special cases. If $q_{11} \in R_3$, $r^3 \neq 1$, and $q_{11}r^2 \neq 1$, then the transformation

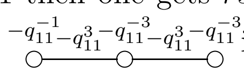


shows that $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system appearing in Step 1.

On the other hand, if $rs = 1$ then the transformations



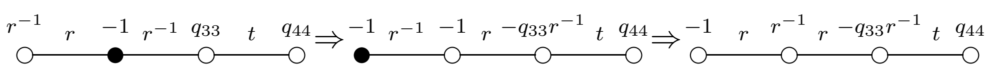
and relation $q_{11}s^{-1} = q_{11}r \neq 1$ show that $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system appearing in Step 2.

Finally, apply Lemma 7(iii) to $\mathbf{\Delta}(\chi; \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1)$ and conclude that either $q_{11}^2 r = 1$, $(s^{-1}q_{11}^2 - 1)(s^{-1}q_{11}^3 + 1) = 0$ or $q_{11} = -s^{-1} \in R_3$, $r \in \{-1, -q_{11}\}$. If $q_{11}^2 r = q_{11}^2 s^{-1} = 1$ then one gets $rs = 1$, and if $q_{11}^2 r = -q_{11}^3 s^{-1} = 1$ then $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ 

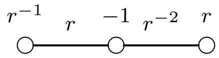
is of infinite Cartan type. It remains to check the case $q_{11} = -s^{-1} \in R_3$, $r^3 \neq 1$. As noted above one can assume also that $q_{11}r^2 = 1$ and $rs \neq 1$. However relations $q_{11} = -s^{-1} \in R_3$ and $q_{11}r^2 = 1$ imply that $(rs-1)(rs+1) = 0$ which is a contradiction to relations $-s \in R_3$, $r^3 \neq 1$ and $rs \neq 1$.

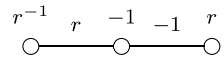
Step 3.2. Suppose now that $q_{11}r = 1$. Then $r^2 = q_{11}^{-2} \neq 1$.

Step 3.2.1. Consider the case when $rs = 1$. Then the transformations

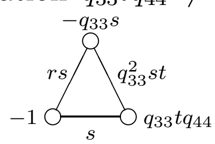


show that if $q_{33} \neq r$ then $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system appearing in Step 2. Therefore one can assume that $q_{33} = r$. Now use that $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ is finite and $q_{33}, q_{44} \neq -1$ and apply [11, Theorem 12]. If $r = t^{-1} = q_{44}$ then $\mathcal{D}_{\chi,E}$ appears in row 10 of Table B. If $q_{44}^2 t = q_{33}t = 1$ then $\mathcal{D}_{\chi,E}$ can be found in row 11 of Table B. If $q_{44}t = q_{33}^2 t = 1$ then $\mathcal{D}_{\chi,E}$

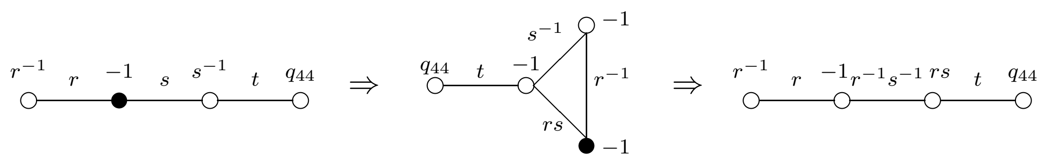
appears in row 12 of Table B. If $q_{44}t = q_{33}^3 t = 1$ then Lemma 7(iii) applied to $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, 2\mathbf{e}_3 + \mathbf{e}_4)$ 

implies that $r \in R_4$ and then one can find $\mathcal{D}_{\chi,E}$ in row 22 of Table B. If $q_{44} = -t = -r^{-1} \in R_3$ then row 19 of Table B contains $\mathcal{D}_{\chi,E}$. Finally, if $r = -t^{-1} = -q_{44} \in R_3$ then Lemma 7(iii) tells that $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$ 

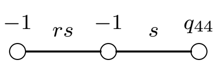
is not finite.

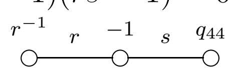
Step 3.2.2. Assume now that $rs \neq 1$. Note that Lemma 6 applied to $\mathbf{\Delta}(\chi; \mathbf{e}_3, \mathbf{e}_4)$ implies also relation $q_{33}tq_{44} \neq -1$. Then Lemma 7(ii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$ 

gives that one of equations $q_{33}^2 st = 1$, $q_{33}s = 1$ holds. Therefore by Lemma 7(vii) for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ one obtains that $q_{33}s = 1$. Further, Lemma 12 and the transformations



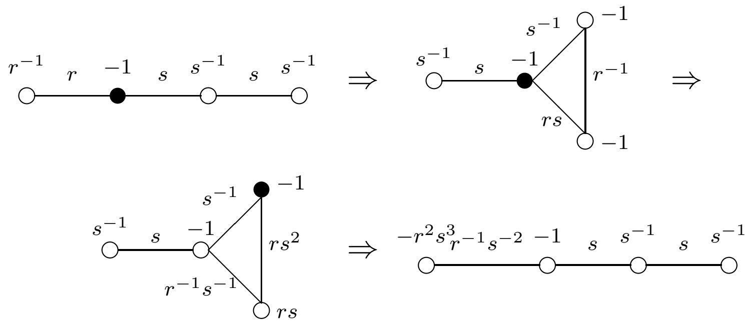
give that $(rst - 1)(s^{-1}t - 1) = 0$, and hence either $rs = -1$ or (by Weyl equivalence) one can assume that $q_{33}t = 1$. Note that relations $rs = -1$ and $s \neq t$ imply that $rst = 1$ and hence $t = -1$ which is a contradiction to $q_{33}, q_{44} \neq -1$ and [11, Lemma 10(i)] for $\mathbf{\Delta}(\chi; \mathbf{e}_3, \mathbf{e}_4)$. Therefore one can suppose that $s = t$.

By relation $rs \neq 1$ and Lemma 7(vi) for $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$ 

one obtains that $(rs+1)(q_{44}s-1)(rs^2-1) = 0$. On the other hand, Lemma 7(iii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$ 

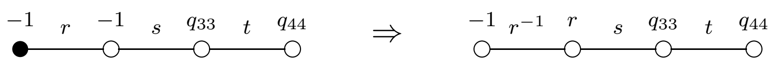
together with relations $s^2 \neq 1$, $rs \neq 1$ implies that either $q_{44}s = 1$ or $q_{44}^2 s = 1$, $q_{44}^3 = -r$. Since $q_{44}^2 \neq 1$, these two conditions give that $q_{44}s = 1$.

If $rs = -1$ or $rs^2 = 1$ or $r^2s^3 = 1$ then $\mathcal{D}_{\chi,E}$ appears in row 14, 13 and 9 of Table B, respectively. Otherwise the transformations



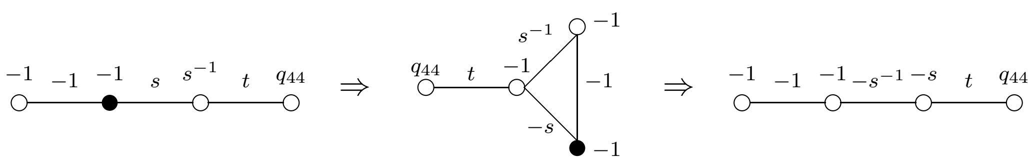
show that $(\boldsymbol{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system appearing in Step 3.1.

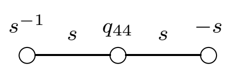
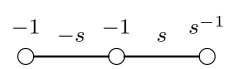
Step 4. $q_{11} = q_{22} = -1$, $q_{33}, q_{44} \neq -1$. One can assume that $r = -1$, because otherwise the transformation

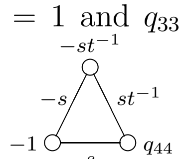


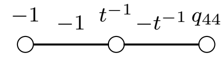
gives an arithmetic root system from Step 2. Further, Lemma 7(vii) for $\boldsymbol{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ implies that equation $(q_{33}s-1)(q_{33}t-1)(q_{33}st+1) = 0$ holds.

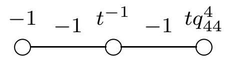
Step 4.1. Consider the case $q_{33}s = 1$. Since $q_{33}^2 \neq 1$, one obtains that $s^2 \neq 1$. Then using Lemma 12 and the first of the transformations



one obtains that $(s-t)(st+1) = 0$ and hence by the second of the above transformations one can assume that $s = t$. Further, Lemma 7(i) applied to $\boldsymbol{\Delta}(\chi; \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$  gives relations $q_{44}s = 1$ and $s \in R_3 \cup R_4 \cup R_6$. Then Lemma 7(vi) for $\boldsymbol{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$  gives that $s \in R_3 \cup R_4$. In this case $\mathcal{D}_{\chi,E}$ appears in row 9 and 13 of Table B, respectively.

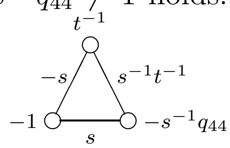
Step 4.2. Assume that $q_{33}t = 1$ and $q_{33}s \neq 1$. Then Lemma 7(ii) for $\boldsymbol{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$  implies that either $s = -1$ or

$-s^3t^{-1} = s^2t^{-1} = 1$. Therefore it remains to consider the case $s = -1$. By Lemma 7(vii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$ 

and since $t^2 \neq 1$ one gets $t \in R_4$. If $q_{44}t \neq 1$ then [11, Lemma 10](i) gives that $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + 2\mathbf{e}_4)$ 

is of infinite Cartan type, a contradiction. Hence one has $q_{44} = t^{-1} \in R_4$. In this case $\mathcal{D}_{\chi,E}$ appears in row 4 of Table B.

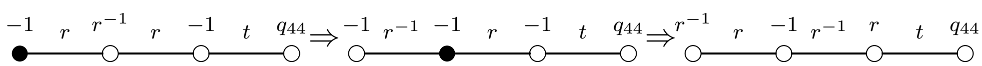
Step 4.3. Suppose now that $q_{33}st = -1$ and $q_{33}s \neq 1 \neq q_{33}t$. Then $s \neq -1$, $t \neq -1$, $st \neq 1$, and by Lemma 6 for $\mathbf{\Delta}(\mathbf{e}_3, \mathbf{e}_4)$ also relation $s^{-1}q_{44} \neq 1$ holds. Hence by Lemma 7(ii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$



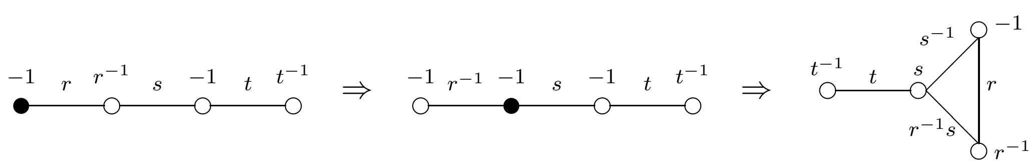
one obtains that $q_{44} = -1$ which is a contradiction.

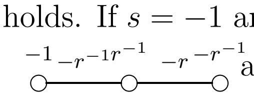
Step 5. Consider the setting $q_{11} = q_{33} = -1$, $q_{22}, q_{44} \neq -1$. Lemma 7(vii) applied to $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ gives that $(q_{22}r - 1)(q_{22}s - 1)(q_{22}rs + 1) = 0$.

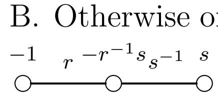
Step 5.1. The case $q_{22}r = 1$. If $q_{22}s = 1$ then the transformations

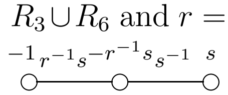


show that $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system in Step 3. Therefore assume that $q_{22}s \neq 1$. In this case [11, Lemma 16] for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ implies that $q_{44}t = 1$. Moreover, the transformations

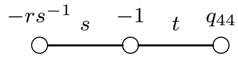


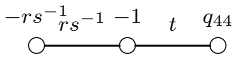
and Lemma 12 show that either equations $s = -1$, $-r^{-1}t = 1$, or equation $st = 1$ holds. If $s = -1$ and $t = -r$ then Lemma 7(vii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$ 

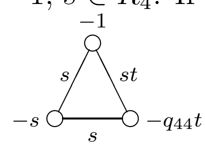
and relation $r^2 = q_{22}^{-2} \neq 1$ imply that $r \in R_4$, and hence $\mathcal{D}_{\chi,E}$ appears in row 22 of Table B. Otherwise one has $st = 1$ and then Lemma 7(vii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$ 

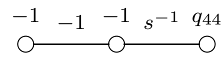
tells that relations $s \in R_3 \cup R_6$ and $r = s^2$ hold. Moreover Lemma 7(vii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$ 

implies that $s \in R_3$, and then $\mathcal{D}_{\chi,E}$ can be found in row 21 of Table B.

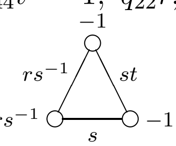
Step 5.2. The case $q_{22}r \neq 1$, $q_{22}s = 1$, $s^2, q_{44}^2 \neq 1$. By [11, Lemma 16] for $\mathbf{\Delta}(\chi; \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ 

and for $\mathbf{\Delta}(\chi; \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_4)$ 

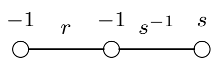
one obtains that either $q_{44}t = 1$ or $r = -1$, $s \in R_4$. If $q_{44}t \neq 1$ then Lemma 7(ii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_4)$ 

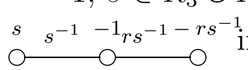
implies that $st = 1$. Therefore Lemma 7(vi) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_4)$ 

gives a contradiction.

Thus it remains to consider the case $q_{22}s = q_{44}t = 1$, $q_{22}r, s^2, t^2 \neq 1$. By Lemma 7(ii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_4)$ 

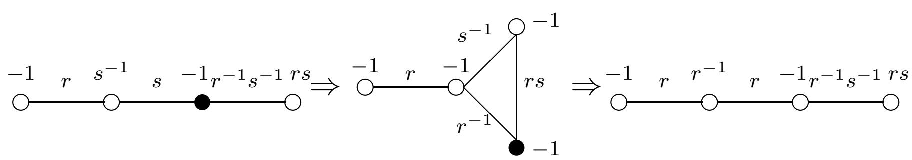
one gets equation $(st-1)(rst-1) = 0$.

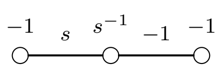
Step 5.2.1. Assume that $q_{22}s = q_{44}t = st = 1$, $r \neq s$, $s^2 \neq 1$. By Lemma 7(viii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ one gets that either $r = -1$, $s \in R_3 \cup R_4 \cup R_6$, or $r^2 = s^{-2} \in R_3$. If $s \in R_6$ and $r^2 = s^{-2}$ then Lemma 7(vi) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_4)$ 

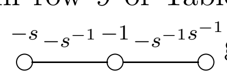
gives a contradiction. Otherwise one has either $s \in R_3$, $r^2 = s$, or $r = -1$, $s \in R_3 \cup R_4 \cup R_6$. Then Lemma 7(iii) for $\mathbf{\Delta}(\chi; \mathbf{e}_4, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2)$ 

implies that $r = -1$ and $s \in R_4$. Then $\mathcal{D}_{\chi,E}$ appears in row 12 of Table B.

Step 5.2.2. Assume that $q_{22}s = q_{44}t = rst = 1$, and $r \neq s$, $s^2 \neq 1$, $r^2s^2 \neq 1$. If $r \neq -1$ then the transformations

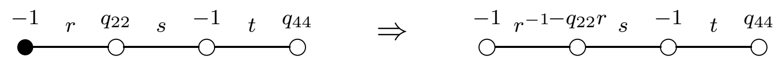


imply that $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system from Step 5.1. Therefore one can assume that $r = -1$. By Lemma 7(viii) for $\mathbf{\Delta}(\chi; \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1)$ 

one gets relation $s \in R_3 \cup R_4 \cup R_6$. If $s \in R_4$ then $\mathcal{D}_{\chi,E}$ can be found in row 9 of Table B. Otherwise Lemma 7(iii) for $\mathbf{\Delta}(\chi; \mathbf{e}_4, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2)$ 

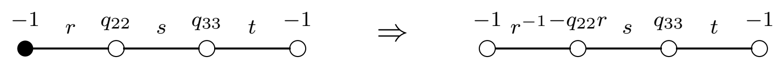
gives a contradiction.

Step 5.3. Assume now that $q_{22}r \neq 1 \neq q_{22}s$ and $q_{22}rs = -1$. Then the transformation



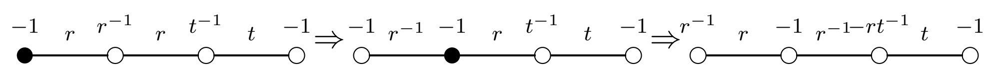
shows that $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system in Step 5.2.

