

On the arithmetic difference of middle Cantor sets

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Abstract: Suppose that \mathcal{C} is the space of all middle Cantor sets. We characterize all triples $(\alpha, \beta, \lambda) \in \mathcal{C} \times \mathcal{C} \times \mathbb{R}^*$ that satisfy $C_\alpha - \lambda C_\beta = [-\lambda, 1]$. Also all triples (that are dense in $\mathcal{C} \times \mathcal{C} \times \mathbb{R}^*$) has been determined such that $C_\alpha - \lambda C_\beta$ forms the attractor of an iterated function system. Then we found a new family of the pair of middle Cantor sets \mathcal{P} in a way that for each $(C_\alpha, C_\beta) \in \mathcal{P}$, there exists a dense subfield $F \subset \mathbb{R}$ such that for each $\mu \in F$, the set $C_\alpha - \mu C_\beta$ contains an interval or has zero Lebesgue measure. In sequel, conditions on the functions f, g and the pair (C_α, C_β) is provided which $f(C_\alpha) - g(C_\beta)$ contains an interval. This leads us to denote another type of stability in the intersection of two Cantor sets. We prove the existence of this stability for regular Cantor sets that have stable intersection and its absence for those which the sum of their Hausdorff dimension is less than one.

At the end, special middle Cantor sets C_α and C_β are introduced. Then the iterated function system corresponding to the attractor $C_\alpha - \frac{2\alpha}{\beta}C_\beta$ is characterized. Some specifications of the attractor has been presented that keep our example as an exception. We also show that $\sqrt{C_\alpha} - \sqrt{C_\beta}$ contains at least one interval.

Keywords: Arithmetic difference, Hausdorff dimension, middle Cantor sets, Palis conjecture.

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1 Introduction

Regular Cantor sets play a fundamental role in dynamical systems and number theory. Intersections of hyperbolic sets with stable and unstable manifolds of its points are often regular Cantor sets. Also, related to diophantine approximations, many Cantor sets given by combinatorial conditions on the continued fraction of real numbers are regular. In studying the homoclinic bifurcations in dynamical systems, also the classical Markov and Lagrange spectra related to diophantine approximations in number theory, we deal with arithmetic difference of regular Cantor sets [2, 3, 12]. Many papers have been

written about metrical and topological properties of sum or difference of regular Cantor sets [6, 13, 15]. Before stating the results of this paper, we establish some notations. A Cantor set K is regular or dynamically defined if:

- (i) there are disjoint compact intervals I_1, I_2, \dots, I_r such that $K \subset I_1 \cup \dots \cup I_r$ and the boundary of each I_i is contained in K ,
- (ii) there is a $C^{1+\epsilon}$ expanding map ψ defined in a neighborhood of set $I_1 \cup I_2 \cup \dots \cup I_r$ such that $\psi(I_i)$ is the convex hull of a finite union of some intervals I_j satisfying:
 - For each i , $1 \leq i \leq r$ and n sufficiently big, $\psi^n(K \cap I_i) = K$,
 - $K = \bigcap_{n=0}^{\infty} \psi^{-n}(I_1 \cup I_2 \cup \dots \cup I_r)$.

The set $\{I_1, I_2, \dots, I_r\}$ is, by definition, a Markov partition for K , and the set $D := \bigcup_{i=1}^r I_i$ is the Markov domain of K .

The Cantor set K is close on the topology $C^{1+\epsilon}$ to a Cantor set \tilde{K} with the Markov partition $\{\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_r\}$ defined by expanding map $\tilde{\psi}$ if and only if the extremes of each I_i are near the corresponding extremes of \tilde{I}_i and supposing $\psi \in C^{1+\epsilon}$ with Holder constant C , we must have $\tilde{\psi} \in C^{1+\tilde{\epsilon}}$ with Holder constant \tilde{C} such that $(\tilde{C}, \tilde{\epsilon})$ is near (C, ϵ) and $\tilde{\psi}$ is close to ψ in the C^1 topology. Regular Cantor sets K and K' have *stable intersection* if for any pair of regular Cantor sets (\tilde{K}, \tilde{K}') near (K, K') , we have $\tilde{K} \cap \tilde{K}' \neq \emptyset$.

The concept of stable intersection has been introduced by Moreira in [6] for the first time. The Cantor sets K and K' that have stable intersection are useful in two points of view. First, in dynamical systems theory, when stable and unstable Cantor sets associated to a homoclinic bifurcation have a stable intersection, they present open sets in the parameter line with positive density at the initial bifurcating value, for which the corresponding diffeomorphisms are not hyperbolic. Second, in number theory, they guarantee the existence of an open set \mathcal{U} including (K, K') such that for each $(\tilde{K}, \tilde{K}') \in \mathcal{U}$, the set $\tilde{K} - \tilde{K}'$ contains an interval. On the other hand, topological and metrical structure of the $K - \lambda K'$ plays a key role in investigation of having stable intersection of regular Cantor sets K and K' [7]. Hence, we can concentrate on the arithmetic difference of regular Cantor sets of this form. Herein, there are several classical results:

- (i) If $\tau(K) \cdot \tau(K') > 1$, then $K - \lambda K'$ contains an interval ([12]),
- (ii) If $HD(K) + HD(K') > 1$, then $K - \lambda K'$ generically contains an interval ([7]),
- (iii) If $HD(K) + HD(K') > 1$, then $|K - \lambda K'| > 0$ for almost every $\lambda \in \mathbb{R}^*$ ([12]),
- (iv) There exist regular Cantor sets K and K' such that $K - K'$ has positive Lebesgue measure, but does not contain any interval ([14]),

(v) If $HD(K) + HD(K') < 1$, then $|K - \lambda K'| = 0$ ([12]).

A regular Cantor set K is affine if $D\psi$ is constant on every interval K_i . Meanwhile, the following conjecture due to Palis is still open: The arithmetic difference of two affine Cantor sets generically, if not always, contains an interval or has zero Lebesgue measure. Many studies have been done on this conjecture [6, 7, 8]. This conjecture can be written for middle Cantor sets too; Cantor set C_α is a middle- $(1 - 2\alpha)$ or in simple words, middle Cantor set, if the convex hull of C_α is $[0, 1]$ and the Markov partition of C_α has exactly two members with $D\psi = \frac{1}{\alpha}$ on their intervals. Regard to above discussion, the morphology of the arithmetic difference $C_\alpha - \lambda C_\beta$ on the mysterious region

$$\Omega := \left\{ (C_\alpha, C_\beta) \mid HD(C_\alpha) + HD(C_\beta) > 1, \tau(C_\alpha) \cdot \tau(C_\beta) < 1 \right\}$$

is unclear. This is our motivation in writing the present paper that is organized as follows:

In Section 2, Theorem 2 shows that for every $(\alpha, \beta) \in \Omega$ we always have:

- $C_\alpha - \lambda C_\beta = [-\lambda, 1]$, if and only if there exists the real number γ which satisfies $\alpha = \gamma^{-n_0}$, $\beta = \gamma^{-m_0}$, and moreover,
- i) $\frac{1}{\gamma} \leq \tau(C_\alpha) \cdot \tau(C_\beta)$,
- ii) $\lambda \in \bigcup_{n=-m_0+1}^{n_0} \gamma^n \cdot \left[\frac{1-2\alpha}{\gamma\beta}, \frac{\alpha}{1-2\beta} \right]$.

In the particular case $\lambda = -1$, we observe that if $\log \alpha$ and $\log \beta$ are not commensurable, then $C_\alpha + C_\beta \neq [0, 2]$ too, and if $\frac{\log \alpha}{\log \beta} = \frac{n_0}{m_0}$, then

$$C_\alpha + C_\beta = [0, 2] \Leftrightarrow \left[n_0 \log_\alpha \frac{\alpha}{1-2\beta} \right] = \left[n_0 \log_\alpha \frac{1-2\alpha}{\beta} \right] + 1$$

This extends Theorem D in [8], (see Corollary 3) .

In Section 3, we first introduce a dense subset $\mathcal{L} \subset \mathcal{C} \times \mathcal{C}$ and we prove a theorem that presents the iterated function systems with their attractors $C_\alpha - \lambda C_\beta$, where $(C_\alpha, C_\beta) \in \mathcal{L}$. We obtain this theorem by the transferred renormalization operators corresponding to a pair (C_α, C_β) on the space $\mathbb{R}^* \times \mathbb{R}$ that explained in [4]. A reason of proposing the theorem is that, tracing and controlling points in \mathbb{R} under suitable compositions of functions which constitute the iterated function system are easier than tracing and controlling points in $\mathbb{R}^* \times \mathbb{R}$ under suitable compositions of transferred operators. Although the number of functions which constitute the iterated function system could be so many, the methods and techniques of the theory of iterated function systems could be profitable. Moreira and Yoccoz in [7] introduced that, a way of having stable

intersection of Cantor sets C_α and C_β is to construct a recurrent compact set of relative configurations corresponding to the renormalization operators. To deal with Palis conjecture, these facts may be a step forward. In this direction, we have found an element of Ω that have stable intersection [10], (see Corollary 4 too). The other applications of the theorem will occur throughout the paper.