Step 6. $q_{11} = q_{44} = -1$, $q_{22}, q_{33} \neq -1$. By Lemma 7(vii) for $\mathbf{\Delta}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ one gets $(q_{22}r - 1)(q_{22}s - 1)(q_{22}rs + 1) = 0$. The transformation

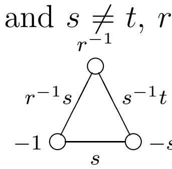


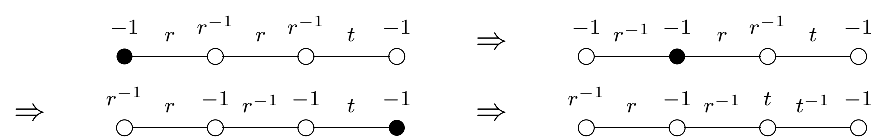
shows that one can assume that $(q_{22}r - 1)(q_{22}s - 1) = 0$. By symmetry one can suppose that equation $(q_{33}s - 1)(q_{33}t - 1) = 0$ holds, too.

Step 6.1. Suppose first that $q_{22}r = q_{33}t = 1$ and $r^2, t^2 \neq 1$. If $q_{22}s = q_{33}s = 1$ then $\mathcal{D}_{\chi,E}$ appears in row 10 of Table B. If $q_{22}s = 1$ and $q_{33}s \neq 1$ then the transformation

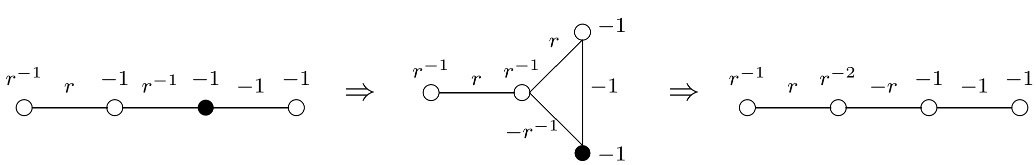


shows that $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system in Step 3. Finally, if $q_{22}s \neq 1$ and $q_{33}s \neq 1$ then one obtains a contradiction to [11, Corollary 13] for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3)$.

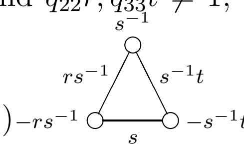
Step 6.2. Assume now that $q_{22}r = q_{33}s = 1$ and $s \neq t$, $r^2, s^2 \neq 1$. Then Lemma 7(ii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$  implies that $r = s$. Therefore if $t \neq -1$ then by the transformations



one can see that $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system from Step 5. Thus suppose that $t = -1$. If $r \in R_4$ then $\mathcal{D}_{\chi,E}$ appears in row 8 of Table B. Otherwise the transformations

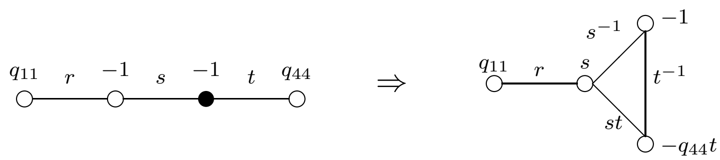


show that $(\boldsymbol{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system from Step 4.

Step 6.3. Suppose now that $q_{22}s = q_{33}s = 1$ and $q_{22}r, q_{33}t \neq 1$, $s^2 \neq 1$. Then Lemma 7(ii) for $\boldsymbol{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4)$ 

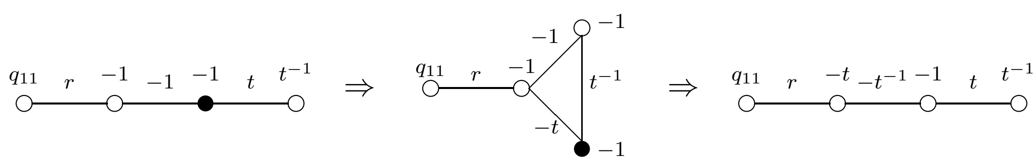
gives a contradiction.

Step 7. $q_{22} = q_{33} = -1$, $q_{11}, q_{44} \neq -1$. If $s \neq -1$ then the transformation



and Lemma 12 give first that either $st = 1$ or $rs = 1$. By twist equivalence one can assume that $st = 1$. Then the above transformation shows also that $(\boldsymbol{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system from Step 3 or Step 4.

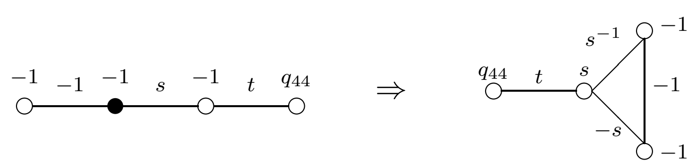
Assume now that $s = -1$. By [11, Lemma 16] for $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$ and twist equivalence one can assume that $q_{44}t = 1$, and hence $t^2 \neq 1$. Therefore the transformation



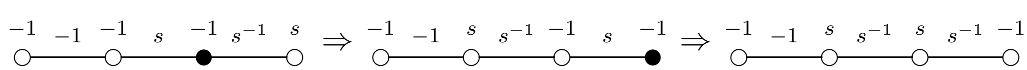
tells that $(\boldsymbol{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system in Step 3.

Step 8. $q_{11} = q_{22} = q_{33} = -1$, $q_{44} \neq -1$. Like at the beginning of Step 4 one can conclude that $r = -1$.

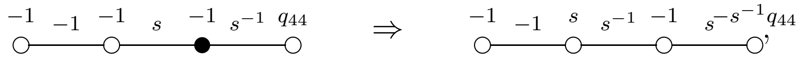
If $s \neq -1$ then Lemma 12 and the transformation



show that $st = 1$. In this case, if $q_{44}t = 1$ or $q_{44}t \neq 1$ then the transformations

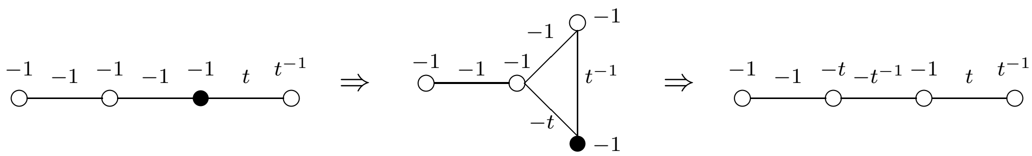


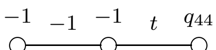
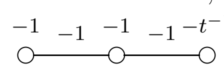
and



give an arithmetic root system from Step 6 and Step 5, respectively.

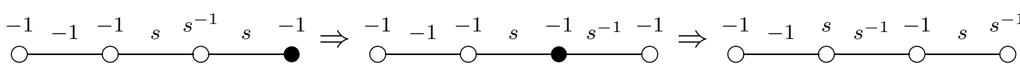
Consider now the case when $s = -1$. If $q_{44}t = 1$ then $t^2 \neq 1$ and hence the transformations



show that $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system in Step 5. On the other hand, if $q_{44}t \neq 1$ then Lemma 7(vi) for $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$  implies that either $q_{44} \in R_3$, $t^2 q_{44} = 1$, or $t = -1$, $q_{44} \in R_4$. In the second case $\mathcal{D}_{\chi,E}$ appears in row 2 of Table B, and in the first case, where $t \in R_3 \cup R_6$, Lemma 7(vi) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4)$  gives a contradiction.

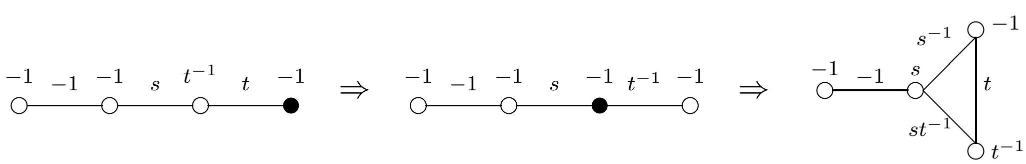
Step 9. $q_{11} = q_{22} = q_{44} = -1$, $q_{33} \neq -1$. By the same arguments as in Steps 4 and 5.3 one can assume that equations $r = -1$, $(q_{33}s-1)(q_{33}t-1) = 0$ hold.

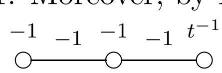
Suppose that $r = -1$, $q_{33}s = 1$, and $s^2 \neq 1$. As in Step 4.1 it suffices to consider the case $t = s$. Then the transformations

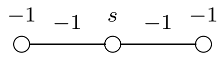


give an arithmetic root system from Step 5.

Assume that $r = -1$, $q_{33}t = 1$, and $s \neq t$, $t^2 \neq 1$. Then Lemma 12 and the transformations



give that $s = -1$. Moreover, by Lemma 7(vi) for the arithmetic root system $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  one has $t \in R_4$, and hence $\mathcal{D}_{\chi,E}$ appears in row 9 of Table B.

Step 10. $q_{11} = q_{22} = q_{33} = q_{44} = -1$. Following the argument at the beginning of Step 4 one can assume that $r = t = -1$. Then Lemma 7(vii) for $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4)$ 

tells that $s = -1$ and hence $\mathcal{D}_{\chi,E}$ appears in row 1 of Table B. ■

5 Simple chains

The results of the previous section show that labeled path graphs play an important role in the study of generalized Dynkin diagrams. As a preparation for Theorem 19 here a special class of such graphs will be studied.

Let E be a basis of $\mathbb{Z}^d$, where $d \geq 2$, and let χ be a bicharacter on $\mathbb{Z}^d$.

Definition 1. Assume that $(\mathbf{\Delta}, \chi, E)$ is an arithmetic root system of rank $d \geq 2$ and $\mathcal{D}_{\chi,E}$ is a labeled path graph. Call this graph a *simple chain* (of length d) if

$$(q_{11}q_{12}q_{21} - 1)(q_{11} + 1) = 0, \quad (q_{dd}q_{d,d-1}q_{d-1,d} - 1)(q_{dd} + 1) = 0,$$
$$q_{ii}^2 q_{i-1,i}q_{i,i-1}q_{i,i+1}q_{i+1,i} = 1 \text{ for } 1 < i < d.$$

By Lemma 7(vii) the latter equations hold if and only if

$$q_{ii} + 1 = q_{i-1,i}q_{i,i-1}q_{i,i+1}q_{i+1,i} - 1 = 0 \text{ or } q_{ii}q_{i-1,i}q_{i,i-1} = q_{ii}q_{i,i+1}q_{i+1,i} = 1. \tag{7}$$

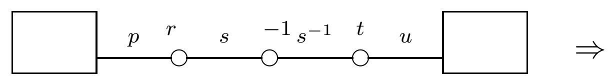
If $(\mathbf{\Delta}_i, \chi_i, E_i)$, where $i \in \{1, 2\}$, are arithmetic root systems, the generalized Dynkin diagrams $\mathcal{D}_{\chi_i,E_i}$ are labeled path graphs, and $E_1 = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subset E_2 = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$, where $2 \leq n < d$, then $\mathcal{D}_{\chi_2,E_2}$ is called a *prolongation of* $\mathcal{D}_{\chi_1,E_1}$ *to the right by a simple chain (of length* $d - n$*)* if

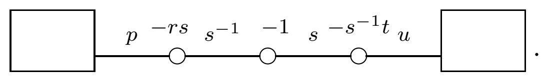
$$\begin{aligned}(q_{dd}q_{d,d-1}q_{d-1,d} - 1)(q_{dd} + 1) &= 0 \text{ and}\\ q_{jj}^2 q_{j-1,j}q_{j,j-1}q_{j,j+1}q_{j+1,j} &= 1 \text{ for } n \leq j < d.\end{aligned}$$

Similarly one defines prolongation of a labeled path graph to the left by a simple chain. Because no other prolongations will be considered, usually the attribute "by a simple chain" will be omitted.

Lemma 17. *If* $\mathcal{D}_{\chi,E}$ *is a simple chain and* $\mathcal{D}_{\chi',E}$ *is Weyl equivalent to* $\mathcal{D}_{\chi,E}$ *then* $\mathcal{D}_{\chi',E}$ *is a simple chain.*

Proof. If $i \in \{1,\dots,d\}$ and $q_{ii} \neq -1$ then Equation (7) for $\mathcal{D}_{\chi,E}$ implies that $\mathcal{D}_{\chi,E} = \mathcal{D}_{\chi',E}$, where $\chi' = \chi \circ (s_{\mathbf{e}_i,E} \times s_{\mathbf{e}_i,E})$. Otherwise application of $s_{\mathbf{e}_i,E}$ gives





Thus if $(rs-1)(r+1)=0$ then $(-rs+1)((-rs)s^{-1}-1)=0$, and if $(pr-1)(r+1)=0$ and $s=1/pr^2$ then

$$(p(-rs)-1)(-rs+1) = (-pr/pr^2-1)(-r/pr^2+1) = 0.$$

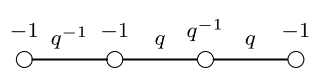
Thus the claim follows by symmetry. ∎

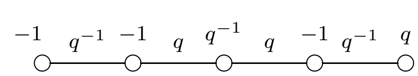
Let $\mathcal{D}_{\chi,E}$ be a simple chain and set $q := q_{dd}^2 q_{d-1,d} q_{d,d-1}$. By (7) this means that if $q_{dd} \neq -1$ then $q = q_{dd}$ and if $q_{dd} = -1$ then $q = q_{d-1,d}q_{d,d-1}$. Equations (7) imply that for all $i \in \{1,\dots,d-1\}$ one has $q_{i,i+1}q_{i+1,1} \in \{q,q^{-1}\}$ and $q_{11}^2 q_{12}q_{21} \in \{q,q^{-1}\}$. Moreover, the knowledge of q and all indices i with $1 \le i \le d$ and $q_{i-1,i}q_{i,i-1} = q$, where $q_{01}q_{10} := 1/q_{11}^2 q_{12}q_{21}$, determines $\mathcal{D}_{\chi,E}$ uniquely. Therefore the symbol

$$\boxed{C(d,q;i_1,\dots,i_j)} \qquad (8)$$

will be used for the simple chain of length d for which $q = q_{d-1,d}q_{d,d-1}q_{dd}^2$ holds, and for which equation $q_{i-1,i}q_{i,i-1} = q$, where $1 \le i \le d$, is valid if and only if $i \in \{i_1,i_2,\dots,i_j\}$. Additionally the convention $1 \le i_1 < i_2 < \ldots < i_j \le d$ is fixed, where $0 \le j \le d$. For brevity, in the running text also the notation $C(d,q;i_1,\dots,i_j)$ will be used.

Example 1. For $q \in k^* \setminus \{1\}$ the generalized Dynkin diagrams





are simple chains of length 4 and 5, respectively. One has $q_{i-1,i}q_{i,i-1} = q$ if and only if $i \in \{1,3,4\}$, and hence for these simple chains the symbols

$$\boxed{C(4,q;1,3,4)} \qquad \boxed{C(5,q;1,3,4)}$$

will be used. ∎

Proposition 18. *Two simple chains $C(d,q;i_1,\dots,i_j)$, where $0\le j\le d$, and $C(d',q';i'_1,i'_2,\dots,i'_{j'})$, where $0\le j'\le d'$, are Weyl equivalent if and only if $d=d'$ and one of the following conditions hold:*

$$(i)\quad q=q',\quad j=j',\qquad\qquad (ii)\quad qq'=1,\quad j+j'=d+1.$$

Proof. If $q_{i-1,i}q_{i,i-1}\ne q_{i,i+1}q_{i+1,i}$ for some $1\le i<d$ then Equation (7) implies that $q_{ii}=-1$ and exactly one of $q_{i-1,i}q_{i,i-1}$, $q_{i,i+1}q_{i+1,i}$ is equal to q. Hence one has either $i\in\{i_1,\dots,i_j\}$ or $i+1\in\{i_1,\dots,i_j\}$, but both relations can not be fulfilled. In this case applying the reflection $s_{\mathbf{e}_i,E}$ is equivalent to replacing the index i respectively $i+1$ in the argument of C by $i+1$ respectively i. Therefore (i) implies that the given simple chains are Weyl equivalent. Further, if $i_j=d$ then $q_{d-1,d}q_{d,d-1}=q$ and $q_{dd}=-1$. Application of $s_{\mathbf{e}_d,E}$ transforms q to q^{-1} and leaves all $q_{i-1,i}q_{i,i-1}$ with $1\le i\le d-1$ invariant. Thus the simple chain $C(d,q;i_1,\dots,i_{j-1},d)$ is Weyl equivalent to $C(d,q^{-1};i'_1,\dots,i'_{d-j},d)$, where relation $\{i_1,\dots,i_{j-1}\}\cup\{i'_1,\dots,i'_{d-j}\}=\{1,2,\dots,d-1\}$ holds. Hence using the first part of the proof one concludes that if condition (ii) holds then the two chains $C(d,q;i_1,\dots,i_j)$ and $C(d,q';i'_1,i'_2,\dots,i'_{j'})$ are Weyl equivalent.

On the other hand, if $q_{i-1,i}q_{i,i-1}=q_{i,i+1}q_{i+1,i}$ for some $1\le i<d$ then the simple chain is invariant under the reflection $s_{\mathbf{e}_i,E}$. Similarly, if $i_j\ne d$ then $q_{d-1,d}q_{d,d-1}q_{dd}=1$ and the simple chain is invariant under the reflection $s_{\mathbf{e}_d,E}$. Therefore the claim of the proposition follows from the first part of the proof. ∎

6 Connected arithmetic root systems of rank higher than four

Having once all arithmetic root systems of rank 4, it is relatively easy to classify those of higher rank too. The methods used are the same as in Section 4. The aim of this section is to prove the following theorem.

Theorem 19. *Let k be a field of characteristic 0. Then twist equivalence classes of connected arithmetic root systems of rank bigger than 4 are in one-to-one correspondence to generalized Dynkin diagrams appearing in Table C. Moreover, two such arithmetic root systems are Weyl equivalent if and only* *if their generalized Dynkin diagrams appear in the same row of Table C and can be presented with the same fixed parameter.*

Again first several special cases will be analyzed before the theorem can be proven.

Let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ be a basis of $\mathbb{Z}^d$, where $d \geq 5$, and let χ be a bicharacter on $\mathbb{Z}^d$ such that $(\boldsymbol{\Delta}, \chi, E)$ is a connected arithmetic root system.

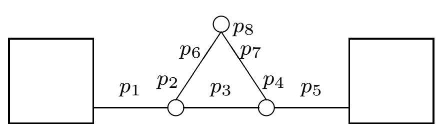
Lemmata 9, 10, and 11 can be generalized as follows.

Lemma 20. *Let $\mathcal{G}$ be a labeled cycle graph which can be obtained from $\mathcal{D}_{\chi,E}$ by omitting some vertices and some edges. Then $\mathcal{G}$ has three vertices.*

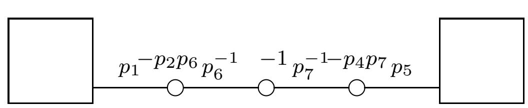
Proof. Assume to the contrary that there exists $r \geq 4$ such that $\prod_{i=1}^{r-1}(q_{i,i+1}q_{i+1,i}-1)(q_{1r}q_{r1}-1) \neq 0$. By [11, Lemma 2] and Proposition 8 one gets $\sum_{i=3}^{r-1}\mathbf{e}_i \in \boldsymbol{\Delta}^+_E$. Thus by Proposition 8 the triple $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \sum_{i=3}^{r-1}\mathbf{e}_i, \mathbf{e}_r)$ is an arithmetic root system. By equations $\chi\chi^{\mathrm{op}}(\mathbf{e}_2, \sum_{i=3}^{r-1}\mathbf{e}_i) = q_{23}q_{32}$ and $\chi\chi^{\mathrm{op}}(\sum_{i=3}^{r-1}\mathbf{e}_i, \mathbf{e}_r) = q_{r-1,r}q_{r,r-1}$ the generalized Dynkin diagram of the arithmetic root system $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \sum_{i=3}^{r-1}\mathbf{e}_i, \mathbf{e}_r)$ contains a labeled cycle with four vertices which is a contradiction to Lemmata 9, 10, and 11. ∎

The following lemmata will be helpful to find $(T, E) \in W^{\mathrm{ext}}_{\chi,E}$ such that the generalized Dynkin diagram $\mathcal{D}_{\chi\circ(T\times T),E}$ of the Weyl equivalent arithmetic root system $(T^{-1}(\boldsymbol{\Delta}), \chi\circ(T\times T), E)$ is of a particularly simple shape.

Lemma 21. *Assume that $\mathcal{D}_{\chi,E}$ is of the form*

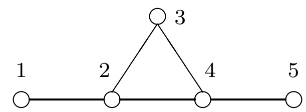


where both boxes may contain an arbitrary finite labeled graph. Then equation $p_8 = -1$ holds, and $(\boldsymbol{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system with generalized Dynkin diagram

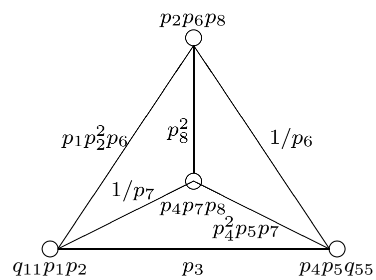


Proof. Without loss of generality assume that the vertex $i \in \{1, 2, \ldots, 5\}$

of the subgraph

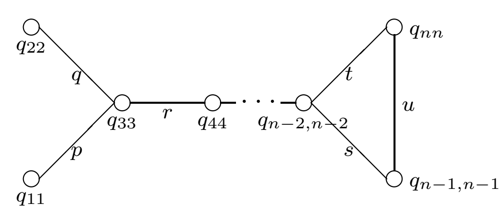


of $\mathcal{D}_{\chi,E}$ corresponds to the basis element $\mathbf{e}_i \in E$. Then $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_4 + \mathbf{e}_5)$ is an arithmetic root system and by Lemma 7(ii) it has the generalized Dynkin diagram



Therefore one has $p_8 = -1$. Indeed, since $p_3, p_6, p_7 \neq 1$, relation $p_8^2 \neq 1$ would contradict to Lemma 20. Now set $T := s_{\mathbf{e}_3,E}$ and observe that the arithmetic root system $(T^{-1}(\mathbf{\Delta}), \chi \circ (T \times T), E)$, which is Weyl equivalent to $(\mathbf{\Delta}, \chi, E)$, satisfies the claim of the lemma. ■

Lemma 22. *For all $d \geq 5$ the graph $\mathcal{D}_{\chi,E}$ does not contain a subgraph of the form*



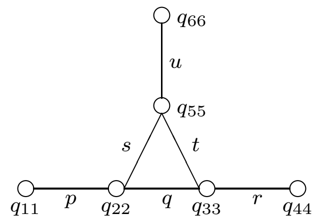
where $p, q, r, s, t \neq 1$ (but u is allowed to be equal to 1), and $5 \leq n \leq d$.

Proof. Assume to the contrary that the claim of the lemma is false. Then $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \sum_{i=3}^{n-2} \mathbf{e}_i, \mathbf{e}_{n-1}, \mathbf{e}_n)$ is an arithmetic root system with generalized Dynkin diagram of the same form but only with five vertices, that is $\mathbf{e}_3 = \mathbf{e}_{n-2}$. Therefore it is sufficient to prove the lemma for $n = 5$.

Set $\mathbf{f}_1 := \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{f}_2 := \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{f}_3 := \mathbf{e}_4$ and $\mathbf{f}_4 := \mathbf{e}_5$. On the one hand $\mathbf{\Delta}(\chi; \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4)$ is finite. On the other hand one has $\chi\chi^{\mathrm{op}}(\mathbf{f}_1, \mathbf{f}_3) = s$,

$\chi\chi^{\mathrm{op}}(\mathbf{f}_3, \mathbf{f}_2) = s$, $\chi\chi^{\mathrm{op}}(\mathbf{f}_2, \mathbf{f}_4) = t$ and $\chi\chi^{\mathrm{op}}(\mathbf{f}_4, \mathbf{f}_1) = t$, which is in view of $s \neq 1$, $t \neq 1$ a contradiction to Lemmata 9, 10 and 11. ■

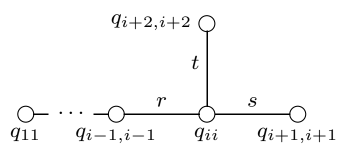
Lemma 23. *The graph $\mathcal{D}_{\chi,E}$ does not contain a subgraph of the form*



where $p, q, r, s, t, u \neq 1$.

Proof. Again one can perform an indirect proof. If $\mathcal{D}_{\chi,E}$ would contain a subgraph as above then $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_5, \mathbf{e}_5 + \mathbf{e}_6)$ would be an arithmetic root system. Note that $\chi\chi^{\mathrm{op}}(\mathbf{e}_1, \mathbf{e}_3 + \mathbf{e}_4) = \chi\chi^{\mathrm{op}}(\mathbf{e}_1, \mathbf{e}_5 + \mathbf{e}_6) = \chi\chi^{\mathrm{op}}(\mathbf{e}_1, \mathbf{e}_3 + \mathbf{e}_5) = 1$ and $\chi\chi^{\mathrm{op}}(\mathbf{e}_2, \mathbf{e}_1)$, $\chi\chi^{\mathrm{op}}(\mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4)$, $\chi\chi^{\mathrm{op}}(\mathbf{e}_2, \mathbf{e}_5 + \mathbf{e}_6)$ and $\chi\chi^{\mathrm{op}}(\mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_5)$ are all different from 1, the latter because of Lemma 7(ii). Depending on the other values of $\chi\chi^{\mathrm{op}}$ this gives a contradiction to one of Lemmata 9, 10, or 22. ■

Lemma 24. *Assume that $\mathcal{D}_{\chi,E}$ contains a subgraph of the form*



(9)

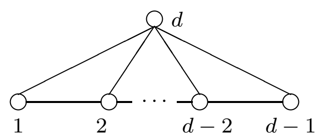
where $i \geq 2$. Then for all $j \leq i + 2$ one has $q_{jj} \in \{t, t^{-1}, -1\}$, and for all $j \leq i$ relation $q_{j,j+1}q_{j+1,j} \in \{t, t^{-1}\}$ holds. Moreover, if $q_{jj} = -1$ for some $j < i$ then $(q_{j+1,j+1} + 1)(q_{j,j+1}q_{j+1,j}q_{j+1,j+1} - 1) = 0$.

Proof. Perform induction on i. If $i = 2$ then the claim can be obtained immediately from Theorem 16. In the general case consider the arithmetic root systems $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_{i+2})$ and $\mathbf{\Delta}(\chi; \mathbf{e}_1, \ldots, \mathbf{e}_{i-2}, \mathbf{e}_{i-1} + \mathbf{e}_i, \mathbf{e}_{i+1}, \mathbf{e}_{i+2})$. To both of them one can apply the induction hypothesis and hence the first part of the claim follows. If $q_{jj} = -1$ then consider the arithmetic root system $(s_{\mathbf{e}_j,E}^{-1}(\mathbf{\Delta}), \chi \circ (s_{\mathbf{e}_j,E} \times s_{\mathbf{e}_j,E}), E)$. By Lemma 22 and the first part of the claim one obtains that $q_{j,j+1}q_{j+1,j} = q_{j+1,j+1} \neq -1$ is not possible. ■

Proposition 25. *Suppose that (χ, E) is not of Cartan type. Then $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system $(\mathbf{\Delta}', \chi', E)$ such that $\mathcal{D}_{\chi',E}$ is a labeled path graph.*

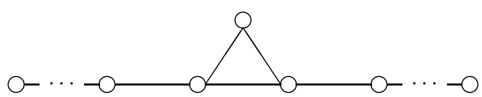
Proof. Note that the proposition holds for arithmetic root systems of rank at most four according to Proposition 15 and the classification results in rank at most 3. The general case will be proven by induction over the rank d.

Assume that $d \geq 5$ and (χ, E) is not of Cartan type. Then there exists a subset $E' \subset E$ with $d-1$ elements such that $(\chi\restriction_{\mathbb{Z}E' \times \mathbb{Z}E'}, E')$ is not of Cartan type and $\mathbf{\Delta}(\chi; E')$ is a connected arithmetic root system. By induction hypothesis it is Weyl equivalent to an arithmetic root system $(\mathbf{\Delta}', \chi', E')$ such that $\mathcal{D}_{\chi', E'}$ is a labeled path graph. Hence, since $W^{\mathrm{ext}}_{\chi,E}$ is full, $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system with unlabeled generalized Dynkin diagram

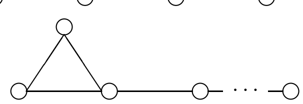


(10)

where it is allowed that some (but not all) nonhorizontal edges in this diagram are omitted. Moreover, according to Lemmata 10, 11 and 20 all nonhorizontal edges in (10) up to either two neighboring or one single edge have to be omitted (that is labeled by 1). If two neighboring nonhorizontal edges are labeled by numbers different from 1,



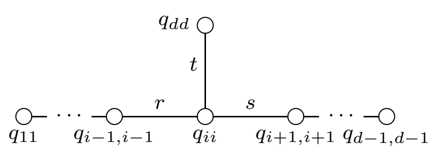
or



(11)

then Lemmata 13 and 21 imply the claim of this proposition.

Thus without loss of generality one can assume that $\mathcal{D}_{\chi,E}$ is a connected labeled graph of the form



(12)

where $1 \leq i \leq d-1$ and $r, s, t \neq 1$, and $\mathbf{\Delta}(\chi; \mathbf{e}_1, \dots, \mathbf{e}_i)$ is not of Cartan type. If $i = 1$ or $i = d-1$ then $\mathcal{D}_{\chi,E}$ is a labeled path graph and we are done. Otherwise the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_{i-1}, \mathbf{e}_i, \mathbf{e}_{i+1}, \mathbf{e}_d)$ and Theorem 16 imply that one of the following is true:

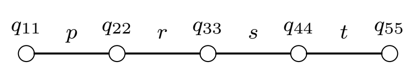
1. $r = s = t = q_{ii} = q_{i-1,i-1} = q_{i+1,i+1} = q_{dd} = -1$.
2. $t \neq -1$, $q_{ii} = -1$, $r, s \in \{t, t^{-1}\}$.
3. $t \neq -1$, $q_{ii} \neq -1$, $r, s \in \{t, t^{-1}\}$, and there exist $m_{i,i-1}, m_{i,i+1}, m_{i,d} \in \{1, 2\}$ such that $q_{ii}^{m_{i,i-1}} r = q_{ii}^{m_{i,i+1}} s = q_{ii}^{m_{id}} t = 1$.

Consider the first case. By the same argument one obtains for $1 < j < i$ by induction on $i - j$ and by considering $\mathbf{\Delta}(\chi; \mathbf{e}_1, \ldots, \mathbf{e}_{j-1}, \sum_{l=j}^{i} \mathbf{e}_l, \mathbf{e}_{i+1}, \mathbf{e}_d)$ that $q_{j-1,j-1} = q_{j-1,j} q_{j,j-1} = -1$. By symmetry this gives that (χ, E) is of Cartan type which is a contradiction to the hypothesis of the proposition.

In the second case for $T = s_{\mathbf{e}_i,E}$ the arithmetic root system $(T^{-1}(\mathbf{\Delta}), \chi \circ (T \times T), E)$ is Weyl equivalent to $(\mathbf{\Delta}, \chi, E)$. Because of Lemmata 9, 10 and 23 its unlabeled generalized Dynkin diagram is of the form as in (11) and hence the claim of the proposition holds following the arguments above.

It remains to prove the proposition in the third case under the assumption that $\mathbf{\Delta}(\chi; \mathbf{e}_1, \ldots, \mathbf{e}_i)$ is not of Cartan type. This means that there exists $j < i$ such that $q_{jj}^m q_{j,j+1} q_{j+1,j} \neq 1$ for all $m \in \mathbb{Z}$ or $q_{jj}^m q_{j,j-1} q_{j-1,j} \neq 1$ for all $m \in \mathbb{Z}$. Obviously, then $q_{jj} \in R_{m+1}$ for some $m \in \mathbb{N}$. Suppose that j is maximal with this property. Then Lemma 24 implies that $q_{jj} = -1$. Indeed, otherwise relations $q_{j,j-1} q_{j-1,j}, q_{j,j+1} q_{j+1,j} \in \{q_{jj}, q_{jj}^{-1}\}$ and $q_{jj} \in R_{m+1}$ give a contradiction to the choice of j. Moreover, $q_{jj} = -1$ and Lemma 24 also imply that $q_{j,j+1} q_{j+1,j} q_{j+1,j+1} = 1$. In this case $(s_{\mathbf{e}_j,E}(\mathbf{\Delta}), \chi \circ (s_{\mathbf{e}_j,E} \times s_{\mathbf{e}_j,E}), E)$ is an arithmetic root system which in view of Lemma 22 has a generalized Dynkin diagram as in (12). Further, the properties of the third case apply, but $\chi'(\mathbf{e}_{j+1}, \mathbf{e}_{j+1}) = -1$ where $\chi' = \chi \circ (s_{\mathbf{e}_j,E} \times s_{\mathbf{e}_j,E})$. Therefore by induction on $i - j$ one can show that $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system as in case 2 and hence the claim of the proposition is proven. ■

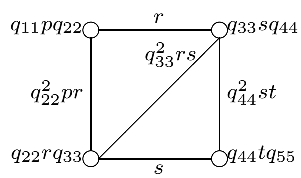
Lemma 26. *If $\mathcal{D}_{\chi,E}$ contains a subdiagram of the form*



where $p, r, s, t \neq 1$, then $(q_{22}^2 pr - 1)(q_{44}^2 st - 1) = 0$. In particular, if for all $i, j \in \{1, 2, \ldots, d\}$ with $|i - j| \geq 2$ one has $q_{ij} q_{ji} = 1$, and I is the set of numbers $i \in \{2, 3, \ldots, d - 1\}$ such that $q_{ii}^2 q_{i,i-1} q_{i-1,i} q_{i,i+1} q_{i+1,i} \neq 1$, then $|I| \leq 2$ and $|i - j| \leq 1$ for all $i, j \in I$.