In Section 4, the first aim is to establish conditions on the function f , g and the pair $(C_\alpha, C_\beta) \in \mathcal{L}$ to ensure the existence of an interval in the set $f(C_\alpha) - g(C_\beta)$. For instance, it will be applied to guarantee the existence of an interval in the sets $C^2 + C^2$ and $\sin C + \cos C$, where C is the middle- $\frac{1}{3}$ Cantor set. This allows us to introduce the concept of “weak stable intersection” that could have a pair of arbitrary Cantor sets embedded in the real line; the pair (K, K') has weak (or geometric) C^r -stable intersection, if for all f and g in a C^r -neighborhood of the identity, we have $f(K) \cap g(K') \neq \emptyset$. Note that, the diffeomorphisms f and g do not change the Hausdorff dimensions of K and K' . To continue, we compare our definition with Moreira’s definition in category of regular Cantor sets and their similar attitude on the classic known regions. Indeed, these two definitions are equivalent on an open and dense subset and it seems that this equivalence happens everywhere.

In Section 5, we introduce $\lambda \in \mathbb{R}^*$ and $(C_\alpha, C_\beta) \in \Omega$ such that $C_\alpha - \lambda C_\beta$ is not an affine Cantor set and $|C_\alpha - \lambda C_\beta| = 0$. Regarding to Theorem 3, we observe that the set $C_\alpha - \lambda C_\beta$ is the attractor of the iterated function system namely $\mathcal{S} := \{S_i \mid S_i(t) = p^{-2}t + b_i\}_{i=1}^{21}$. It is not clear at this point that the Hausdorff dimension of the attractor is smaller than one since we have twenty one different affine maps with slopes $p^2 = 17, 94\dots < 21$. Nevertheless, we calculate its exact value in a different manner and then we present some other results about these Cantor sets. In the context of regular Cantor sets K and K' that $K - K'$ is not an affine Cantor set and $HD(K - K') < 1 < HD(K) + HD(K')$, there exist some properties that prove our example as an exceptional one among others:

- The set $C_\alpha - \lambda C_\beta$ forms the attractor of an iterated function system that is of finite type.
- The similarity dimension of \mathcal{S} is bigger than one and also it is not obvious to determine $HD(C_\alpha - \lambda C_\beta) < 1$ on the lower steps of the construction $C_\alpha \times C_\beta$. In fact, if we do twelve steps in the construction of C_α and eighteen steps in the construction of C_β , then we can select an iterated function system on $I \times I \subset \mathbb{R}^2$ of Hausdorff dimension smaller than one, such that $C_\alpha - \lambda C_\beta$ becomes the projection of its attractor under angle $\cot^{-1} \lambda$. While, this method does not apply for the lower steps of the construction $C_\alpha \times C_\beta$, (see Corollary 3 and the Remark 2 of [15]). Another purpose in presenting this method is to find an upper bound of Hausdorff dimension and it may be useful for the situations that the iterated function system is not of finite type.

- $\dim_H(C_\alpha - \lambda C_\beta) = \dim_B(C_\alpha - \lambda C_\beta)$, that is computable and $C_\alpha - \lambda C_\beta$ is a s-set.
- There exists a dense subfield F of the real numbers such that for each $\mu \in F$, the set $C_\alpha - \mu C_\beta$ contains an interval or has zero Lebesgue measure.
- We can not put them in a non constant continuous curve from the pair of regular Cantor sets that Hausdorff dimension of their arithmetic difference is less than one.

2 Renormalization operators

We begin this section by posing a basic lemma on the difference set of two affine Cantor set in their most general form. Any element in this set is related to one or several (and perhaps infinite) certain relatively compact sequence. The expanding maps appeared in Markov partition corresponding to these pair of Cantor sets can justify all underling sequences directly.

Suppose that K is an affine Cantor set with the Markov domain $\{I_0, I_1, \dots, I_r\}$ and expanding map ψ defined by $\psi|_{I_i} =: p_i x + e_i$ and also K' is an affine Cantor set with the Markov domain $\{I'_0, I'_1, \dots, I'_{r'}\}$ and expanding map ψ' defined by $\psi'|_{I'_j} =: q_j x + f_j$. For every $0 \leq i \leq r$ and $0 \leq j \leq r'$, consider the maps T_i and T'_j from $\mathbb{R}^* \times \mathbb{R}$ into itself given by

$$(s, t) \xrightarrow{T_i} (p_i s, p_i t + e_i) \quad (s, t) \xrightarrow{T'_j} \left(\frac{s}{q_j}, t - \frac{f_j}{q_j} s\right) \quad (2.1)$$

We call each one of these maps a “transferred renormalization operator” or in simple word “operator”. By these maps we can characterize all renormalization operators of relative configurations of limited geometries corresponding to affine Cantor sets K and K' , and that’s why we selected this name, as explained at the end of section 3 in [4].

Note, for given $(s, t) \in \mathbb{R}^+ \times \mathbb{R}$, the operator T_i pushes the point forward in direction to the line that passes through this point and point $(0, -\frac{e_i}{p_i})$, and also the operator

T'_j returns the point (s, t) in direction to the line that passes with slop $\frac{f_j}{q_j - 1}$ among this point. In the other word, the operators T_i are invariant on all the lines that passes among the point $(0, -\frac{e_i}{p_i})$ and the operators T'_j are invariant on all lines $t = \frac{f_j}{q_j - 1} s + t_0$.

Another important property of the operators (2.1) is $T_i \circ T'_j = T'_j \circ T_i$.

Lemma 1. $t \in K - sK'$, if and only if there exists a sequence (s_n, t_n) relatively compact in the space $\mathbb{R}^* \times \mathbb{R}$ with $(s_1, t_1) = (s, t)$ such that the point (s_{n+1}, t_{n+1}) obtained from (s_n, t_n) by applying one of transferred renormalization operators.

Proof. In this proof we take same notations used in [7], then the following assertions are equivalent:

- (i) $t \in K - sK'$,
- (ii) configuration $(id|_K, sid|_{K'} + t)$ is intersection. Here id means the identity map,
- (iii) for every $\underline{\theta} \in \Sigma^-$ and $\underline{\theta}' \in \Sigma'^-$, there exists a sequence $[(\underline{\theta}, id), (\underline{\theta}', s_n id + t_n)]$ relatively compact in the space of all the relative configurations of the limit geometries \mathcal{C} , such that $[(\underline{\theta}, id), (\underline{\theta}', s_{n+1} id + t_{n+1})]$ is obtained from $[(\underline{\theta}, id), (\underline{\theta}', s_n id + t_n)]$ by applying one of the renormalization operators, see page 51 of [7],
- (iv) sequence (s_{n+1}, t_{n+1}) obtained from (s_n, t_n) by applying one of the transferred renormalization operators is relatively compact in the space $\mathbb{R}^* \times \mathbb{R}$, see Theorem 1 of [4] and the explanation after that.

The assertion is obtained by considering these facts together.

□

Regard to Lemma 1, we observe that an easy way to show $t \in K - sK'$ is finding a bounded subset and far from zero $R \subset \mathbb{R}^* \times \mathbb{R}$ containing (s, t) such that for every element of R there exist suitable composites of transferred renormalization operators, which transfers that element to R . Indeed, idea appeared in proof of Theorems 2 and is to find a suitable subset such that for every their element there exist suitable composites of transferred renormalization operators, which transfers that element to the subset.

The compact set $R \subset \mathbb{R}^* \times \mathbb{R}$ is called a *recurrent* set for transferred renormalization operators if for every element of R there exists suitable composites of transferred renormalization operators, which transfers that element to R° . The existence of a non empty compact recurrent set guarantees that the underling affine Cantor sets have stable intersection ([4]). We construct a compact recurrent set for affine Cantor sets with Markov partitions exactly with two elements. Thus, if Newhouse's thickness condition satisfies in these cases we obtain a stronger assertion rather than Newhouse's thickness theorem.

Proposition 1. *If $\tau(K) \cdot \tau(K') > 1$, then Cantor sets K and K' carry a compact recurrent set.*

Proof. Without loss of generality we take Cantor set K and K' with convex hulls $[0, a]$ and $[0, b]$, receptively. Thus, $e_0 = f_0 = 0$ and

$$I_0 = [0, \frac{a}{p_0}], \quad I_1 = [\frac{-e_1}{p_1}, \frac{a - e_1}{p_1}], \quad I'_0 = [0, \frac{b}{q_0}], \quad I'_1 = [\frac{-f_1}{q_1}, \frac{b - f_1}{q_1}].$$

As $\tau(K) \cdot \tau(K') > 1$, we can choose the numbers $0 < \epsilon \ll 1$ and $\delta < a\epsilon$ that satisfy

$$\frac{\min\{\frac{a}{p_0}, \frac{a}{p_1}\}}{-\frac{e_1}{p_1} - \frac{a}{p_0} + \frac{\delta}{p_1}} \cdot \frac{\min\{\frac{b}{q_0}, \frac{b}{q_1}\}}{-\frac{f_1}{q_1} - \frac{b}{q_0}} > \frac{1}{1 - \epsilon} \quad (2.2)$$

Let $s_0 = -\frac{q_0 q_1 a}{q_0 f_1 + q_1 b}$ that is the first coordinate of the point obtained from the intersection of the lines $L_0 : t = -\frac{b}{q_0} s$ and $L_1 : t = \frac{f_1}{q_1} s + a$, see Figure 1. Denote

$$R = \left\{ (s, t) \mid -bs + \delta \leq t \leq a - \delta, \quad \frac{(1 - \epsilon)s_0}{\max\{p_0, p_1\} \cdot \max\{q_0, q_1\}} \leq s \leq (1 - \epsilon)s_0 \right\}$$

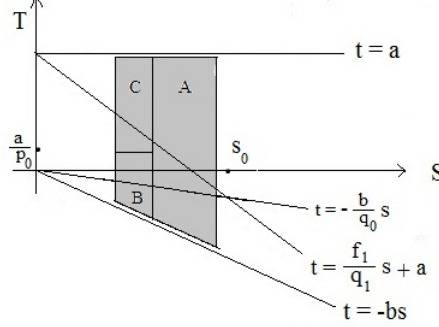


Figure 1: Gray region illustrates the recurrent set R .

We split the region R in below subsets:

- $A = \left\{ (s, t) \mid -bs + \delta \leq t \leq a - \delta, \quad \frac{(1 - \epsilon)s_0}{\max\{p_0, p_1\}} < s \leq (1 - \epsilon)s_0 \right\}$,
- $B = \left\{ (s, t) \mid -bs + \delta \leq t < \frac{a}{p_0}, \quad \frac{(1 - \epsilon)s_0}{\max\{p_0, p_1\} \cdot \max\{q_0, q_1\}} \leq s \leq \frac{(1 - \epsilon)s_0}{\max\{p_0, p_1\}} \right\}$,
- $C = \left\{ (s, t) \mid \frac{a}{p_0} \leq t \leq a - \delta, \quad \frac{(1 - \epsilon)s_0}{\max\{p_0, p_1\} \cdot \max\{q_0, q_1\}} \leq s \leq \frac{(1 - \epsilon)s_0}{\max\{p_0, p_1\}} \right\}$.

For the points of the subset R , we have below facts.

Case 1. $(s, t) \in A$, here we use the operators T'_0 and T'_1 (perhaps several time) and transfer all points A to the sets B and C . Indeed, we observe that $T'_0(L_0) = \{(s, t) \mid t = -bs\}$ and $T'_1(L_1) = \{(s, t) \mid t = a\}$, moreover T'_0 is invariant on the horizontal lines and T'_1 on the lines $\{(s, t) \mid t + bs = c\}$ and for $s \leq (1 - \epsilon)s_0$ diameter of the subset $\{(s, t) \in R \mid -\frac{b}{q_0}s < t < \frac{f_1}{q_1}s + a\}$ is more than δ .

Case 2. $(s, t) \in B$, if $t < \frac{a - \delta}{p_0}$, we use the operator T_0 and transfer the point $(s, t) \in B$ to R° since $T_0(\{(s, t) \mid t = \frac{a - \delta}{p_0}\}) = \{(s, t) \mid t = a - \delta\}$. For other points, at first we use the operator T_0 and then we use the operators T'_0 and T_1 respectively to ensure that the point is transferred to R° . Note that for each $s \in \left[\frac{(1 - \epsilon)s_0}{\max\{p_0, p_1\} \cdot \max\{q_0, q_1\}}, (1 - \epsilon)s_0 \right]$, always one of numbers $p_1 s$ or $\frac{s}{q_0}$ stays in interior of this interval.

Case 3. $(s, t) \in C$, here we use the operator T_1 and we transfer all points C to R° since $T_1\left(\left(\frac{(1-\epsilon)s_0}{\max\{p_0, p_1\} \cdot \max\{q_0, q_1\}}, \frac{a}{p_0}\right)\right) \in R^\circ$. Note that this relation is equivalent to

$$-b \cdot \frac{p_1(1-\epsilon)s_0}{\max\{p_0, p_1\} \cdot \max\{q_0, q_1\}} + \delta < p_1 \frac{a}{p_0} + e_1$$

and this relation is equivalent to relation (2.2).

Above cases ensure us that every element of R transfers to R° by the operators (2.1). Thus compact set R is a recurrent set and this complete the proof. \square

In the cases that Newhouse's thickness condition of C_α and C_β fails, Theorem 3 could be a suitable tool in the investigation of having stable intersection them. In [C], we applied one aspect of this theorem for a special pair of middle Cantor sets to show that they have stable intersection. Finding a dense subset from this type of pairs in Ω gives a positive answer to Palis conjecture in the context of middle Cantor sets. Also it is not easy for us to answer to above theorem in the context of affine Cantor sets with Markov partitions more than two elements. In the rest, we pose below definition.

Definition 1. *We say that the pair (s, t) is a difference point for Cantor sets K and K' , if there exists a sequence (s_n, t_n) relatively compact in the space $\mathbb{R}^* \times \mathbb{R}$ with $(s_1, t_1) = (s, t)$ such that the point (s_{n+1}, t_{n+1}) obtained from (s_n, t_n) by applying one of transferred renormalization operators.*

By using Lemma 1, the point (s, t) is a difference point for Cantor sets K and K' if and only if $t \in K - sK'$. Take K and K' same Cantor sets selected in Proposition 1 and $s_1 := \frac{1}{b}\left(\frac{-e_1}{p_1} - \frac{a}{p_0}\right)$, then by planning the arguments similar to what employed in the proof of the proposition, we can show that if

$$R := \left\{ (s, t) \mid s \in [s_1, s_0], \quad -bs \leq t \leq a \right\},$$

see figure 1, then each point of subset R is a difference point, since

$$T_1\left(s_1, \frac{a}{p_0}\right) = \left(p_1 s_1, p_1 \frac{a}{p_0} + e_1\right) = \left(p_1 s_1, -bp_1 s_1\right).$$

In contrast, let

$$E := \{(s, t) \mid s > 0, \quad t > a \text{ or } t < -bs\}.$$

The explanations before Lemma 1 express that any relatively compact sequence (s_{n+1}, t_{n+1}) starting from point $(s, t) \in E$ obtained from (s_n, t_n) by applying one of the transferred renormalization operators is unbounded and so (s, t) is not a difference point.

The operators T'_0 and T'_1 send the points of set

$$F := \left\{ (\lambda, t) \mid \lambda > s_0, \quad \frac{f_1}{q_1} \lambda + a < t < -\frac{b}{q_0} \lambda \right\}$$

to E . Also, the operators T_0 and T_1 send the points of set

$$G := \{(\lambda, t) \mid 0 < \lambda < s_1, \quad \frac{a}{p_0} < t < -bs + a - \frac{a}{p_1}\}$$

to E (in case of middle cantor sets, see the subset E, F and G in figure 2). Thus, none of the points of subsets F and G are a difference point.

Above facts have proved below theorem.

Theorem 1. *If the affine Cantor sets K and K' are with convex hulls $[0, a]$ and $[0, b]$, respectively and Markov partitions with two elements that satisfy $\tau(K) \cdot \tau(K') > 1$, then $K - \lambda K' = [-\lambda b, a]$, if and only if*

$$s_1 = \frac{1}{b} \left(\frac{-e_1}{p_1} - \frac{a}{p_0} \right) \leq \lambda \leq \frac{a}{\frac{-f_1}{q_1} - \frac{b}{q_0}} = s_0,$$

In the absence of Newhouse's thickness condition for the affine Cantor sets K and K' , it is not easy for us to characterize all λ 's that $K - \lambda K' = [-\lambda b, a]$. But in context of middle cantor sets, we yield below result.

Theorem 2. *Suppose that (C_α, C_β) are a pair of middle Cantor sets that satisfy $\tau(C_\alpha) \cdot \tau(C_\beta) < 1$. Also,*

I) *let γ be a real number that satisfies $\alpha = \gamma^{-n_0}$ and $\beta = \gamma^{-m_0}$, then*

1. *if $\frac{1}{\gamma} \leq \tau(C_\alpha) \cdot \tau(C_\beta)$, then $\lambda \in \bigcup_{n=-m_0+1}^{n_0} \gamma^n \cdot \left[\frac{1-2\alpha}{\gamma\beta}, \frac{\alpha}{1-2\beta} \right]$, if and only if $C_\alpha - \lambda C_\beta = [-\lambda, 1]$,*
2. *if $\frac{1}{\gamma} > \tau(C_\alpha) \cdot \tau(C_\beta)$, then for every real number λ , we have $C_\alpha - \lambda C_\beta \neq [-\lambda, 1]$,*

II) *let $\frac{\log \alpha}{\log \beta}$ be irrational, then for every real number λ , we have $C_\alpha - \lambda C_\beta \neq [-\lambda, 1]$,*

Proof. As before, take $\alpha = \frac{1}{p}$ and $\beta = \frac{1}{q}$. Put $p_0 = p_1 = p$ and $q_0 = q_1 = q$ in above discussion we obtain $s_0 = \frac{q}{q-2}$, $s_1 = \frac{p-2}{p}$ and the discussion before theorem says that for every $\lambda \notin [s_1, s_0]$ we have $C_\alpha - \lambda C_\beta \neq [-\lambda, 1]$.

Since $\tau(C_\alpha) \cdot \tau(C_\beta) < 1$, then $\frac{q}{p(q-2)} < \frac{q(p-2)}{p}$. For simplicity, put

$$I := \left(\frac{q}{p(q-2)}, \frac{q(p-2)}{p} \right) = \left(\frac{1}{p}s_0, qs_1 \right)$$

If $\lambda \in I$, then non of $p\lambda$ and $\frac{1}{q}\lambda$ does not belong to $[s_1, s_0]$ and so $C_\alpha - \lambda C_\beta \neq [-\lambda, 1]$.