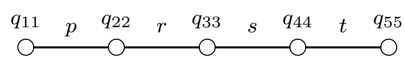
Proof. Consider $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_4 + \mathbf{e}_5)$. Its generalized

Dynkin diagram is



and hence Lemmata 10 and 11 imply the first part of the claim. For the second part, suppose that $|I| \geq 2$ and $i, j \in I$ with $j - i \geq 2$. Then the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_{i-1}, \mathbf{e}_i, \sum_{l=i+1}^{j-1} \mathbf{e}_l, \mathbf{e}_j, \mathbf{e}_{j+1})$ contradicts the first part of the proof. ■

Lemma 27. *If $\mathcal{D}_{\chi,E}$ contains a subdiagram of the form*



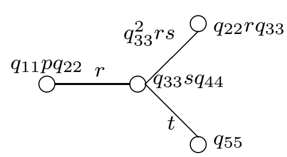
such that $q_{22}^2pr = q_{44}^2st = 1$, $q_{33}^2rs \neq 1$, and $p, r, s, t \neq 1$ then the following relations hold:

$$(p - s)(ps - 1) = (r - t)(rt - 1) = 0, \qquad q_{33}^2rs \in \{r, r^{-1}\}, \tag{13}$$

$$q_{22}rq_{33}, q_{33}sq_{44} \in \{-1, 1/q_{33}^2rs\}. \tag{14}$$

Moreover, if $q_{33}sq_{44} = t$ then equation $q_{55}t = 1$ holds.

Proof. The arithmetic root system $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_5)$ has generalized Dynkin diagram



and hence by Theorem 16 one gets relations $(r - t)(rt - 1) = 1$, $q_{33}^2rs \in \{r, r^{-1}\}$, and $q_{22}rq_{33} \in \{-1, 1/q_{33}^2rs\}$. By the same reason one gets $q_{55}t = 1$ whenever $q_{33}sq_{44} = t$. The remaining relations follow by symmetry. ■

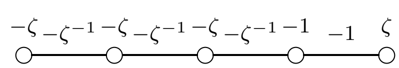
Proof of Theorem 19. As in the proof of Theorem 16 one shows that if $\mathcal{D}_{\chi,E}$ appears in Table C then $W_{\chi,E}^{\mathrm{ext}}$ is finite, and two arithmetic root systems corresponding to generalized Dynkin diagrams in Table C are Weyl equivalent if and only if these diagrams appear in the same row of Table C and can be presented with the same fixed parameter. For the list of elements $(T, E) \in W_{\chi,E}$ mentioned in Proposition 5 confer Appendix C. In order to determine the Weyl equivalence classes Proposition 18 is helpful.

It remains to prove that if $W^{\text{ext}}_{\chi,E}$ is full and finite then $\mathcal{D}_{\chi,E}$ appears in Table C. This will be done by induction over d. Further, Proposition 25 is used to reduce to the case where $\mathcal{D}_{\chi,E}$ is a labeled path graph. More precisely, assume that $q_{ij}q_{ji} = 1$ whenever $|i-j| \geq 2$. Moreover, [13, Theorem 1] tells that if (χ, E) is of Cartan type then $W^{\text{ext}}_{\chi,E}$ is full and finite if and only if (χ, E) is of finite Cartan type. These examples appear in rows 1, 3, 7, 8, 16, 20, and 22 of Table C. Thus it suffices to consider pairs (χ, E) such that $W^{\text{ext}}_{\chi,E}$ is full and finite but (χ, E) is not of Cartan type.

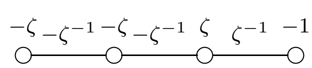
Since $\boldsymbol{\Delta}(\chi; \mathbf{e}_i, \mathbf{e}_{i+1}, \mathbf{e}_{i+2}, \mathbf{e}_{i+3})$ is finite for all $i \leq d-3$, by Theorem 16 any connected subgraph of $\mathcal{D}_{\chi,E}$ with four verices has to appear in Table B. Using this fact it is sufficient to consider prolongations of the labeled path graphs in Table B by (not necessarily simple) chains. This will be done in several steps.

Step 1. Assume that $(q_{dd}+1)(q_{dd}q_{d,d-1}q_{d-1,d}-1) \neq 1$. Then, since $\boldsymbol{\Delta}(\chi; \sum_{j=1}^{i} \mathbf{e}_j, \mathbf{e}_{i+1}, \sum_{j=i+2}^{d-1} \mathbf{e}_j, \mathbf{e}_d)$ is finite for all $i \leq d-3$, Theorem 16 implies that the generalized Dynkin diagram of $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d-2})$ is a simple chain. Moreover, since $\boldsymbol{\Delta}(\chi; \mathbf{e}_{d-3}, \mathbf{e}_{d-2}, \mathbf{e}_{d-1}, \mathbf{e}_d)$ is finite, either $\mathcal{D}_{\chi,E}$ appears in one of rows 4, 5, and 6 of Table C, or it is a prolongation of the first or the second graph in row 17 of Table B to the left by a simple chain. Using the reflection $s_{\mathbf{e}_d,E}$ the two latter cases are Weyl equivalent, and hence it is sufficient to consider the second diagram in row 17 of Table B.

Assume that $d = 5$. Since $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5)$ is finite, $\mathcal{D}_{\chi,E}$ is of the form



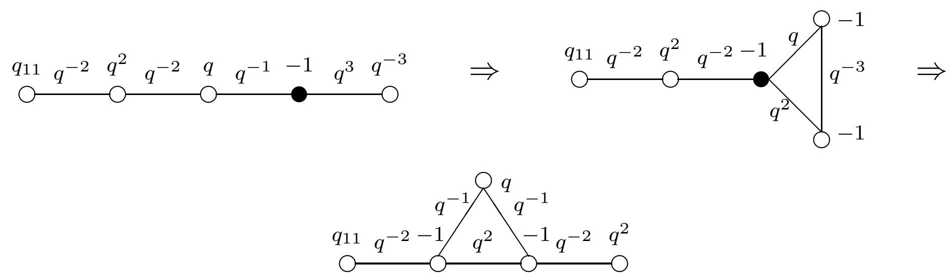
where $\zeta \in R_3$. Then $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_4+\mathbf{e}_5)$ has generalized Dynkin diagram



which gives a contradiction to Theorem 16.

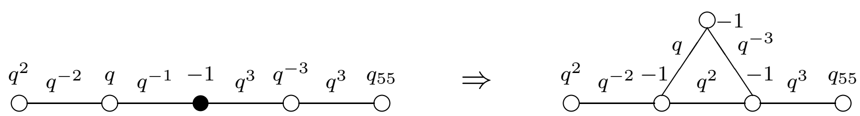
Step 2. Assume that equations $(q_{11}+1)(q_{11}q_{12}q_{21}-1) = 0$ and $(q_{dd}+1)(q_{dd}q_{d,d-1}q_{d-1,d}-1) = 0$ hold, and there exist at least two integers i with $1 < i < d$ such that $q_{ii}^2 q_{i,i-1}q_{i-1,i}q_{i,i+1}q_{i+1,i} \neq 1$. Then by Lemma 26 there exist exactly two such numbers and they differ by 1. Thus by Theorem 16 $\mathcal{D}_{\chi,E}$ is a prolongation of one of the following diagrams in Table B to the left and/or to the right by simple chains: diagram 6 in row 9, diagram 4 in row 14, diagram 5 in row 17, diagram 6 in row 22, diagram 7 in row 22.

Step 2.1. Prolongations of diagram 6 in row 9 of Table B. One has $q \in k^*$ and $q^2, q^3 \neq 1$. To the left:



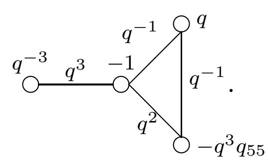
This is a contradiction to $q \neq -1$ and Lemma 21.

Prolongations to the right:



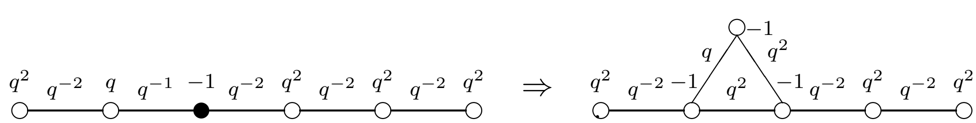
(15)

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ correspond to the four vertices of the last diagram lying on the same straight line, and let $\mathbf{e}_5$ correspond to the fifth vertex. The generalized Dynkin diagram of $\boldsymbol{\Delta}(\chi; \mathbf{e}_3 + \mathbf{e}_5, \mathbf{e}_2 + \mathbf{e}_5, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_2)$ is



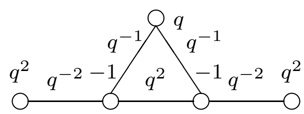
By Lemma 13 one obtains that $q_{55} = q^{-3}$, and Lemma 12 gives that $q \in R_5$. Then the first diagram in (15) coincides with the first diagram in row 15 of Table C.

In a further prolongation to the right by a simple chain of length 1 one has $q \in R_5$. Moreover, since $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 + \mathbf{e}_5, \mathbf{e}_6)$ is finite, one can only have



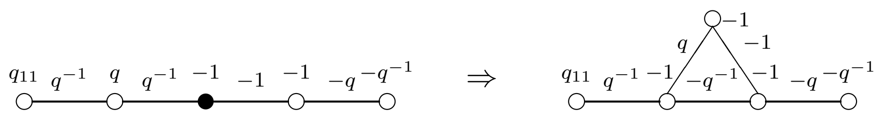
(16)

However for the last diagram, where $\mathbf{e}_1, \dots, \mathbf{e}_5$ correspond to the vertices lying on the same straight line and $\mathbf{e}_6$ corresponds to the remaining vertex, the generalized Dynkin diagram of $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_6, \mathbf{e}_3 + \mathbf{e}_6, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_5)$ has the form



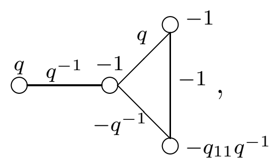
which is a contradiction to Lemma 21 and $q \in R_5$.

Step 2.2. Prolongations of diagram 4 in row 14 of Table B. By symmetry it is enough to prolong to the left or to both directions by simple chains. Assume that $q^2 \neq 1$ and $(q_{11} + 1)(q_{11} - q) = 0$. Then Weyl equivalence gives

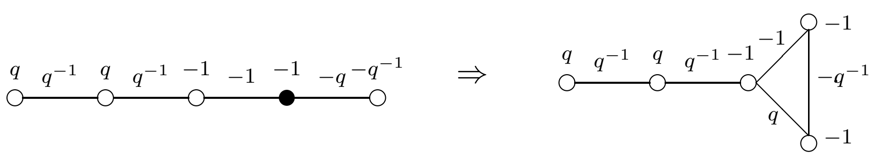


(17)

With the same notation as below (15) the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_2 + \mathbf{e}_5, \mathbf{e}_3 + \mathbf{e}_5, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_2)$, which has the generalized Dynkin diagram

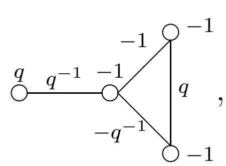


and Theorem 16 imply that $q_{11} \neq -1$. Therefore it remains to check the case when $q_{11} = q$. Starting again with the original diagram, Weyl equivalence gives also



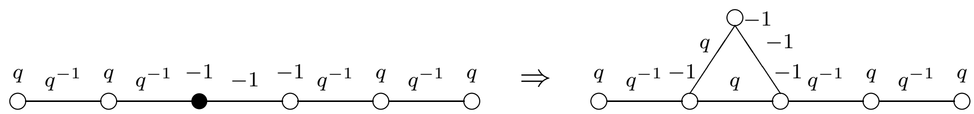
(18)

Assign the nodes of the right diagram from left to right and top to bottom to the basis vectors $\mathbf{e}_i$, where $1 \leq i \leq 5$. Applying Lemma 12 to $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_5)$, which has as generalized Dynkin diagram the labeled graph

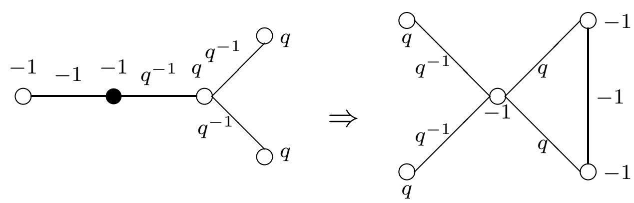


one obtains that $q \in R_4$. In this case (see the left graph in (18)) $\mathcal{D}_{\chi,E}$ appears in row 14 of Table C.

Next it will be shown that there is no prolongation of the left graph in (18) to the right. To this end note that since $\mathbf{\Delta}(\chi; \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6)$ is finite, by the previous arguments any prolongation to the right has to be of the form

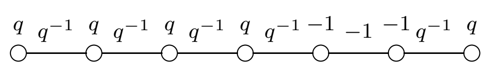


where $q \in R_4$. Using the conventions below (16) for the last graph, the generalized Dynkin diagram of $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_6, \mathbf{e}_4, \mathbf{e}_5)$ has the form

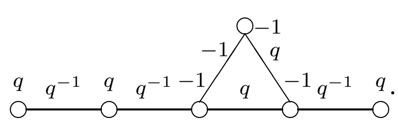


which is a contradiction to Lemma 22.

It remains to consider prolongations of the left graph in (18) to the left, where $q \in R_4$. Since $\mathbf{\Delta}(\chi; E \setminus \{\mathbf{e}_2\})$ is finite, one has to have $q_{11} = q$ and $q_{12}q_{21} = q^{-1}$. This prolongation of length 1 is the first diagram in row 19 of Table C. The prolongation of length 2 looks like



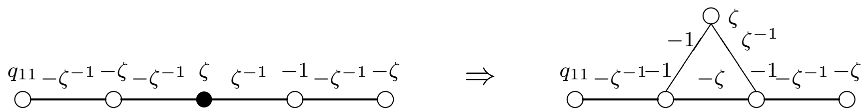
in which case the generalized Dynkin diagram of $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_4 + \mathbf{e}_5, \mathbf{e}_5 + \mathbf{e}_6, \mathbf{e}_6 + \mathbf{e}_7)$ is



This is a contradiction to Theorem 19 for $d = 6$.

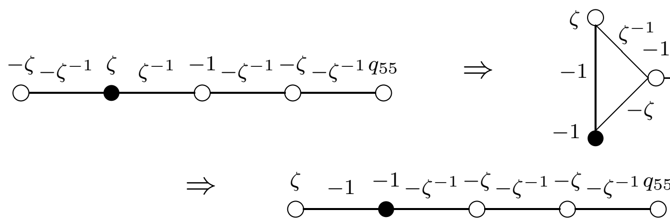
Step 2.3. Prolongations of diagram 5 in row 17 of Table B. Let $\zeta \in R_3$.

Prolongations to the left:



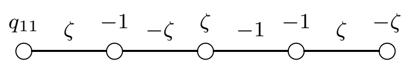
This is a contradiction to Lemma 21.

To the right:

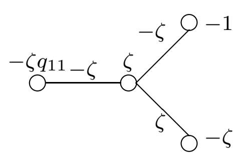


The latter diagram does not correspond to an arithmetic root system, as shown in Step 1.

Step 2.4. Prolongations of diagram 6 in row 22 of Table B. Assume that $\zeta \in R_4$. For a prolongation to the left by a simple chain of length 1

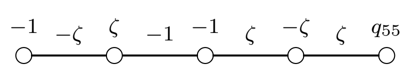


$\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_5)$ has the generalized Dynkin diagram

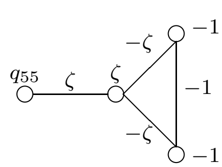


which is a contradiction to Theorem 16.

For a prolongation to the right by a simple chain of length 1



$\mathbf{\Delta}(\chi; \mathbf{e}_4 + \mathbf{e}_5, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2)$ has the generalized Dynkin diagram

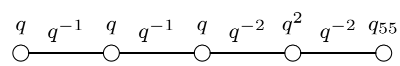


which is again a contradiction to Theorem 16.

Step 2.5. Prolongations of diagram 7 in row 22 of Table B. In a prolongation to the left by a simple chain of length 1 apply $s_{\mathbf{e}_2,E}$, and in a prolongation to the right by a simple chain apply $s_{\mathbf{e}_1,E}$. Then one obtains a diagram which was already considered in Step 2.4.