To see this, let

$$\Delta_1 := \left\{ (s, t) \mid t + \left(1 - \frac{1}{q}\right)s > 1, \quad t > \frac{1}{p}, \quad t + \frac{1}{q}s < 1 - \frac{1}{p} \right\}$$

and

$$\Delta_2 := \left\{ (s, t) \mid t + s < 1 - \frac{1}{p}, \quad t + \frac{1}{q}s < 0, \quad t + \left(1 - \frac{1}{q}\right)s > \frac{1}{p} \right\},$$

Triangles Δ_1 and Δ_2 are non empty and both project to the same horizontal interval I . Also, $T_1(\Delta_1)$ and $T_0(\Delta_2)$ are subsets of F and $T_0(\Delta_1)$ and $T_1(\Delta_2)$ are subsets of E , (note, here $a = b = 1, e_1 = 1 - p$ and $f_1 = 1 - q$). Also, $T'_0(\Delta_1)$ and $T'_1(\Delta_2)$ are subsets of G and $T'_1(\Delta_1)$ and $T'_0(\Delta_2)$ are subsets of E . These facts say that there is not any relatively compact sequence (s_{n+1}, t_{n+1}) obtained from (s_n, t_n) by applying one of the transferred renormalization operators that starts from the points Δ_1 and Δ_2 .

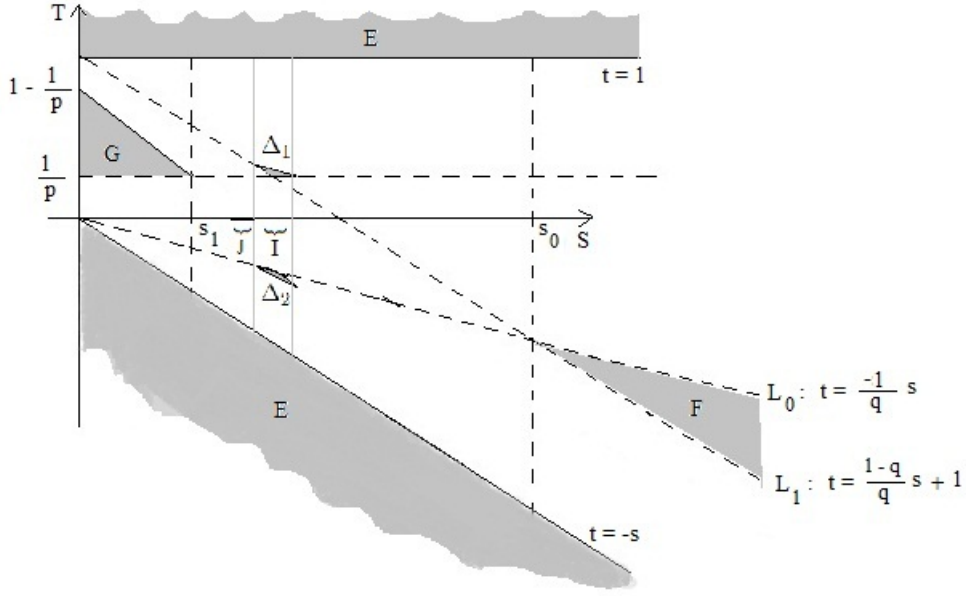


Figure 2: Gray region illustrates the recurrent set R .

We prove the theorem via three claims below.

Claim 1. For each $\lambda \in [s_1, s_0]$, we always have

- 1) if $q\lambda \leq s_0$, then $C_\alpha - \lambda C_\beta = [-\lambda, 1]$ if and only if $C_\alpha - q\lambda C_\beta = [-q\lambda, 1]$,
- 2) if $s_1 \leq \frac{1}{p}\lambda$, then $C_\alpha - \lambda C_\beta = [-\lambda, 1]$ if and only if $C_\alpha - \frac{1}{p}\lambda C_\beta = [-\frac{1}{p}\lambda, 1]$.

Prove (1) : Firstly, assume that $C_\alpha - \lambda C_\beta = [-\lambda, 1]$ and $t \in [-q\lambda, 1]$. Thus, at least one of the points $T'_0(q\lambda, t)$ or $T'_1(q\lambda, t)$ belong to set $\{\lambda\} \times [-\lambda, 1]$, since

$$T'_0(L_0) = \{ (s, t) \mid t = -s \}, \quad T'_1(L_1) = \{ (s, t) \mid t = 1 \}$$

and $q\lambda \leq s_0$, see Figure 1. By using Lemma 1, we observe $t \in C_\alpha - q\lambda C_\beta$ and so $C_\alpha - q\lambda C_\beta = [-q\lambda, 1]$.

Now we prove the converse of the assertion (1). Firstly for each given (s, t) , we obtain that

$$(s, t) = T'_0(qs, t) = T'_1(qs, t - (q-1)s).$$

On the other hand, the relation $(p-2)(q-2) > 1$ is equivalent to

$$\frac{p-2}{p} > \frac{1}{2q-3}$$

If $s \geq \frac{p-2}{p}$, thus $s > \frac{1}{2q-3}$ and this relation is equivalent to

$$\left(-\frac{q-1}{q}(qs) + 1\right) - \left(-\frac{1}{q}(qs)\right) < (q-1)s.$$

Therefore, one of the points (qs, t) or $(qs, t - (q-1)s)$ never take place in triangle obtained by the lines L_0, L_1 and $s = \frac{p-2}{p}$ that is the region

$$\left\{ (s, t) \mid s_1 \leq s \leq s_0, \quad -\frac{1}{q}s \leq t \leq -\frac{q-1}{q}s + 1 \right\}.$$

Now suppose that the assertion does not hold. Thus there exist $t \in [-\lambda, 1]$ such that $t \notin C_\alpha - \lambda C_\beta$. Above discussion (note $s_1 \leq \lambda$) and Lemma 1 guarantees that one of the points t or $t - (q-1)\lambda$ does not belong to $C_\alpha - q\lambda C_\beta$, (note t and $t - (q-1)\lambda$ belong to the interval $[-q\lambda, 1]$). This is a contradiction.

Proof (2): We have

$$T_0\left(\frac{\lambda}{p} \times \left[-\frac{\lambda}{p}, \frac{1}{p}\right]\right) = \{\lambda\} \times [-\lambda, 1] = T_1\left(\frac{\lambda}{p} \times \left[\frac{p-1}{p} - \frac{\lambda}{p}, 1\right]\right)$$

But $\frac{p-1}{p} - \frac{\lambda}{p} < \frac{1}{p}$, since $s_1 \leq \frac{1}{p}\lambda$. Since each element of the set $\{\lambda\} \times [-\lambda, 1]$ is a difference point. Thus each element of the set $\frac{\lambda}{p} \times \left[-\frac{\lambda}{p}, 1\right]$ is a difference point.

Now we prove the converse of the assertion (2). Take $t \in [-\lambda, 1]$, we have

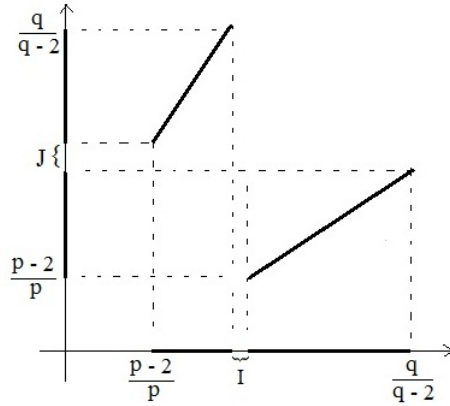
$$T_0\left(\frac{\lambda}{p}, \frac{t}{p}\right) = (\lambda, t) = T_1\left(\frac{\lambda}{p}, \frac{p-1}{p} + \frac{t}{p}\right)$$

$$T_1\left(\frac{\lambda}{p}, \frac{t}{p}\right) = (\lambda, t - p + 1), \quad T_0\left(\frac{\lambda}{p}, \frac{p-1}{p} + \frac{t}{p}\right) = (\lambda, t + p - 1)$$

One of the points $(\lambda, t - p + 1)$ and $(\lambda, t + p - 1)$ does not belong to set $\{\lambda\} \times [-\lambda, 1]$, Now suppose that the assertion does not hold. Thus there exists $t \in [-\lambda, 1]$ such that $t \notin C_\alpha - \lambda C_\beta$. Above discussion and Lemma 1 guarantees that one of the points $\frac{t}{p}$ or $\frac{p-1}{p} + \frac{t}{p}$ (both stay in $[-\frac{\lambda}{p}, 1]$) does not belong to $C_\alpha - \frac{\lambda}{p}C_\beta$. This is a contradiction and completes our claim.

Note that we may take the case $I_0 \subset I$. Indeed, if $s_0 \leq qs_1$ happens, by using the second part of Claim 1 (perhaps several times), we observe $C_\alpha - \lambda C_\beta \neq [-\lambda, 1]$, for every λ . The similar situation can be treated for $\frac{1}{p}s_0 \leq s_1$.

$$\text{Claim 2. Let } T(x) := \begin{cases} px & x \in \left[\frac{p-2}{p}, \frac{q}{p(q-2)}\right] \\ \frac{1}{q}x & x \in \left[\frac{q(p-2)}{p}, \frac{q}{q-2}\right] \end{cases},$$



then

$$s \in [s_1, s_0] - \bigcup_{n=0}^{\infty} T^{-n}(I) \Leftrightarrow C_\alpha - sC_\beta = [-s, 1].$$

For given $0 \leq n \leq m_0 + n_0$, it is not hard to show that for each x that stays in domain T^n , there exists $0 \leq i \leq m_0$ and $0 \leq j \leq n_0$ such that $T^n(x) = \frac{p^i}{q^j}x$.