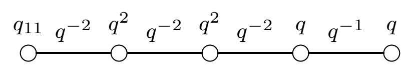
Step 3. Assume that $(q_{11}+1)(q_{11}q_{12}q_{21}-1)=0$, $(q_{dd}+1)(q_{dd}q_{d,d-1}q_{d-1,d}-1)=0$, and there exists exactly one integer i with $1<i<d$ such that $q_{ii}^2q_{i,i-1}q_{i-1,i}q_{i,i+1}q_{i+1,i}\neq 1$. Again it is sufficient to consider prolongations of those labeled path graphs by simple chains which appear in Table B and satisfy equations $(q_{11}+1)(q_{11}q_{12}q_{21}-1)=0$ and $(q_{44}+1)(q_{44}q_{34}q_{43}-1)=0$.

Step 3.1. Prolongations of the diagram in row 3 of Table B. To the left: this prolongation appears in row 7 or row 9 of Table C. To the right: one gets the diagram

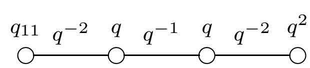


where $q_{55}\in\{q^2,-1\}$ and $q^2\neq 1$. If $q_{55}=q^2$ then (χ,E) is of infinite Cartan type. If $q_{55}=-1$ then by Theorem 16 $\mathbf{\Delta}(\chi;\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5)$ is finite if and only if $q^2=-1$ in which case again equation $q_{55}=q^2$ holds.

Step 3.2. Prolongations of the diagram in row 4 of Table B. To the left: one obtains the diagram

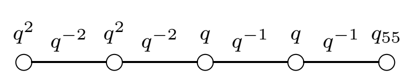


where $q_{11}\in\{q^2,-1\}$ and $q^2\neq 1$. In this case $\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_3+\mathbf{e}_4,\mathbf{e}_4+\mathbf{e}_5)$ has generalized Dynkin diagram



which contradicts Theorem 16.

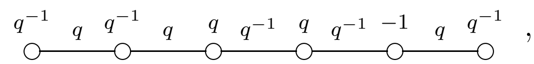
To the right: if in the diagram



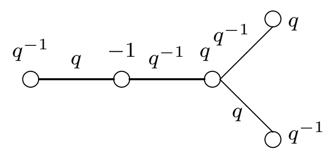
(19)

where $(q_{55}+1)(q_{55}-q)=0$, equation $q_{55}=q$ holds, then $(\mathbf{\Delta},\chi,E)$ is of infinite Cartan type. On the other hand, if $q_{55}=-1$ then Lemma 27 implies that $q\in R_3$. In this case $\mathcal{D}_{\chi,E}$ is the last graph in row 13 of Table C.

By the first part of Step 3.2 there exists no prolongation of (19) to the left. A prolongation of length 1 to the right has to be of the form



where $q \in R_3$, because of the finiteness of $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5 + \mathbf{e}_6)$. In this case however one obtains a contradiction to the finiteness of $\boldsymbol{\Delta}(\chi; \mathbf{e}_6, \mathbf{e}_5, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2)$ with generalized Dynkin diagram

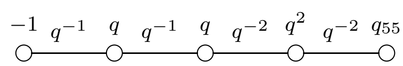


and to Theorem 19 with $d = 5$.

Step 3.3. Prolongations of the labeled path graphs in row 8 of Table B. Note that the first and second diagrams in row 8 are Weyl equivalent via $s_{\mathbf{e}_1,E}$, and the second and third are Weyl equivalent via $s_{\mathbf{e}_2,E}$. Therefore any prolongation of one of these generalized Dynkin diagrams is Weyl equivalent to a prolongation of another one. Thus it is sufficient to consider the first diagram in row 8.

Prolongations to the left: these appear in rows 9 and 10 of Table C.

Prolongations to the right: one obtains the diagram

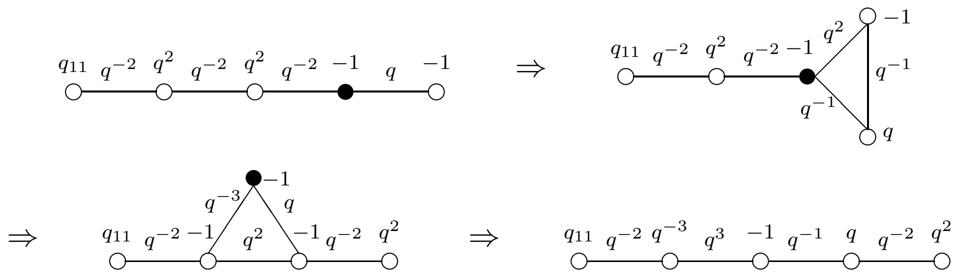


(20)

where $(q_{55} + 1)(q_{55} - q^2) = 0$, and $q^2 \neq 1$. Then Lemma 27 implies that $q \in R_3$ and $q_{55} = q^{-1}$. Thus diagram (20) and its prolongations were already considered in Step 3.2 below graph (19).

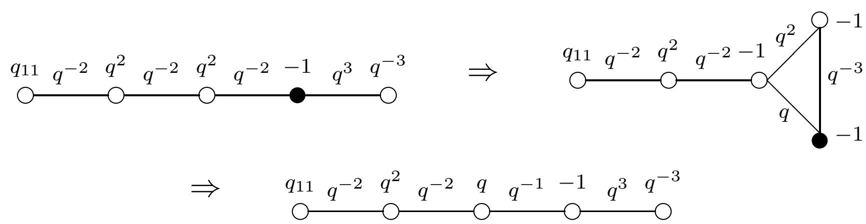
Step 3.4. Prolongations of the labeled path graphs in row 9 of Table B. As in Step 3 Weyl equivalence allows to reduce to the consideration of the second and fifth diagrams. Suppose that $q^2, q^3 \neq 1$.

Prolongations to the left: assume that $(q_{11} + 1)(q_{11} - q^2) = 0$. By Weyl equivalence one gets



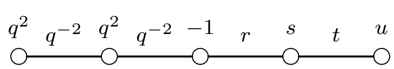
By Lemma 26 this implies that $q \in R_5$. This diagram and its prolongations to the left were already considered in Step 2.1.

Similarly, for a prolongation of the fifth diagram to the left one obtains



In Step 2.1 it was shown that the latter diagram does not correspond to an arithmetic root system.

Prolongations to the right: if $\mathcal{D}_{\chi,E}$ is of the form

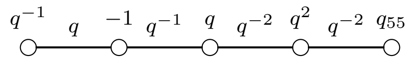


where $q^2 \neq 1$, $q^3 \neq 1$, and one of the systems of equations

$$r - q = s + 1 = t - q^{-1} = (u+1)(u-q) = 0,$$
$$r - q^3 = s - q^{-3} = t - q^3 = (u+1)(uq^3 - 1) = 0$$

holds, then Lemma 27 implies that $\{r, r^{-1}\} = \{q^2, q^{-2}\} = \{q^{-2}r, q^2r^{-1}\}$. However if $r = q$ then $r \notin \{q^2, q^{-2}\}$ and if $r = q^3$ then $q^{-2}r \notin \{q^2, q^{-2}\}$. Therefore there exist no prolongations to the right of the diagrams 2 and 5 in row 9 of Table B.

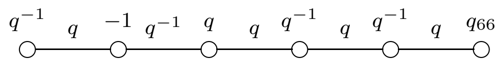
Step 3.5. Prolongations of the labeled path graphs in row 12 of Table B. Again Weyl equivalence allows to reduce to the consideration of the first diagram. The prolongations to the left appear in rows 9 and 10 of Table C. A prolongation of length 1 to the right takes the form



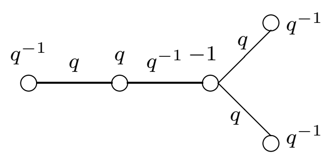
(21)

where $q^2 \neq 1$ and $(q_{55}+1)(q_{55}-q^2)=0$. Thus Lemma 27 gives that $q \in R_3$ and $q_{55}=q^2$. This diagram coincides with the last one in row 12 of Table C.

A prolongation of length one to the right of the diagram (21), where $q \in R_3$ and $q_{55}=q^2$, has to be of the form

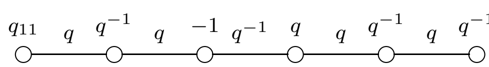


where $q_{66} = q^{-1}$, since $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4+\mathbf{e}_5, \mathbf{e}_6)$ is finite. In this case $\mathbf{\Delta}(\chi; \mathbf{e}_5+\mathbf{e}_6, \mathbf{e}_4+\mathbf{e}_5, \mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_1)$ has the generalized Dynkin diagram



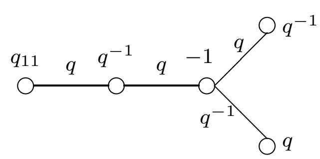
which is a contradiction to Theorem 19 for $d=5$.

It remains to consider prolongations of (21) to the left, where $q \in R_3$ and $q_{55}=q^{-1}$. Assume that $\mathcal{D}_{\chi,E}$ is of the form

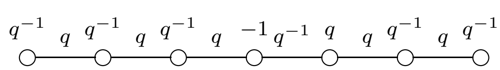


(22)

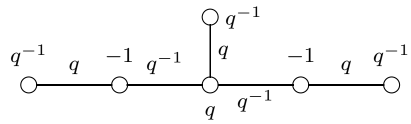
where $q \in R_3$ and $(q_{11}+1)(q_{11}q-1)=0$. Then $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_4+\mathbf{e}_5, \mathbf{e}_5+\mathbf{e}_6)$ has generalized Dynkin diagram



and hence Theorem 19 for $d=5$ implies that $q_{11}=q^{-1}$. Thus (22) is the fourth last diagram in row 18 of Table C. For a prolongation of length one to the left of (22) one again has the only possibility

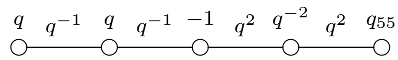


since $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7)$ has to be finite. However in this case $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_4 + \mathbf{e}_5, \mathbf{e}_5 + \mathbf{e}_6, \mathbf{e}_7)$ has to be finite but it has generalized Dynkin diagram



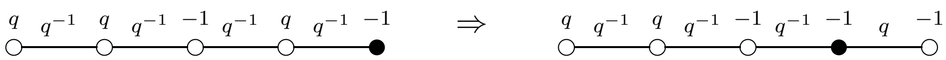
which is a contradiction to Theorem 19 with $d = 6$.

Step 3.6. Prolongations of the first diagram in row 13 of Table B. The prolongations to the left appear in rows 9 and 10 of Table C. A prolongation of length 1 to the right takes the form



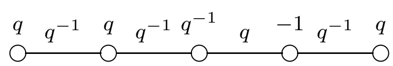
(23)

where $q^2 \neq 1$ and $(q_{55}+1)(q_{55}q^2 - 1) = 0$. Thus Lemma 27 gives that $q \in R_3$. If $q_{55} = -1$ then Weyl equivalence gives



(24)

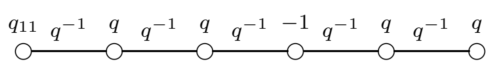
⇒



(25)

This diagram and all of its prolongations were already considered in Step 3.5 and hence for $q_{55} = -1$ (23) and all of its prolongations are Weyl equivalent to generalized Dynkin diagrams appearing in Step 3.5.

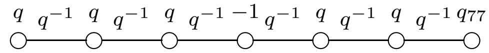
If $q_{55} = q \in R_3$ then (23) is the first diagram in row 11 of Table C. Because of its symmetry it is sufficient to consider ist prolongations to the left and to both directions, respectively. So assume next that $\mathcal{D}_{\chi,E}$ takes the form



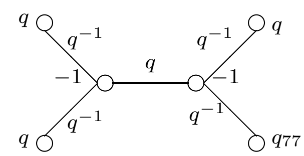
(26)

where $q \in R_3$ and $(q_{11}+1)(q_{11} - q) = 0$. If $q_{11} = -1$ then similarly to the transformations in (24) and (25) one obtains a prolongation of (25) which was already considered in Step 3.5. On the other hand, if $q_{11} = q$ then $\mathcal{D}_{\chi,E}$ appears in row 17 of Table C.

The diagram (26) has no prolongation to the right. Indeed, if $\mathcal{D}_{\chi,E}$ is of the form

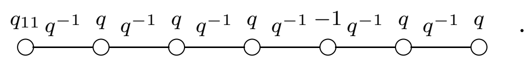


with $q \in R_3$ and $(q_{77}+1)(q_{77}-q)=0$ then the finiteness of $\boldsymbol{\Delta}(\chi; \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_4+\mathbf{e}_5, \mathbf{e}_5+\mathbf{e}_6, \mathbf{e}_6+\mathbf{e}_7)$

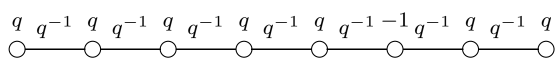


is a contradiction to Lemma 22. Thus it remains to consider prolongations of (26) to the left.

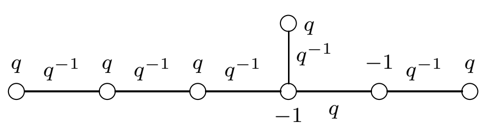
Suppose that $q \in R_3$, $(q_{11}+1)(q_{11}-q)=0$ and $\mathcal{D}_{\chi,E}$ is of the form



If $q_{11}=-1$ then similarly to the transformations in (24) and (25) one obtains a prolongation of (25) which was already considered in Step 3.5. On the other hand, if $q_{11}=q$ then $\mathcal{D}_{\chi,E}$ appears in row 21 of Table C. For an additional prolongation to the left one can again assume that $\mathcal{D}_{\chi,E}$ is of the form



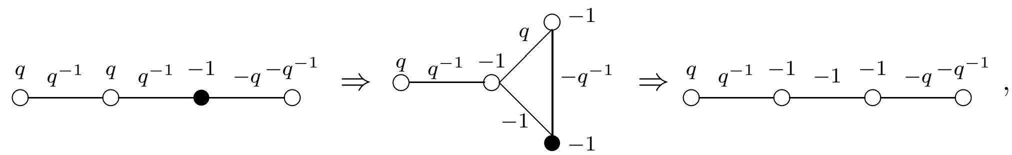
where $q \in R_3$. In this case $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_4+\mathbf{e}_5, \mathbf{e}_5+\mathbf{e}_6, \mathbf{e}_6+\mathbf{e}_7, \mathbf{e}_8)$ has generalized Dynkin diagram



which is a contradiction to Theorem 19 for $d=7$.

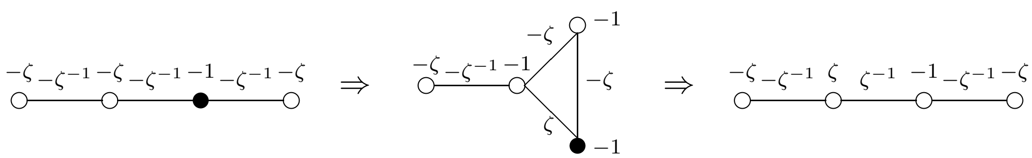
Step 3.7. Prolongations of the first and last diagrams in row 14 of Table B. The last diagram in row 14 of Table B can be obtained from the first one by replacing q by $-q^{-1}$, and hence it is sufficient to consider prolongations of the first graph.

By Lemma 27 for any prolongation to the right one has to have the relation $\{-q,-q^{-1}\}=\{q,q^{-1}\}=\{-1\}$ which is a contradiction to $q^2 \neq 1$. Moreover, since the first diagram of row 14 is Weyl equivalent to the fourth one via



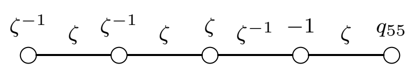
any prolongation of the first diagram to the left is Weyl equivalent to a diagram which was already considered in Step 2.2.

Step 3.8. Prolongations of the last diagram in row 17 of Table B. By Lemma 27 for any prolongation to the right one has to have the relation $\{-\zeta, -\zeta^{-1}\} = \{\zeta, \zeta^{-1}\}$ which is a contradiction to $\zeta \in R_3$. Moreover, since the last diagram of row 17 is Weyl equivalent to the fifth one via



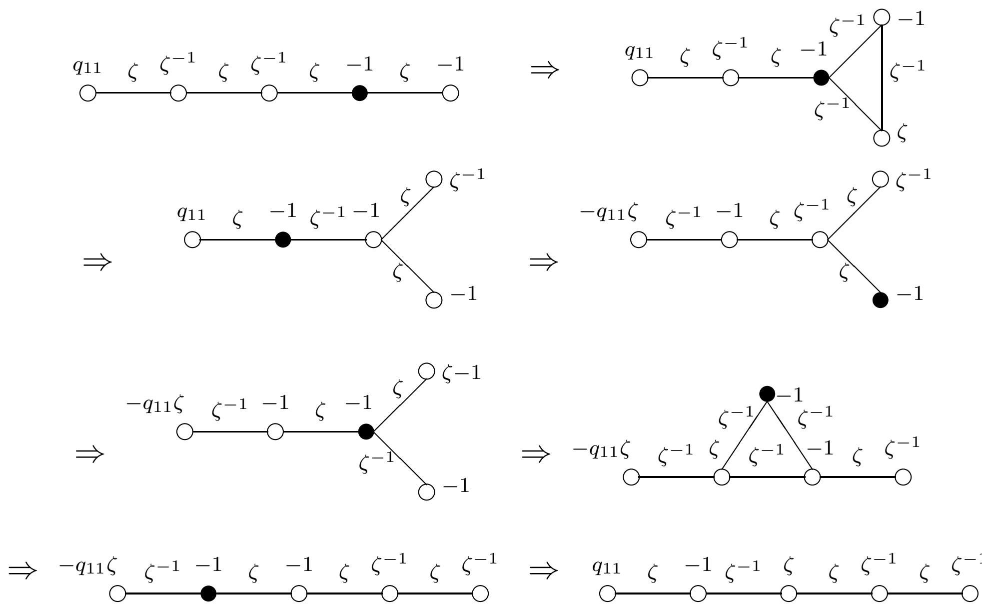
any prolongation of the last diagram of row 17 to the left is Weyl equivalent to a diagram which was already considered in Step 2.3.

Step 3.9. Prolongations of the first two diagrams in row 18 of Table B. By Weyl equivalence it is sufficient to consider one of these diagrams. Prolongations to the right of the first diagram: assume first that $\mathcal{D}_{\chi,E}$ takes the form



where $\zeta \in R_3$ and $(q_{55}+1)(q_{55}\zeta - 1) = 0$. If $q_{55} = \zeta^{-1}$ then this diagram was already considered in Step 3.5. If $q_{55} = -1$ then up to Weyl equivalence this diagram and its prolongations were already considered in Step 3.2, see (19) and below.

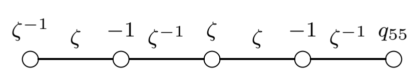
Assume now that $\zeta \in R_3$ and $\mathcal{D}_{\chi,E}$ is a prolongation of length 1 of the second diagram in row 18 of Table B. Then Weyl equivalence gives



The latter diagram was already considered in the first part of this step. In the same way one can see that any prolongation to the left of the second diagram in row 18 of Table B is Weyl equivalent to a prolongation of the first diagram to the right and hence it was already considered in the first part of this step.

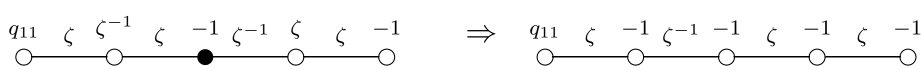
Step 3.10. Prolongations of the first six diagrams in row 20 of Table B. One can see that such prolongations are Weyl equivalent to prolongations of the first diagram.

Assume first that $\mathcal{D}_{\chi,E}$ is of the form



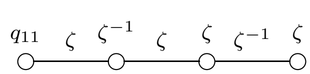
where $\zeta \in R_3$. Then one obtains a contradiction to (14).

Now suppose that $\mathcal{D}_{\chi,E}$ is a prolongation of length one to the left, that is



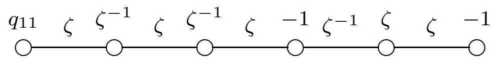
(27)

where $\zeta \in R_3$. Then $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_4+\mathbf{e}_5)$ with respect to the last diagram has generalized Dynkin diagram

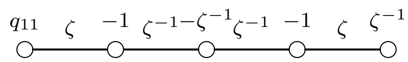


and hence Theorem 16 implies that $q_{11} = \zeta^{-1}$. In this case the first diagram of (27) coincides with the eleventh diagram in row 13 of Table C.

By the above argumentations a prolongation to the left of length 2 of the first diagram in row 20 of Table B has to take the form



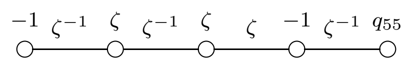
where $\zeta \in R_3$ and $(q_{11}+1)(q_{11}\zeta - 1) = 0$. However in this case the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_4 + \mathbf{e}_5, \mathbf{e}_5 + \mathbf{e}_6)$



gives a contradiction to Theorem 16.

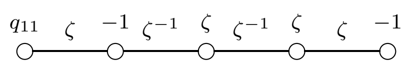
Step 3.11. Prolongations of the first six diagrams in row 21 of Table B. One can see that such prolongations are Weyl equivalent to prolongations of the first diagram.

Assume first that $\mathcal{D}_{\chi,E}$ is of the form



where $\zeta \in R_3$. Then one obtains a contradiction to (14).

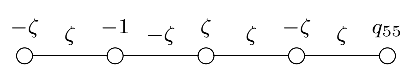
Now suppose that $\mathcal{D}_{\chi,E}$ is a prolongation of length one to the left of the first diagram in row 21 of Table B, that is



(28)

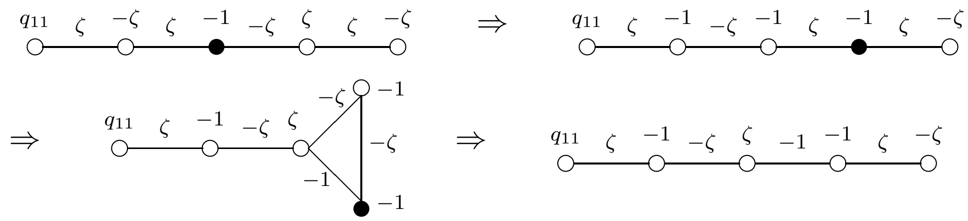
where $\zeta \in R_3$ and $(q_{11}+1)(q_{11}\zeta - 1) = 0$. If $q_{11} = -1$ then the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5)$ gives a contradiction to Theorem 16. On the other hand, if $q_{11} = \zeta^{-1}$ then the transformation $s_{\mathbf{e}'_1,E'} s_{\mathbf{e}_2,E}$, where $E' = s_{\mathbf{e}_2,E}(E)$ and $\mathbf{e}'_1 = \mathbf{e}_1 + \mathbf{e}_2 = s_{\mathbf{e}_2,E}(\mathbf{e}_1)$, gives a prolongation to the left of the first diagram in row 20 of Table B. Therefore (28) with $q_{11} = \zeta^{-1}$ and all of its prolongations to the left were already considered in Step 3.10.

Step 3.12. Prolongations of the first three diagrams in row 22 of Table B. Again all such prolongations are Weyl equivalent to prolongations of the first diagram. Moreover, a prolongation of length one to the right of the first diagram in row 22 of Table B would be of the form



where $\zeta \in R_4$, which is a contradiction to (13).

Finally, a prolongation of length one to the left of the first diagram in row 22 of Table B is via the transformations given below Weyl equivalent to a diagram which was already considered in Step 2.4.



Prolongations to the left of length bigger than one can be handled with the analogous transformations.

Step 4. Assume that equations $(q_{11}+1)(q_{11}q_{12}q_{21}-1)=0$ and $(q_{dd}+1)(q_{dd}q_{d,d-1}q_{d-1,d}-1)=0$ hold and one has $q_{ii}^2 q_{i,i-1}q_{i-1,i}q_{i,i+1}q_{i+1,i}=1$ for all i with $1<i<d$. Then $\mathcal{D}_{\chi,E}$ is a simple chain and it appears in row 1 or row 2 of Table C.

With Steps 1–4 the analysis of generalized Dynkin diagrams which are labeled path graphs is finished and the theorem is proven. ■

A On the finiteness of the Weyl groupoid

According to the proofs of Theorem 16 and 19 for generalized Dynkin diagrams $\mathcal{D}_{\chi,E}$ of Tables B and C elements $(T,E)\in W^{\mathrm{ext}}_{\chi,E}$ are given such that $T(E)$ consists of $d-1$ elements of E and one element of $-\boldsymbol{\Delta}^+_E$. It suffices to consider one single representant in each Weyl equivalence class, and generalized Dynkin diagrams of Cartan type can be ignored.

The starting point in each row is the first diagram which is a labeled path graph. The numbering of the vertices of this diagram is from left to right by 1 to d.

Abbreviate $m_1\mathbf{e}_1+m_2\mathbf{e}_2+m_3\mathbf{e}_3+m_4\mathbf{e}_4$ by $1^{m_1}2^{m_2}3^{m_3}4^{m_4}$, where i^{m_i} is omitted if $m_i=0$ and i^{m_i} is replaced by i if $m_i=1$. For the map $s_{\mathbf{f}_i,F}$, where $\mathbf{f}_i$ is the ith element of the basis F, the symbol s_i will be used. Lower indices to bases of $\mathbb{Z}^d$ indicate the diagram corresponding to this basis.

A.1 Computations for Table B

Rows 6 and 10: $T := s_4s_3s_2s_1$.

$$E = (1,2,3,4) \mapsto (-1,12,3,4) \mapsto (2,-12,123,4) \mapsto (2,3,-123,1234) \mapsto (2,3,4,-1234)$$

Rows 7,11,15,16 and 19: $T = s_1s_2s_3s_4s_3s_2s_1$.

$$(1,2,3,4) \mapsto (-1,12,3,4) \mapsto (2,-12,123,4) \mapsto (2,3,-123,1234) \mapsto (2,3,1234^2,-1234) \mapsto (2,123^24^2,-1234^2,4) \mapsto (12^23^24^2,-123^24^2,3,4) \mapsto (-12^23^24^2,2,3,4)$$

Row 8: $T = s_1s_2s_4s_3s_2s_1$.

$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_2 \mapsto (2,-12,123,4)_3 \mapsto (2,3,-123,1234)_4 \mapsto (2,123^24,4,-1234)_3 \mapsto (12^23^24,-123^24,4,3)_2 \mapsto (-12^23^24,2,4,3)_1$$

Row 9: $T = s_1s_4s_2s_3s_4s_1s_2s_3s_2s_1$.

$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_1 \mapsto (2,-12,123,4)_1 \mapsto (2,123^2,-123,1234)_1 \mapsto (12^23^2,-123^2,3,1234)_1 \mapsto (-12^23^2,2,3,1234)_1 \mapsto (-12^23^2,2,123^24,-1234)_2 \mapsto (-12^23^2,12^23^24,-123^24,3)_3 \mapsto (4,-12^23^24,2,12^23^34)_4 \mapsto (12^23^34^2,3,2,-12^23^34)_6 \mapsto (-12^23^34^2,3,2,4)_6$$

Row 12: $T = s_1s_2s_4s_3s_2s_1$.

$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_1 \mapsto (2,-12,123,4)_2 \mapsto (2,3,-123,1234)_4 \mapsto (2,123^24,4,-1234)_2 \mapsto (12^23^24,-123^24,4,3)_1 \mapsto (-12^23^24,2,4,3)_1$$

Row 13: $T = s_1s_2s_4s_3s_2s_1$.

$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_1 \mapsto (2,-12,123,4)_1 \mapsto (2,3,-123,1234)_2 \mapsto (2,123^24,4,-1234)_1 \mapsto (12^23^24,-123^24,4,3)_1 \mapsto (-12^23^24,2,4,3)_1$$

Row 14: $T = s_3s_4s_1s_2s_4s_3s_2s_1$.

$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_1 \mapsto (2,-12,123,4)_1 \mapsto (2,3,-123,1234)_2 \mapsto (2,123^24,4,-1234)_4 \mapsto (12^23^24,-123^24,4,3)_2 \mapsto (-12^23^24,2,4,12^23^34)_5 \mapsto (3,2,12^23^34^2,-12^23^34)_5 \mapsto (3,2,-12^23^34^2,4)_5$$

Row 17: $T = s_1s_2s_3s_4s_3s_2s_1s_4s_1s_2s_3s_4s_3s_2s_1$.

$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_1 \mapsto (2,-12,123,4)_1 \mapsto (2,3,-123,1234)_1 \mapsto (2,3,1234^2,-1234)_2 \mapsto (2,123^24^2,-1234^2,4)_3 \mapsto (12^23^24^2,-123^24^2,3,123^24^3)_4 \mapsto (-12^23^24^2,2,3,1^22^33^44^5)_6 \mapsto (123^24^3,2,3,-1^22^33^44^5)_6 \mapsto (-123^24^3,12^23^24^3,3,-12^23^24^2)_4 \mapsto (2,-12^23^24^3,12^23^34^3,4)_3 \mapsto (2,3,-12^23^34^3,12^23^34^4)_2 \mapsto (2,3,12^23^34^5,-12^23^34^4)_1 \mapsto (2,12^23^44^5,-12^23^34^5,4)_1 \mapsto (12^33^44^5,-12^23^44^5,3,4)_1 \mapsto (-12^33^44^5,2,3,4)_1$$

Row 18: $T = s_4s_2s_3s_4s_1s_2s_3s_2s_1$.

$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_1 \mapsto (2,-12,123,4)_1 \mapsto (2,123^2,-123,1234)_1 \mapsto (12^23^2,-123^2,3,1234)_1 \mapsto (-12^23^2,2,3,1234)_1 \mapsto (-12^23^2,2,123^24,-1234)_2 \mapsto (-12^23^2,12^23^24,-123^24,3)_3 \mapsto (4,-12^23^24,2,12^23^34)_4 \mapsto (4,3,2,-12^23^34)_5$$

Row 20: $T = s_1s_2s_3s_4s_2s_3s_1s_2s_4s_3s_2s_1$.

$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_1 \mapsto (2,-12,123,4)_3 \mapsto (2,3,-123,1234)_7 \mapsto (2,123^24,4,-1234)_7 \mapsto (12^23^24,-123^24,123^24^2,3)_7 \mapsto (-12^23^24,2,123^24^2,3)_8 \mapsto (-12^23^24,12^23^24^2,-123^24^2,123^34^2)_4 \mapsto (4,-12^23^24^2,2,123^34^2)_4 \mapsto (4,-12^23^24^2,12^23^34^2,-123^34^2)_6 \mapsto (4,12^23^44^2,-12^23^34^2,2)_6 \mapsto (12^23^44^3,-12^23^44^2,3,2)_6 \mapsto (-12^23^44^3,4,3,2)_5$$

Row 21: $T = s_1s_2s_3s_4s_2s_3s_4s_1s_2s_3s_4s_2s_3s_1s_2s_4s_3s_2s_1$.

$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_3 \mapsto (2,-12,123,4)_6 \mapsto (2,3,-123,1234)_7 \mapsto (2,123^24,4,-1234)_7 \mapsto (12^23^24,-123^24,123^24^2,3)_7 \mapsto (-12^23^24,2,123^24^2,3)_7 \mapsto (-12^23^24,12^23^24^2,-123^24^2,123^34^2)_6 \mapsto (4,-12^23^24^2,2,123^34^2)_3 \mapsto (4,-12^23^24^2,12^23^34^2,-123^34^2)_4 \mapsto (4,12^23^44^2,-12^23^34^2,12^33^34^2)_4 \mapsto (12^23^44^3,-12^23^44^2,3,12^33^34^2)_5 \mapsto (-12^23^44^3,4,3,12^33^34^2)_5 \mapsto (-12^23^44^3,4,12^33^44^2,-12^33^34^2)_6 \mapsto (-12^23^44^3,12^33^44^3,-12^33^44^2,3)_7 \mapsto (-12^23^44^3,12^33^44^3,-12^33^44^2,3)_7 \mapsto (2,-12^33^44^3,4,12^33^54^3)_7 \mapsto (2,3,12^33^54^4,-12^33^54^3)_7 \mapsto (2,12^33^64^4,-12^33^54^4,4)_6 \mapsto (12^43^64^4,-12^33^64^4,3,4)_3 \mapsto (-12^43^64^4,2,3,4)_1$$

Row 22: $T = s_1s_2s_3s_4s_2s_4s_1s_2s_3s_2s_4s_1s_2s_4s_3s_2s_1$.