Regard to explanations after Lemma 1, we introduce a subset $R \subset \mathbb{R}^* \times \mathbb{R}$ as below bounded and far from zero such that for every element of R there exist suitable composites of transferred renormalization operators, which transfers that element to R .

Let $\Lambda := [s_1, s_0] - \bigcup_{n=0}^{\infty} T^{-n}(I)$ and

$$R := \{ (s, t) \mid s \in \Lambda, -s \leq t \leq 1 \}.$$

It is straightforward to show that $T(\Lambda) \subset \Lambda$, that we mean Λ is invariant for the map T . For given each $(s, t) \in R$,

- if $s_1 \leq s \leq \frac{1}{p}s_0$, then point $T_0(s, t) = (ps, pt)$ falls above line $\{ (s, t) \mid t = -s \}$ and point $T_1(s, t) = (ps, pt - p + 1)$ falls below line $\{ (s, t) \mid t = 1 \}$. Thus, one of the points $T_0(s, t)$ and $T_1(s, t)$ situated in R , since $s_1 \leq s$ and Λ is invariant for the map T .

- if $qs_1 \leq s \leq s_0$, then one of the operators T'_0 and T'_1 transfer the point to the sets R . since $T'_0(L_0) = \{ (s, t) \mid t = -s \}$, $T'_1(L_1) = \{ (s, t) \mid t = 1 \}$, $s \leq s_0$ and Λ is invariant for the map T .

Conversely, for each $(s, t) \in R$, we always have $C_\alpha - sC_\beta = [-s, 1]$.

Conversely, assume that $C_\alpha - sC_\beta = [-s, 1]$ and $s \notin \Lambda$. Take number minimum n that $T^n(s) \in I$. Thus, $C_\alpha - T^n(s)C_\beta \neq [-T^n(s), 1]$. By applying Claim (1), perhaps several times, we see that $C_\alpha - sC_\beta \neq [-s, 1]$. This is a contradiction and completes the proof of the claim.

proof I. 1 . let γ be a real number that satisfies $p = \gamma^{n_0}$, $q = \gamma^{m_0}$ and $1 < (p-2)(q-2) \leq \gamma$. Put

$$J := \left[\frac{q(p-2)}{\gamma p}, \frac{q}{p(q-2)} \right]$$

then

$$[s_1, s_0] - \bigcup_{n=0}^{\infty} T^{-n}(I) = \bigcup_{n=0}^{n=n_0+m_0-1} T^n(J)$$

We first show $\bigcup_{n=0}^{n=n_0+m_0-1} T^n(J) \subset \Lambda$. To do this, it is enough to show that for each $\lambda \in J$ and $n \in \mathbb{N}$, the point $T^n(\lambda)$ does not belong to the interval I . Indeed, if for any $\lambda \in J$, the point $T^n(\lambda)$ belongs to the interval I , then

$$\frac{1}{\gamma} = \frac{\frac{q(p-2)}{\gamma p}}{\frac{q(p-2)}{p}} < \frac{\lambda}{T^n(\lambda)} < 1.$$

On the other hand, there exist i and j such that $\frac{\lambda}{T^n(\lambda)} = \frac{q^i}{p^j} = \gamma^{m_0 i - n_0 j}$ and this is a contradiction, since the relation $\frac{1}{\gamma} < \gamma^{m_0 i - n_0 j} < 1$ never takes place.

Conversely, for simplicity let

$$S(x) := T^{-1}(x) = \begin{cases} qx & x \in \left[\frac{p-2}{p}, \frac{1}{q-2} \right] \\ \frac{1}{p}x & x \in \left[p-2, \frac{q}{q-2} \right] \end{cases},$$

We need to show that for each $\lambda \in I$ and $1 \leq n < n_0 + m_0 - 2$, the point $S^n(\lambda)$ is well defined and does not belong to the interval $\left(\frac{1}{q-2}, p-2 \right)$. Indeed, if $S^n(\lambda) \in \left(\frac{1}{q-2}, p-2 \right)$, then there exists $0 \leq i \leq m_0 - 1$ and $0 \leq j \leq n_0 - 1$ with $n = i + j$ such

that $\frac{q^j}{p^i}\lambda \in (\frac{1}{q-2}, p-2)$. On the other hand, $\frac{q^{m_0-1}}{p^{n_0-1}}\lambda \in (\frac{1}{q-2}, p-2)$. Thus,

$$\frac{1}{\gamma} \leq \frac{1}{p-2} < \frac{\frac{q^j}{p^i}\lambda}{\frac{q^{m_0-1}}{p^{n_0-1}}\lambda} < \frac{p-2}{\frac{1}{q-2}} < \gamma$$

Hence, $i = m_0 - 1$. and $j = n_0 - 1$. This says that $n = m_0 + n_0 - 2$.

For every $0 \leq n, m \leq n_0 + m_0 - 1$ with $n \neq m$, we have $T^n(J) \cap T^m(J) = \emptyset$. Otherwise, there exist $x, y \in J$ such that $T^n(x) = T^m(y)$. Thus, there exist $0 \leq i_1, i_2 \leq m_0$ and $0 \leq j_1, j_2 \leq n_0$ with $n = i_1 + j_1$ and $m = i_2 + j_2$ such that $\frac{p^{i_1}}{q^{j_1}}x = \frac{p^{i_2}}{q^{j_2}}y$. This says that $i_1 = i_2$ and $j_1 = j_2$, since $\frac{1}{\gamma} < \frac{x}{y} < \gamma$. Hence, $n = m$ that is a contradiction.

For every $0 \leq n, m \leq n_0 + m_0 - 2$ with $n \neq m$, we have $S^n(I) \cap S^m(I) = \emptyset$. Otherwise, there exist $x, y \in I$ such that $S^n(x) = S^m(y) := z$. Thus, $x = T^n(z)$ and $y = T^m(z)$. Thus, there exist $0 \leq i_1, i_2 \leq m_0$ and $0 \leq j_1, j_2 \leq n_0$ with $n = i_1 + j_1$ and $m = i_2 + j_2$ such that $x = \frac{p^{i_1}}{q^{j_1}}z$ and $y = \frac{p^{i_2}}{q^{j_2}}z$. This says that $i_1 = i_2$ and $j_1 = j_2$, since $\frac{1}{\gamma} < \frac{x}{y} < \gamma$. Hence, $n = m$ that is a contradiction.

Also, for every $0 \leq n \leq n_0 + m_0 - 1$ and $0 \leq m \leq n_0 + m_0 - 2$, we have $T^n(J) \cap S^m(I) = \emptyset$. Indeed, if there exist $x \in J$ and $y \in I$ such that $T^n(x) = S^m(y)$. Since $T^{m_0+n_0}(x) = x$, then for every $k \in \mathbb{Z}$, the point $T^k(x)$ stays inside the interval $[\frac{p-2}{p}, \frac{q}{q-2}]$. This says that for every $k \in \mathbb{N}$, the point $S^k(y)$ stays inside the interval $[\frac{p-2}{p}, \frac{q}{q-2}]$. While $S^{m_0+n_0-2}(y) \in (\frac{1}{q-2}, p-2)$ and so $S^{m_0+n_0-1}(y)$ is not defined. This makes a contradiction.

Now we show that

$$[s_1, s_0] = (\cup_{n=0}^{n=n_0+m_0-1} T^n(J)) \cup (\cup_{n=0}^{n=n_0+m_0-2} S^n(I))$$

Since for each $0 \leq n \leq n_0 + m_0 - 1$, $T^{n-1}(J) \cap I = \emptyset$, the subsets $T^n(J)$ are intervals inside $[s_1, s_0]$. Hence, for each $-m_0 + 1 \leq i \leq n_0$, there exists $0 \leq n \leq n_0 + m_0 - 1$ such that $|T^n(J)| = \gamma^i |J|$. Also, for each $0 \leq n \leq n_0 + m_0 - 1$, there exists $-m_0 + 1 \leq i \leq n_0$ such that $|T^n(J)| = \gamma^i |J|$. This fact can be written for the intervals $S^n(I)$, where

$0 \leq n \leq n_0 + m_0 - 2$. Hence,

$$\begin{aligned}
\sum_{n=0}^{n_0+m_0-1} |T^n(J)| + \sum_{n=0}^{n_0+m_0-2} |S^n(I)| &= |J| \sum_{i=-m_0+1}^{n_0} \gamma^i + |I| \sum_{i=-m_0+1}^{n_0-1} \gamma^i \\
&= (|J| + |I|) \sum_{i=-m_0+1}^{n_0} \gamma^i - |I| \gamma^{n_0} \\
&= \frac{|J| + |I|}{\gamma^{m_0-1}} \cdot \frac{1 - \gamma^{n_0+m_0}}{1 - \gamma} - |I| \gamma^{n_0} \\
&= \frac{q}{q-2} - \frac{p-2}{p} \\
&= |[s_1, s_0]|,
\end{aligned}$$

since $|I| = \frac{q}{p}(p-2 - \frac{1}{q-2})$ and $|I| + |J| = \frac{q}{p}(p-2)(1 - \frac{1}{\gamma})$.