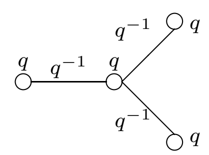
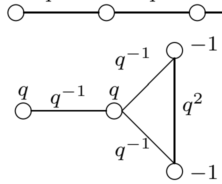
$$(1,2,3,4)_1 \mapsto (-1,12,3,4)_1 \mapsto (2,-12,123,4)_2 \mapsto (2,3,-123,1234)_4 \mapsto (2,123^24,4,-1234)_6 \mapsto (12^23^24,-123^24,4,123^34)_6 \mapsto (-12^23^24,2,4,123^34)_7 \mapsto (-12^23^24,12^23^34,123^34^2,-123^34)_5 \mapsto (3,-12^23^34,1^22^33^64^3,2)_8 \mapsto (3,123^34^2,-1^22^33^64^3,2)_8 \mapsto (123^44^2,-123^34^2,-12^23^34,12^23^34^2)_5 \mapsto (-123^44^2,3,-12^23^34,12^23^34^2)_4 \mapsto (-123^44^2,12^23^44^2,4,-12^23^34^2)_6 \mapsto (2,-12^23^44^2,4,12^23^54^2)_6 \mapsto (2,3,12^23^54^3,-12^23^54^2)_4 \mapsto (2,12^23^64^3,-12^23^54^3,4)_2 \mapsto (12^33^64^3,-12^23^64^3,3,4)_1 \mapsto (-12^33^64^3,2,3,4)_1$$

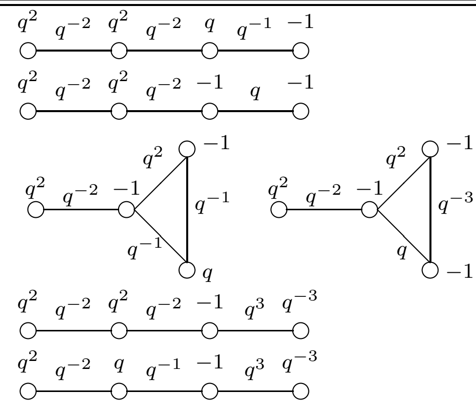
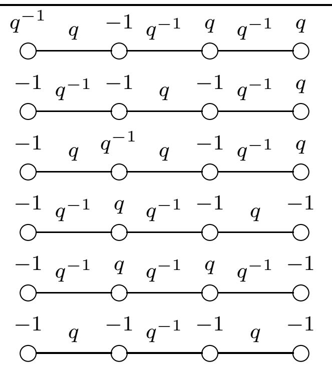
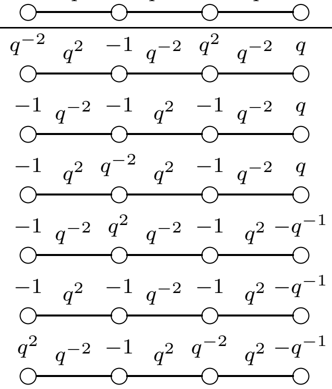
A.2 Computations for Table C

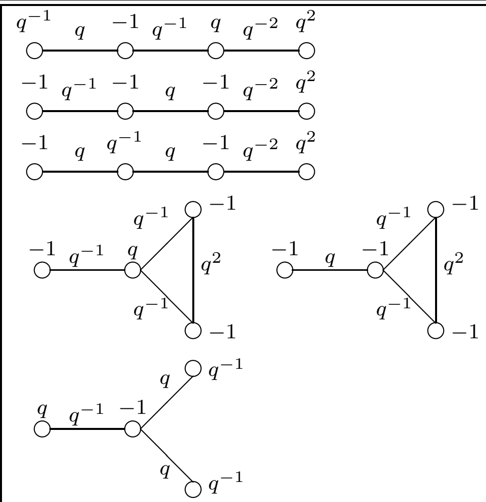
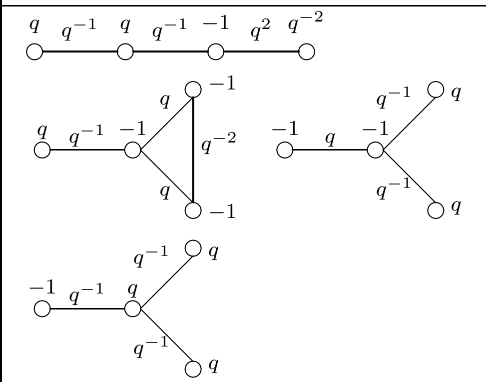
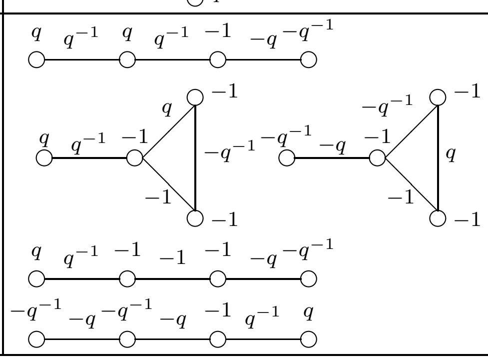
Row 2: $T := s_d \cdots s_2s_1$.

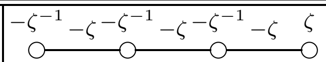
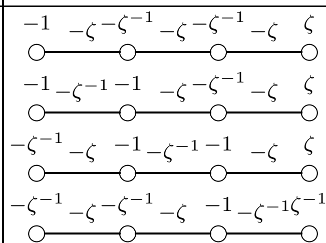
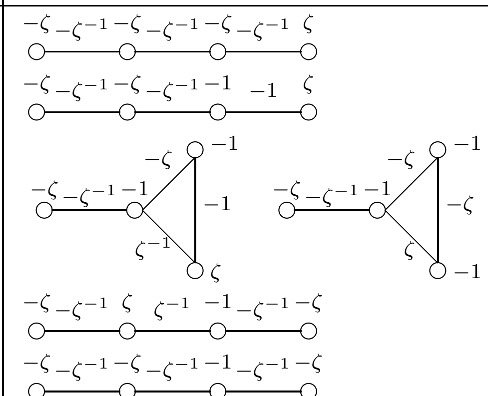
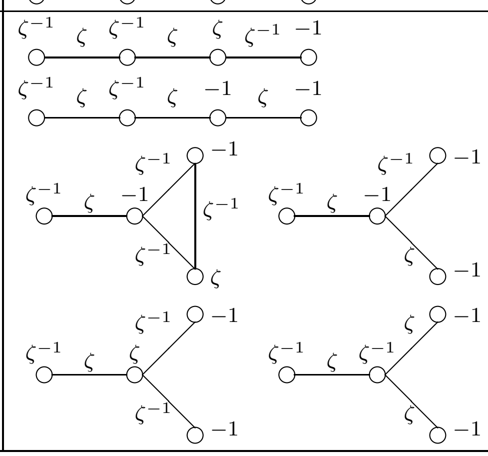
Rows 4,5 and 6: $T := s_1 s_2 \cdot \cdots \cdot s_{d-1} s_d s_{d-1} \cdots s_2 s_1$.
Row 8, $i_1 = 1$: $T := s_1 s_2 \cdots s_{d-3} s_{d-2} s_d s_{d-1} s_{d-2} \cdots s_2 s_1$.
Row 9, $(i_1, \ldots, i_j) = (1, \ldots, j)$: $T := s_1 s_2 \cdots s_{d-3} s_{d-2} s_d s_{d-1} s_{d-2} \cdots s_2 s_1$.
Row 10: $T := s_5 s_4 s_3 s_2 s_4 s_5 s_1 s_2 s_4 s_3 s_2 s_1$.
Row 11: $T := s_1 s_2 s_4 s_3 s_5 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_3 s_2 s_1$.
Row 12: $T := s_1 s_2 s_4 s_3 s_5 s_4 s_3 s_2 s_4 s_3 s_5 s_2 s_4 s_3 s_1 s_2 s_4 s_2 s_3 s_5 s_1 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 \times s_3 s_2 s_1$.
Row 13: $T := s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_3 s_5 s_1 s_2 s_4 s_3 s_2 s_1$.
Row 14: $T := s_1 s_2 s_3 s_4 s_5 s_2 s_4 s_3 s_5 s_1 s_2 s_4 s_5 s_2 s_4 s_3 s_5 s_1 s_2 s_4 s_3 s_2 s_1$.
Row 16: $T := s_1 s_2 s_3 s_4 s_5 s_6 s_3 s_5 s_4 s_2 s_3 s_5 s_6 s_1 s_2 s_3 s_5 s_4 s_3 s_2 s_1$.
Row 17: $T := s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_5 s_3 s_4 s_6 s_2 s_3 s_4 s_1 s_2 s_3 s_5 s_3 s_4 s_6 s_2 s_3 s_4 s_5 s_6 s_1 s_2 s_3 \times s_4 s_5 s_4 s_3 s_2 s_1$.
Row 18: $T := s_1 s_2 s_3 s_4 s_5 s_6 s_3 s_5 s_4 s_6 s_2 s_3 s_5 s_1 s_2 s_3 s_4 s_3 s_5 s_6 s_2 s_3 s_5 s_4 s_6 s_1 s_2 s_3 s_5 \times s_4 s_3 s_2 s_1$.
Row 20: $T := s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_4 s_6 s_5 s_3 s_4 s_6 s_7 s_2 s_3 s_4 s_6 s_1 s_2 s_3 s_4 s_5 s_4 s_6 s_7 s_3 s_4 s_6 \times s_5 s_2 s_3 s_4 s_6 s_7 s_1 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1$.

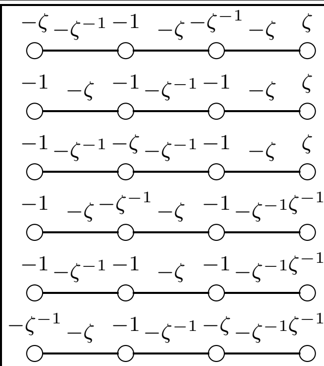
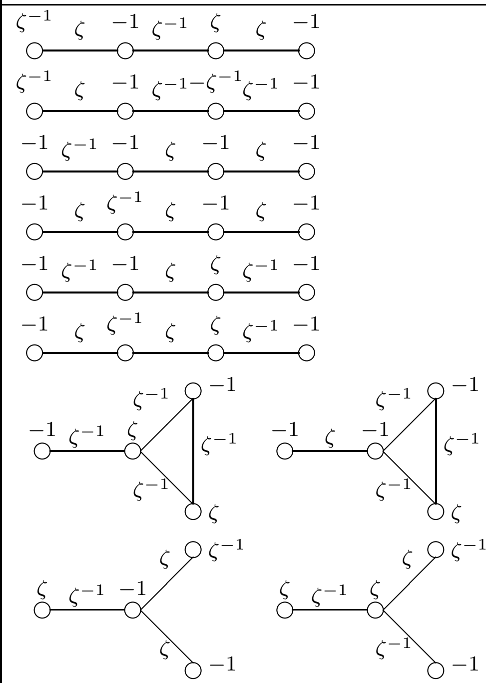
B Connected arithmetic root systems of rank four

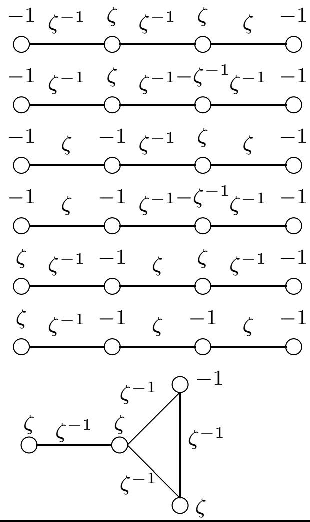
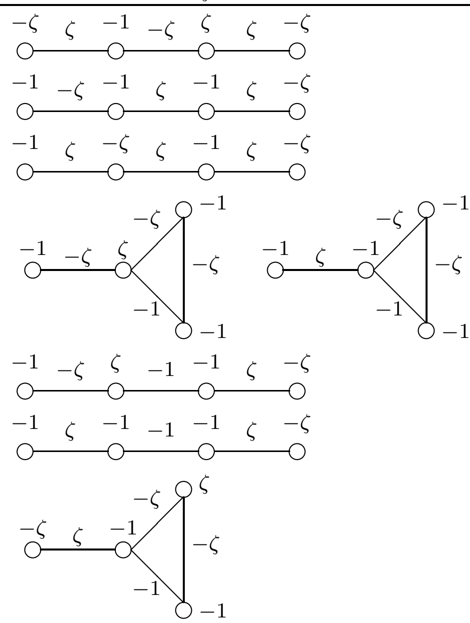
row	gener. Dynkin diagrams	fixed parameters
1	q —q^{-1}— q —q^{-1}— q —q^{-1}— q	$q \in k^* \setminus \{1\}$
2	q^2 —q^{-2}— q^2 —q^{-2}— q^2 —q^{-2}— q	$q \in k^* \setminus \{-1, 1\}$
3	q —q^{-1}— q —q^{-1}— q —q^{-2}— q^2	$q \in k^* \setminus \{-1, 1\}$
4	q^2 —q^{-2}— q^2 —q^{-2}— q —q^{-1}— q	$q \in k^* \setminus \{-1, 1\}$
5		$q \in k^* \setminus \{1\}$
6	-1 —q^{-1}— q —q^{-1}— q —q^{-1}— q -1 —q— -1 —q^{-1}— q —q^{-1}— q q —q^{-1}— -1 —q— -1 —q^{-1}— q	$q \in k^* \setminus \{-1, 1\}$
7	-1 —q^{-2}— q^2 —q^{-2}— q^2 —q^{-2}— q -1 —q^2— -1 —q^{-2}— q^2 —q^{-2}— q q^2 —q^{-2}— -1 —q^2— -1 —q^{-2}— q q^2 —q^{-2}— q^2 —q^{-2}— -1 —q^2— $-q^{-1}$	$q \in k^*$, $q^4 \neq 1$
8	-1 —q^{-1}— q —q^{-1}— q —q^{-2}— q^2 -1 —q— -1 —q^{-1}— q —q^{-2}— q^2 q —q^{-1}— -1 —q— -1 —q^{-2}— q^2 	$q \in k^* \setminus \{-1, 1\}$

row	gener. Dynkin diagrams	fixed param.
9		$q \in k^*$, $q^2, q^3 \neq 1$
10		$q \in k^* \setminus \{-1, 1\}$
11		$q \in k^*$, $q^4 \neq 1$

row	gener. Dynkin diagrams	fixed param.
12		$q \in k^* \setminus \{-1, 1\}$
13		$q \in k^* \setminus \{-1, 1\}$
14		$q \in k^* \setminus \{-1, 1\}$

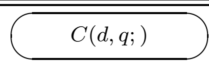
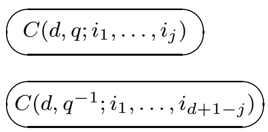
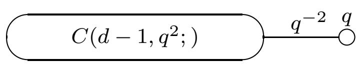
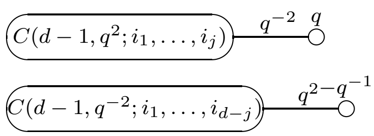
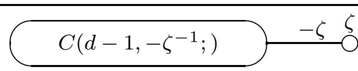
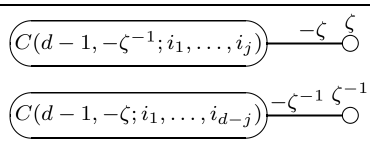
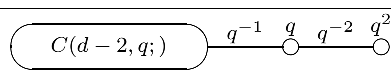
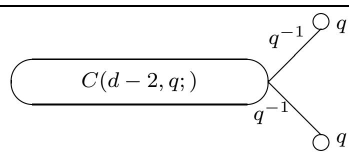
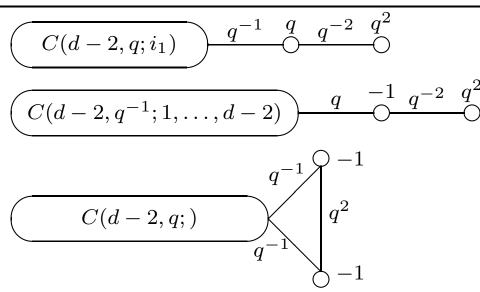
row	gener. Dynkin diagrams	fixed param.
15		$\zeta \in R_3$
16		$\zeta \in R_3$
17		$\zeta \in R_3$
18		$\zeta \in R_3$

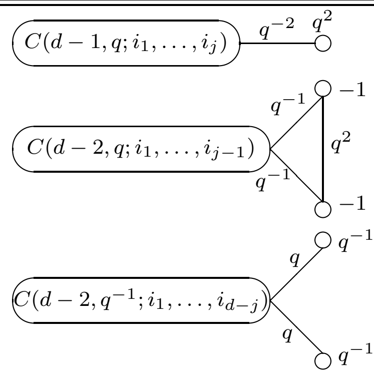
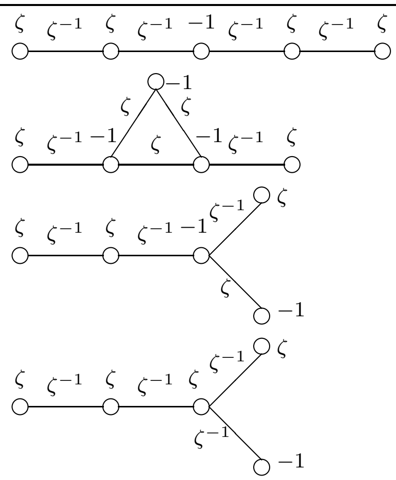
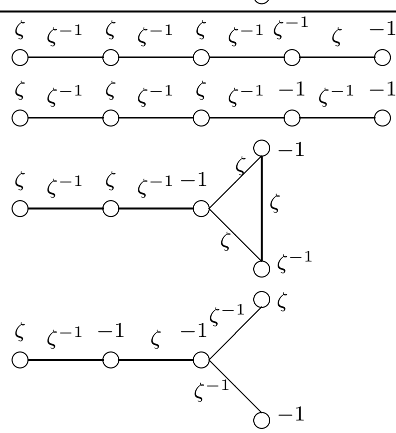
row	gener. Dynkin diagrams	fixed param.
19		$\zeta \in R_3$
20		$\zeta \in R_3$

row	gener. Dynkin diagrams	fixed param.
21		$\zeta \in R_3$
22		$\zeta \in R_4$

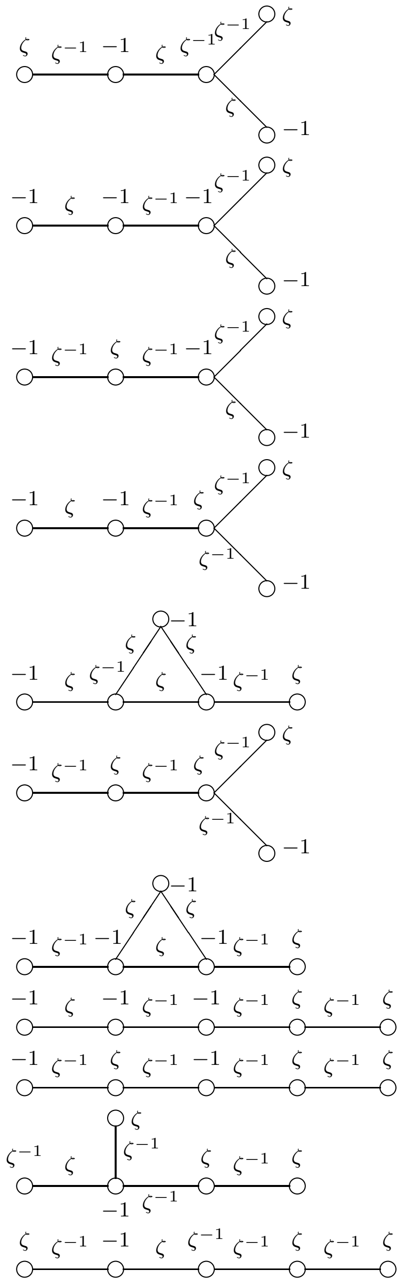
C Connected arithmetic root systems of rank $d \geq 5$