Therefore, the set $\cup_{n=0}^{n_0+m_0-1} T^n(J)$ is equal to $[s_1, s_0] - \cup_{n=0}^{\infty} T^{-n}(I)$.

Above discussion says that for $x \in J$, the set $\{T^n(x) \mid 0 \leq n \leq n_0 + m_0 - 1\}$ is exactly $\{\gamma^n \mid -m_0 + 1 \leq n \leq n_0\}$ and so $T^n(J) = T^n J$. Now the assertion is obtained since the invariant set

$$\bigcup_{n=0}^{n_0+m_0-1} T^n(J) = \bigcup_{n=-m_0+1}^{n_0} \gamma^n \cdot \left[\frac{q(p-2)}{\gamma p}, \frac{q}{p(q-2)} \right] = \bigcup_{n=-m_0+1}^{n_0} \gamma^n \cdot \left[\frac{1-2\alpha}{\gamma\beta}, \frac{\alpha}{1-2\beta} \right]$$

is minimal for the map T .

proof I. 2 . we show that $\Lambda = \emptyset$. Suppose that $s \in \Lambda$, then $s \in [s_1, s_0]$ and $T^n(s) \notin I$, for each n . Let $orb(s) := \{T^n(s) \mid n \in \mathbb{N}\}$. This set have exactly $m_0 + n_0 - 1$ element and for each two disjoint points $s', s'' \in orb(s)$, we have $\frac{s'}{s''} \notin (\frac{1}{\gamma}, \gamma)$. But, $\tau(C_\alpha) \cdot \tau(C_\beta) < \frac{1}{\gamma}$ give $\frac{1}{\gamma^{m_0+n_0+1}} \frac{q}{q-2} < \frac{p-2}{p}$. Thus, $orb(s) \cap I \neq \emptyset$, since $\frac{1}{\gamma p} s_0 < q s_1$. By using Claim 1, the assertion obtains.

II. Consider $p \geq q$, and $p < q$ can be treated similar. Here also we show that $\Lambda = \emptyset$. Suppose that $s \in \Lambda$, then $s \in [s_1, s_0]$ and $T^n(s) \notin I$, for each n . This and the condition $p \geq q$ say that for each $i \in \mathbb{N}$, there exists the greatest natural number j_i and smallest natural number k_i such that $T^{i+j}(s) = \frac{p^i}{q^j} s \in [s_1, s_0] - I$, for each $k_i \leq j \leq j_i$. As $\frac{\log p}{\log q}$ is irrational, then $\{m \log p - n \log q \mid m, n \in \mathbb{N}\}$ is a dense subset of \mathbb{R} . Thus, $\{\frac{p^m}{q^n} \mid m, n \in \mathbb{N}\}$ is a dense subset of \mathbb{R}^+ . Fix natural numbers i and j such that $\frac{p^i}{q^j} s \in I$. Thus $k_i \leq j \leq j_i$ and this is a contradiction.

Here the proof of lemma is completed. \square

All of our results about arithmetic difference of affine Cantor sets can be interpreted for their sum too. Indeed we can write

$$C_\alpha + \lambda C_\beta = C_\alpha + \lambda(1 - C_\beta) = C_\alpha - \lambda C_\beta + \lambda.$$

For instance, when $\lambda = 1$ above theorem can be written as below.

Corollary 1. *Suppose that (C_α, C_β) are a pair of middle Cantor sets that satisfy $\tau(C_\alpha) \cdot \tau(C_\beta) < 1$.*

i) *if $\frac{\log \alpha}{\log \beta}$ be irrational, then $C_\alpha + C_\beta \neq [0, 2]$,*

ii) *if $\frac{\log \alpha}{\log \beta} = \frac{n_0}{m_0}$, then*

$$C_\alpha + C_\beta = [0, 2] \Leftrightarrow \lceil n_0 \log_\alpha \frac{\alpha}{1-2\beta} \rceil = \lfloor n_0 \log_\alpha \frac{1-2\alpha}{\beta} \rfloor + 1$$

Proof. The assertion (i) obtains from the part (II) of Theorem 2. For (ii), firstly we observe that the relation

$$1 \in \bigcup_{n=-m_0+1}^{n=n_0} \gamma^n \cdot \left[\frac{q(p-2)}{\gamma p}, \frac{q}{p(q-2)} \right]$$

means that there exists $-m_0 + 1 \leq n \leq n_0$ such that

$$\gamma^{-n} \in \left[\frac{q(p-2)}{\gamma p}, \frac{q}{p(q-2)} \right].$$

By putting $\alpha = \frac{1}{p}$ and $\beta = \frac{1}{q}$

$$\left[\log_\gamma \frac{1-2\beta}{\alpha}, 1 - \log_\gamma \frac{1-2\alpha}{\beta} \right] \cap \{-m_0 + 1, \dots, -1, 0, 1, \dots, n_0\} \neq \emptyset$$

As $\tau(C_\alpha) \cdot \tau(C_\beta) < 1$, this is equivalent

$$\lceil \log_\gamma \frac{1-2\beta}{\alpha} \rceil = \lfloor 1 - \log_\gamma \frac{1-2\alpha}{\beta} \rfloor$$

Now if $C_\alpha + C_\beta = [0, 2]$, then by using part (I.2) of Theorem 2 we have $\frac{1}{\gamma} \leq \tau(C_\alpha) \cdot \tau(C_\beta)$. The part (I.1) of the theorem and plugging $\alpha = \gamma^{-n_0}$ yields the result.

Conversely, if the relation

$$\lceil n_0 \log_\alpha \frac{\alpha}{1-2\beta} \rceil = \lfloor n_0 \log_\alpha \frac{1-2\alpha}{\beta} \rfloor + 1$$

holds, then

$$n_0 \log_\alpha \frac{\alpha}{1-2\beta} \leq n_0 \log_\alpha \frac{1-2\alpha}{\beta} + 1$$

So $\frac{1}{\gamma} \leq \tau(C_\alpha) \cdot \tau(C_\beta)$. By getting $\alpha = \gamma^{-n_0}$ the assertion yields by using part (I.1) of Theorem 2. \square

Fix $m, n \in \mathbb{N}$. Let α_1 that satisfies $\alpha_1^{\frac{m}{n}} = 1 - 2\alpha_1$ that means the point $(\alpha_1, \alpha_1^{\frac{m}{n}})$ is on the line $\beta = 1 - 2\alpha$. Also, let α_2 satisfies

$$\frac{\alpha_2}{1 - 2\alpha_2^{\frac{m}{n}}} \cdot \frac{\alpha_2^{\frac{m}{n}}}{1 - 2\alpha_2} = 1$$

and this means the point $(\alpha_2, \alpha_2^{\frac{m}{n}})$ is on the curve $\frac{\alpha}{1 - 2\beta} \cdot \frac{\beta}{1 - 2\alpha} = \tau(C_\alpha) \cdot \tau(C_\beta) = 1$.

Always suppose that $\beta < \alpha$ with $\beta = \alpha^{\frac{m}{n}}$.

Corollary 2. *Under the above notations, there is a unique $\alpha_0 \in [\alpha_1, \alpha_2]$ such that $C_\alpha + C_{\alpha^{\frac{m}{n}}} = [0, 2]$, if and only if $\alpha \in [\alpha_0, \alpha_2]$. Moreover,*

- i) if $\frac{m}{n}$ be irrational, then $C_\alpha + C_{\alpha^{\frac{m}{n}}} \neq [0, 2]$,
- ii) $n = 1$ or $m = n + 1$, if and only if $\alpha_0 = \alpha_1$.

Proof. On the interval $(0, \alpha_2]$, define the functions

$$f_1(\alpha) = n \log_\alpha \frac{\alpha}{1 - 2\alpha^{\frac{m}{n}}}, \quad f_2(\alpha) = n \log_\alpha \frac{1 - 2\alpha}{\alpha^{\frac{m}{n}}} + 1.$$

The function f_1 is increasing and f_2 is non increasing. Also $f_1(\alpha_2) + 1 = f_2(\alpha_2)$. Let

$$\alpha_0 := \inf \{ \alpha \mid \lceil f_1(\alpha) \rceil = \lfloor f_2(\alpha) \rfloor, \quad \alpha \in (0, \alpha_2] \}$$

The mentioned properties of these functions in above expresses $\lceil f_1(\alpha_0) \rceil = \lfloor f_2(\alpha_0) \rfloor$. Put $k = \lceil n \log_{\alpha_0} \frac{\alpha_0}{1 - 2\beta_0} \rceil$ with $\beta_0 = \alpha_0^{\frac{m}{n}}$. As $\tau(C_{\alpha_0}) \cdot \tau(C_{\beta_0}) < 1$ and $\beta_0 < \alpha_0$, then $\alpha_0 < 1 - 2\beta_0$. This says $k \in \mathbb{N}$, since $\alpha_0 < 1$. So $\lfloor n \log_{\alpha_0} \frac{1 - 2\alpha_0}{\alpha_0^{\frac{m}{n}}} + 1 \rfloor \in \mathbb{N}$. This says $1 - 2\alpha_0 \leq \beta_0$ and so $\alpha_1 \leq \alpha_0$. Now the first assertion obtains by using the part (ii) of Corollary 1.

The assertion (i) obtains by using the part (i) of Corollary 1.

We prove (ii) via claim below.

- $\alpha_0 = \alpha_1$ if and only if $\frac{1}{\sqrt[n]{\alpha_1}} + \frac{1}{\alpha_1} \geq 4$,

if $\alpha_0 = \alpha_1$, then $\beta_0 = 1 - 2\alpha_0$. so $k = 1$. This say

$$0 < n \log_{\alpha_0} \frac{\alpha_0}{1 - 2\beta_0} \leq 1$$

that is equivalent

$$\frac{\frac{n-1}{\alpha_0^n}}{\frac{m}{1-2\alpha_0^n}} \geq 1$$

Thus,

$$\frac{n-1}{\alpha_0^n} + 2\alpha_0^n \geq \frac{m}{1-2\alpha_0^n}$$

Put $\frac{m}{\alpha_0^n} = 1 - 2\alpha_0$ so

$$\frac{-1}{\alpha_0^n} \geq \frac{4\alpha_0 - 1}{\alpha_0}$$

By putting $\alpha_0 = \alpha_1$ the assertion obtains.

Conversely,

$$\frac{-1}{\alpha_1^n} \geq \frac{4\alpha_1 - 1}{\alpha_1}$$

As above

$$0 < n \log_{\alpha_1} \frac{\alpha_1}{1-2\alpha_1^n} \leq 1$$

Thus, $\lceil n \log_{\alpha_1} \frac{\alpha_1}{1-2\alpha_1^n} \rceil = 1$. Also, $\lfloor n \log_{\alpha_1} \frac{1-2\alpha_1}{\alpha_1^n} + 1 \rfloor = 1$. Consequently, $\lceil f_1(\alpha_1) \rceil = \lfloor f_2(\alpha_1) \rfloor$. Thus, $\alpha_1 = \alpha_0$ since, $\alpha_1 \leq \alpha_0$.

if $n = 1$, the relation holds, since $\alpha_1 < \frac{1}{2}$.

if $m = n + 1$, then $\frac{1}{\sqrt[n]{\alpha_1}} = \frac{a_1}{1-2\alpha_1}$, since $\alpha_1^{\frac{n+1}{n}} = 1 - 2\alpha_1$. Hence,

$$\frac{1}{\sqrt[n]{\alpha_1}} + \frac{1}{\alpha_1} = \frac{a_1}{1-2\alpha_1} + \frac{1}{\alpha_1} \geq 4$$

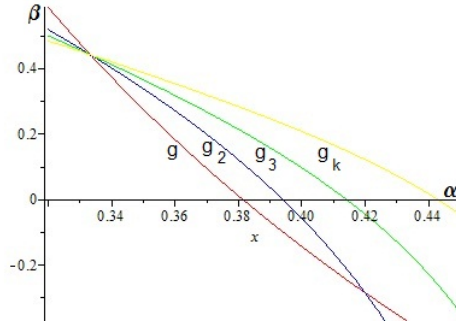
since $\alpha_1 \geq \frac{1}{3}$.

if $m = n + k$ with $(n, n+k) = 1$ and $k \geq 2$, then $\frac{1}{\sqrt[n]{\alpha_1}} = \sqrt[k]{\frac{a_1}{1-2\alpha_1}}$, since

$\alpha_1^{\frac{n+k}{n}} = 1 - 2\alpha_1$. On the interval $[\frac{1}{3}, \frac{1}{2})$ take functions

$$g(\alpha) = \frac{1}{\sqrt[n]{\alpha}} + \frac{1}{\alpha} - 4, \quad g_k(\alpha) = \frac{1}{\sqrt[n]{\alpha}} - \sqrt[k]{\frac{a}{1-2\alpha}}$$

The functions g and g_k are increasing and $g_k(\frac{1}{3}) = g(\frac{1}{3}) \geq 0$. Moreover, $g_2 \leq g_k$. Also, $g(\alpha) \leq g_2(\alpha)$ on $[\frac{1}{3}, \frac{7+\sqrt{5}}{22}]$ and $g_2(\frac{7+\sqrt{5}}{22}) = g(\frac{7+\sqrt{5}}{22})$.



On the other hand,

$$g\left(\frac{7 + \sqrt{5}}{22}\right) \leq \sqrt{\frac{22}{7 + \sqrt{5}} + \frac{22}{7 + \sqrt{5}}} - 4 \cong -0.0746 < 0$$

These facts tells that $g(\alpha_1) < 0$. Thus, $\alpha_0 \neq \alpha_1$. □

Fix $m, n \in \mathbb{N}$. Define interval

$$I_{m,n} = \left\{ \gamma \mid \lceil \log_{\gamma}(\gamma^m - 2) \rceil = \lfloor -\log_{\gamma}(\gamma^n - 2) \rfloor + 1 \right\}.$$

With putting $\alpha = \gamma^{-n}$ and $\beta = \gamma^{-m}$ in corollary 1 we obtain

$$C_{\gamma^{-n}} + C_{\gamma^{-m}} = [0, 2] \iff \gamma \in I_{m,n}.$$

Let

$$\mathcal{R} = \left\{ (\alpha, \beta) \mid HD(C_{\alpha}) + HD(C_{\beta}) > 1, \tau(C_{\alpha}) \cdot \tau(C_{\beta}) < 1 \right\}.$$

Thus, the curves

$$\begin{aligned} r_{m,n} : I_{m,n} &\mapsto \mathcal{R} \\ \gamma &\rightarrow (\gamma^{-n}, \gamma^{-m}) \end{aligned}$$

will determine all the pairs of middle Cantor sets whose arithmetic sum is $[0, 2]$, see Figure 3. Also they stay in the space $\left\{ (\alpha, \beta) \mid \beta \geq 1 - 2\alpha, \tau(C_{\alpha}) \cdot \tau(C_{\beta}) < 1 \right\}$. The curve $r_{m,n}$ separate this region, if and only if $n = 1$ or $m = n + 1$. This is a generalization of Theorem D of [8].

3 Iterated function system

We first concentrate on a dense subset \mathcal{L} of the space of the pair of middle Cantor sets. Then apply Lemma 1 to translate the arithmetic difference set of each element \mathcal{L} as the attractor of an iterated function system with equal contraction ratios.

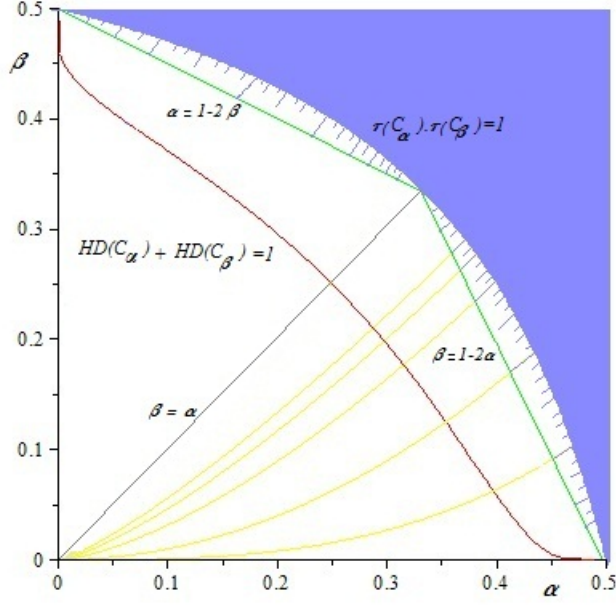


Figure 3: Blue region and lines determine all α, β that $C_\alpha + C_\beta = [0, 2]$.

An element (C_α, C_β) belongs to \mathcal{L} if and only if $\frac{\log \alpha}{\log \beta} \in \mathbb{Q}$. Obviously, \mathcal{L} is a dense subset in the space $\mathcal{C} \times \mathcal{C}$. Let $p, q > 2$ and pair $(C_\alpha, C_\beta) := (C_{\frac{1}{p}}, C_{\frac{1}{q}})$ be middle Cantor sets

$$C_\alpha : \begin{array}{c} \frac{1}{p} \\ \text{---} \\ \frac{1}{p} \end{array} \quad C_\beta : \begin{array}{c} \frac{1}{q} \\ \text{---} \\ \frac{1}{q} \end{array}$$

with the convex hull $[0, 1]$ and expanding maps ϕ and ϕ' as

$$\phi(x) := \begin{cases} px & x \in [0, \frac{1}{p}] \\ px - p + 1 & x \in [1 - \frac{1}{p}, 1] \end{cases} \quad \phi'(x) := \begin{cases} qx & x \in [0, \frac{1}{q}] \\ qx - q + 1 & x \in [1 - \frac{1}{q}, 1] \end{cases}.$$

Thus, transferred renormalization operators corresponding to pair $(C_\alpha, C_\beta) := (C_{\frac{1}{p}}, C_{\frac{1}{q}})$ given by

$$\begin{aligned} (s, t) &\xrightarrow{T_0} (ps, pt) & (s, t) &\xrightarrow{T_1} (ps, pt - p + 1) \\ (s, t) &\xrightarrow{T'_0} (\frac{s}{q}, t) & (s, t) &\xrightarrow{T'_1} (\frac{s}{q}, t + \frac{q-1}{q}s) \end{aligned} \quad (3.1)$$

If $\frac{\log p}{\log q} =: \frac{n_0}{m_0} \in \mathbb{Q}$ with $(m_0, n_0) = 1$, then every vertical line $s = \lambda =: \cot \theta$ passes over itself with suitable compositions of the operators (2). Hence, we can transfer the these operators on vertical lines by

Theorem 3. Let $\lambda \in \mathbb{R}^*$ and $\{a_k\}_{k=0}^{m_0-1}$, $\{b_k\}_{k=0}^{n_0-1}$ be two finite sequences of numbers 0 and 1. Then the maps

$$T_\lambda(t) := p^{m_0}t + a_\lambda, \quad a_\lambda := -(p-1)p^{m_0-1} \left(\sum_{k=0}^{m_0-1} \frac{a_k}{p^k} - \frac{p(q-1)}{q(p-1)} \lambda \sum_{k=0}^{n_0-1} \frac{b_k}{q^k} \right) \quad (3.2)$$

are return maps to the vertical line $s = \lambda$ and the attractor of iterated function system $\{T_\lambda^{-1}\}$ is $C_\alpha - \lambda C_\beta$.