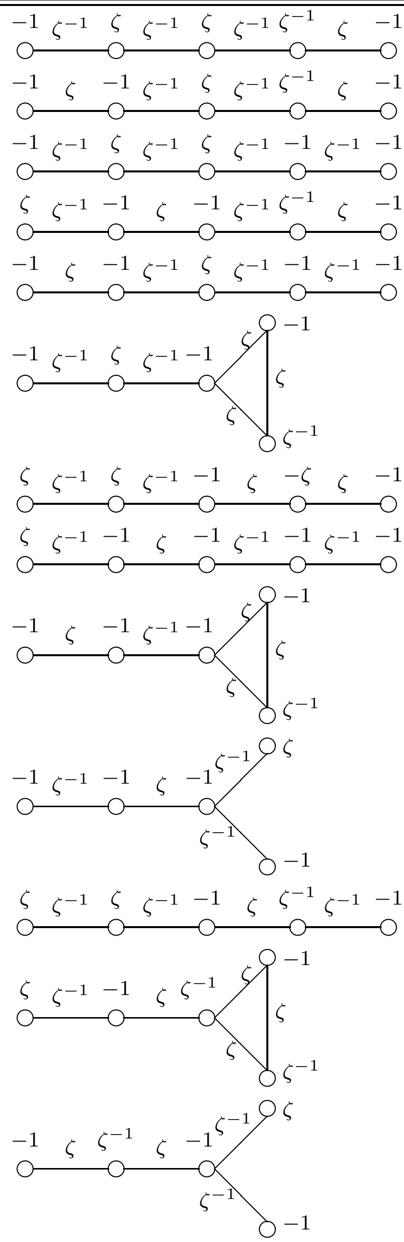
Recall the notation for simple chains in Section 5.

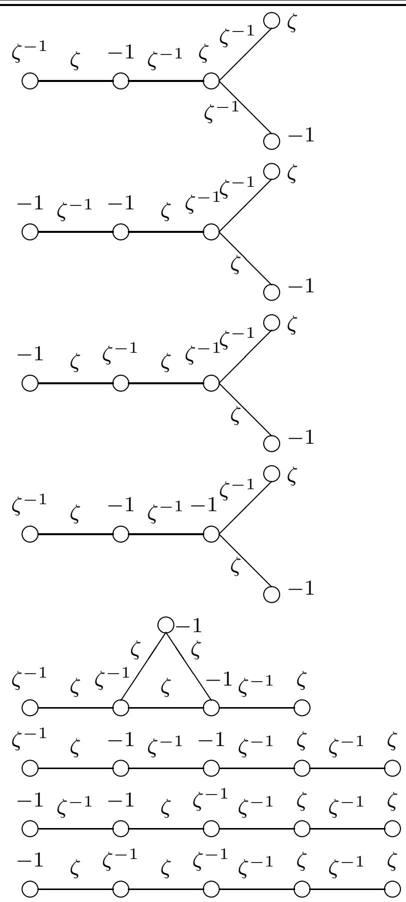
row	gener. Dynkin diagrams	fixed parameters
1		$q \in k^* \setminus \{1\}$
2		$q \in k^* \setminus \{-1, 1\}$ $1 \leq j \leq (d+1)/2$
3		$q \in k^* \setminus \{-1, 1\}$
4		$q \in k^*$, $q^4 \neq 1$ $1 \leq j \leq d-1$
5		$\zeta \in R_3$
6		$\zeta \in R_3$ $1 \leq j \leq d-1$
7		$q \in k^* \setminus \{-1, 1\}$
8		$q \in k^* \setminus \{1\}$
9		$q \in k^* \setminus \{-1, 1\}$

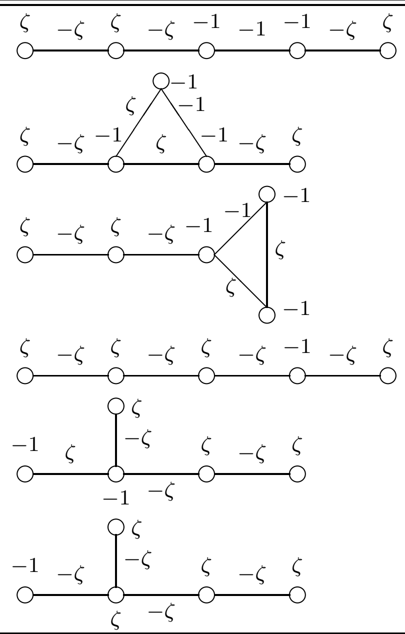
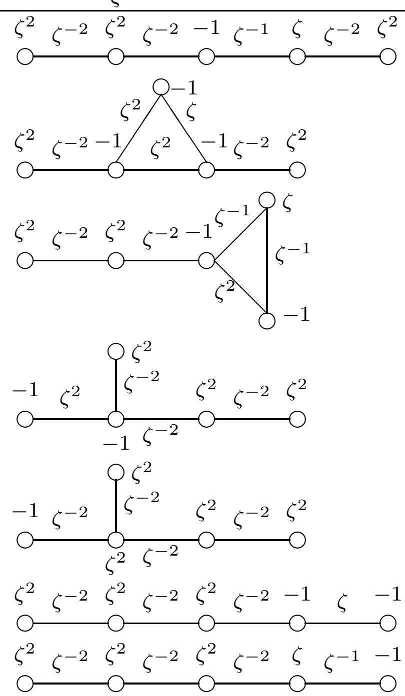
<table>
<tr><th>row</th><th>gener. Dynkin diagrams</th><th>fixed parameters</th></tr>
<tr><td>10</td><td>
</td><td>$q \in k^* \setminus \{-1, 1\}$
$2 \le j \le d-1$</td></tr>
<tr><td>11</td><td>
</td><td>$\zeta \in R_3$</td></tr>
<tr><td>12</td><td>

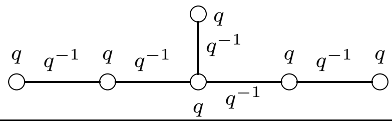
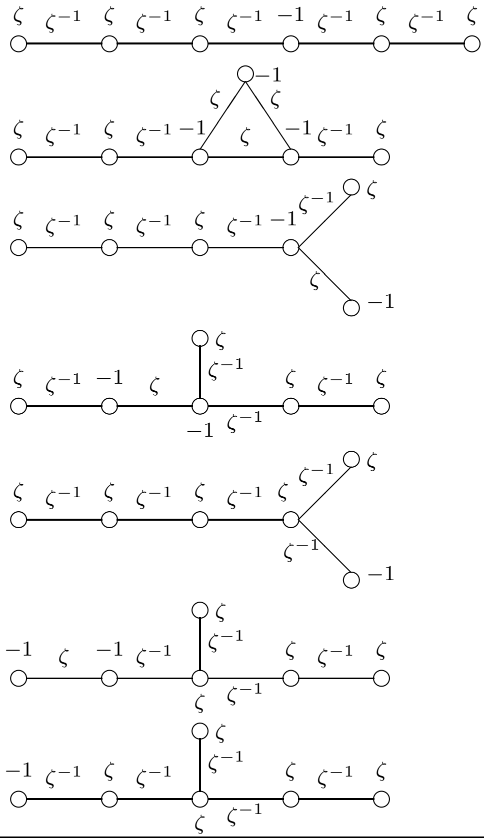
row continues on next page</td><td>$\zeta \in R_3$</td></tr>
</table>

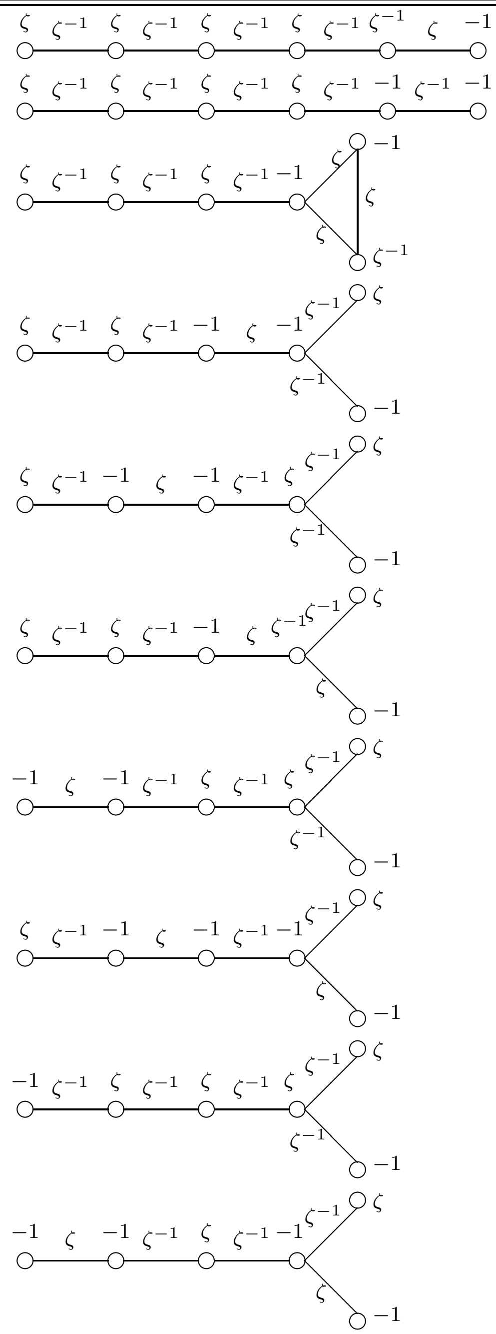
row	gener. Dynkin diagrams	fixed parameters
		

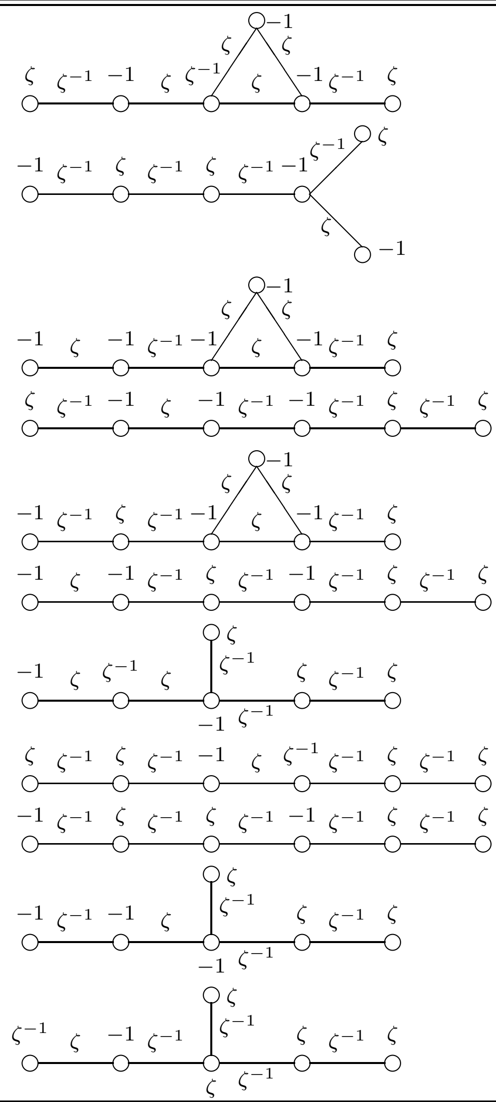
row	gener. Dynkin diagrams	fixed parameters
13	 row continues on next page	$\zeta \in R_3$

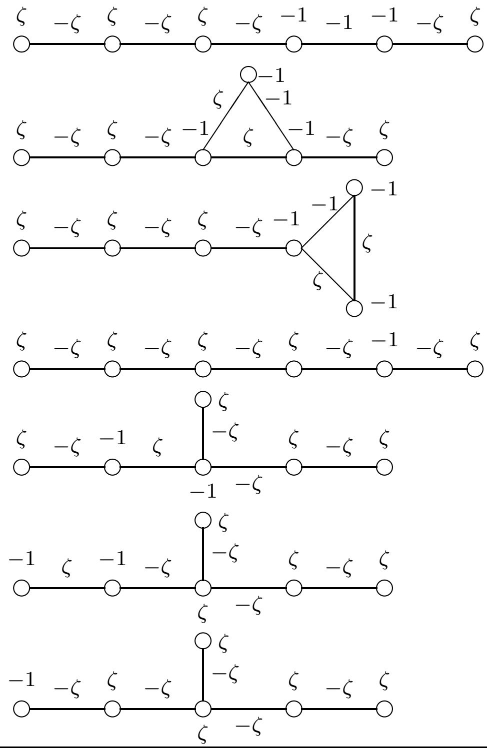
row	gener. Dynkin diagrams	fixed parameters
		

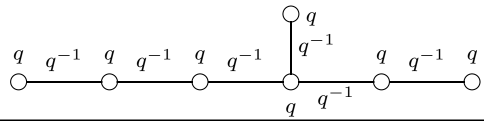
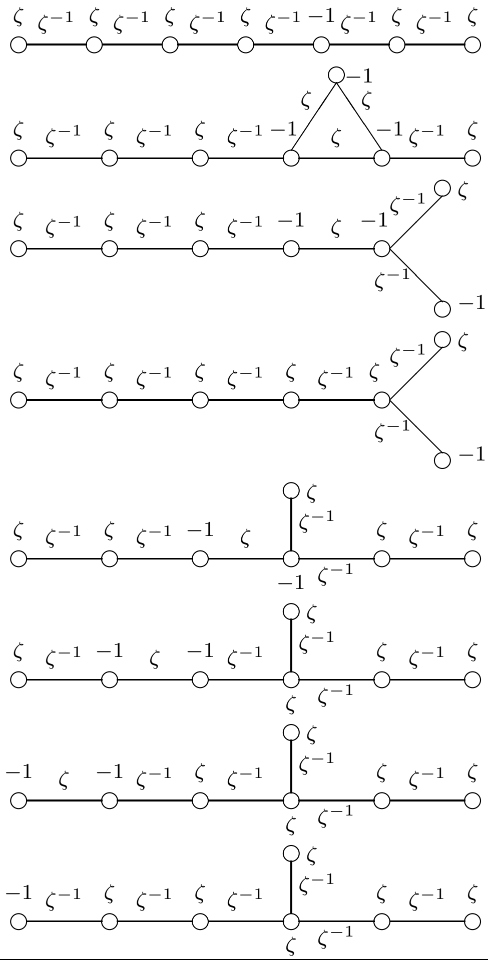
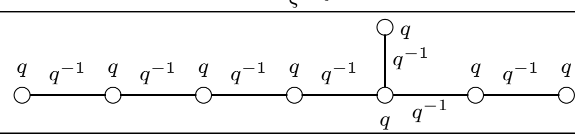
row	gener. Dynkin diagrams	fixed parameters
14		$\zeta \in R_4$
15		$\zeta \in R_5$

row	gener. Dynkin diagrams	fixed parameters
16		$q \in k^* \setminus \{1\}$
17		$\zeta \in R_3$

row	gener. Dynkin diagrams	fixed parameters
18		$\zeta \in R_3$
	row continues on next page	

row	gener. Dynkin diagrams	fixed parameters
		

row	gener. Dynkin diagrams	fixed parameters
19		$\zeta \in R_4$

row	gener. Dynkin diagrams	fixed parameters
20		$q \in k^* \setminus \{1\}$
21		$\zeta \in R_3$
22		$q \in k^* \setminus \{1\}$

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2000), vol. 294 ser. *Contemp. Math.*, pp. 1–57. Amer. Math. Soc. (2002)

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