Proof. Suppose that $\{b_k\}_{k=0}^\infty$ and $\{a_k\}_{k=0}^\infty$ are two arbitrary sequences of numbers 0 and 1. For every a_k and b_k , we can write the operators (2) in form

$$T_{a_k}(s, t) := (ps, pt - (p-1)a_k) \quad T'_{b_k}(s, t) := \left(\frac{s}{q}, t + \left(\frac{q-1}{q} \right) b_k s \right).$$

Let $m, n \in \mathbb{N}$, then we claim that

- (i) $T_{a_{m-1}} \circ \dots \circ T_{a_0}(s, t) = (p^m s, p^m t - (p-1) \sum_{k=0}^{m-1} a_k p^{m-1-k})$,
- (ii) $T'_{b_{n-1}} \circ \dots \circ T'_{b_0}(s, t) = \left(\frac{s}{q^n}, t + \frac{s}{q^n} (q-1) \sum_{k=0}^{n-1} b_k q^{n-1-k} \right)$.

To prove the claim, we use induction. The case $m = n = 1$ is true. Assume that formulas are valid for the cases $m = i$ and $n = j$, then we have

$$\begin{aligned} T_{a_i} \circ T_{a_{i-1}} \circ \dots \circ T_{a_0}(s, t) &= \left(p^{i+1} s, p^{i+1} t - (p-1) \sum_{k=0}^{i-1} a_k p^{i-k} - (p-1) a_i \right) \\ &= \left(p^{i+1} s, p^{i+1} t - (p-1) \sum_{k=0}^i a_k p^{i-k} \right), \\ T'_{b_j} \circ T'_{b_{j-1}} \circ \dots \circ T'_{b_0}(s, t) &= \left(\frac{s}{q^{j+1}}, t + \frac{s}{q^j} (q-1) \sum_{k=0}^{j-1} b_k q^{j-1-k} + b_j \frac{(q-1)}{q} \cdot \frac{s}{q^j} \right) \\ &= \left(\frac{s}{q^{j+1}}, t + \frac{s}{q^{j+1}} (q-1) \sum_{k=0}^j b_k q^{j-k} \right) \end{aligned}$$

and we see the validity of the relations (i) and (ii) for the cases $m = i + 1$ and $n = j + 1$. Put $s = \lambda$, $m = m_0$ and $n = n_0$ in the relations (i) and (ii). Then we obtain the maps (3), since

$$\begin{aligned} T'_{b_{n_0-1}} \circ \dots \circ T'_{b_0} \circ T_{a_{m_0-1}} \circ \dots \circ T_{a_0}(\lambda, t) &= \left(\frac{p^{m_0}}{q^{n_0}} \lambda, p^{m_0} t - (p-1) \sum_{k=0}^{m_0-1} a_k p^{m_0-1-k} \right) \\ &\quad + \frac{p^{m_0} \lambda}{q^{n_0}} (q-1) \sum_{k=0}^{n_0-1} b_k q^{n_0-1-k} \\ &= \left(\lambda, p^{m_0} t - (p-1) p^{m_0-1} \left(\sum_{k=0}^{m_0-1} \frac{a_k}{p^k} - \frac{p(q-1)}{q(p-1)} \lambda \sum_{k=0}^{n_0-1} \frac{b_k}{q^k} \right) \right). \end{aligned}$$

Note that, $(m_0, n_0) = 1$ and the operators T_{a_i} and T'_{b_j} commute together. Hence, the operators $T'_{b_{n_0-1}} \circ \dots \circ T'_{b_0} \circ T_{a_{m_0-1}} \circ \dots \circ T_{a_0}$ are all maps that for the first time transfer the points of the line $s = \lambda$ to itself. Moreover, every relatively compact sequence (s_{n+1}, t_{n+1}) that start with (s, t) and is obtained from (s_n, t_n) by applying one of these operators can be characterized relatively compact sequence (s_{n+1}, t_{n+1}) that start with (s, t) and is obtained from (s_n, t_n) by applying one of operators (2). Also, every relatively compact sequence (s_{n+1}, t_{n+1}) that start with (s, t) and is obtained from (s_n, t_n) by applying one of operators (2) carry a relatively compact subsequence $(s_{k_{n+1}}, t_{k_{n+1}})$ that start with (s, t) and is obtained from (s_{k_n}, t_{k_n}) by applying one of these operators. Basic facts in theory of iterated function systems and regarding to lemma 1 establish the maps (3) an iterated function system with contractions $T_\lambda^{-1}(t) = p^{-m_0}t + b_\lambda := p^{-m_0}t - \frac{a_\lambda}{p^{m_0}}$ whose attractor is $C_\alpha - \lambda C_\beta$. This completes the proof. \square

For a given $\lambda \in \mathbb{R}^*$, rename the maps (3) to T_λ^i with $1 \leq i \leq 2^{m_0+n_0}$, (sometimes they are less than this number) and let $S_\lambda^i := (T_\lambda^i)^{-1}$. We call $\mathcal{S}_\lambda := \{S_\lambda^i\}_{i=1}^{2^{m_0+n_0}}$ the iterated function systems corresponding to the pair $(C_\alpha, C_\beta) \in \mathcal{L}$, or the attractors $C_\alpha - \lambda C_\beta$. Indeed, the set $C_\alpha - \lambda C_\beta$ forms a uniformly contracting self-similar set that obeys from the formula

$$C_\alpha - \lambda C_\beta = \bigcap_{i \in \mathbb{N}} \mathcal{S}_\lambda^i([- \lambda, 1]).$$

Let $\Pi_\theta := \text{Proj}_\theta$ be the map that projects the points of \mathbb{R}^2 onto $\mathbb{R} \times \{0\}$ in the direction of lines with angle θ . It is easy to see $\Pi_\theta(x, y) = x - sy$, for each $(x, y) \in \mathbb{R}^2$. If we do m_0 steps in the construction of C_α and n_0 steps in the construction of C_β , then the squares that obtain from their Cartesian product are called the first step of the construction of $C_\alpha \times C_\beta$. The number of the squares are $2^{m_0+n_0}$ and each of them has length p^{-m_0} . Let C be one of these squares. Therefore the affine map that sends the interval $[- \lambda, 1]$ to the interval $\Pi_\theta(C)$ is one of the maps (3). To do this, let $\{c_i\}_{i=1}^{m_0}$ be ordered set of the initial points of intervals appeared in the m_0 -th step of the construction of C_α and let $\{d_i\}_{i=1}^{n_0}$ be ordered set of the initial points of intervals appeared in the n_0 -th step of the construction of C_β . Let $C_{ij} := [c_i, c_i + \frac{1}{p^{m_0}}] \times [d_j, d_j + \frac{1}{q^{n_0}}]$ be the same square that its lower left vertice the point (c_i, d_j) with $c_i := (1 - \frac{1}{p}) \sum_{k=0}^{m_0-1} \frac{a_{ik}}{p^k}$ and $d_j := (1 - \frac{1}{q}) \sum_{k=0}^{n_0-1} \frac{b_{jk}}{q^k}$. Hence, for $s > 0$ we obtain

$$\begin{aligned} \Pi_\theta(C_{ij}) &= [\Pi_\theta(c_i, d_j + \frac{1}{p^{m_0}}), \Pi_\theta(c_i + \frac{1}{p^{m_0}}, d_j)] \\ &= [c_i - sd_j - \frac{s}{p^{m_0}}, c_i + \frac{1}{p^{m_0}} - sd_j]. \end{aligned}$$

On the other hand, the affine map T_{ij} that sends the interval $\Pi_\theta(C_{ij})$ to the interval $[-s, 1]$ is $T_{ij}(t) = p^{m_0}t + p^{m_0}(sd_j - c_i)$. After putting c_i and d_j , we see that T_{ij} forms a

map of the maps (3) that we corresponds it to the square C_{ij} . Basically, the calculation of the maps (3) is easier in this way. Here, we present two corollaries that tell us how effective can be the iterated function systems corresponding to elements \mathcal{L} .

Corollary 3. *Under the above notations above,*

- (i) *If $\lambda = \frac{q(p-1)}{p(q-1)}$ and $(m_0 + n_0) \log 2 + \log \frac{3}{4} < m_0 \log p$, then $|C_\alpha - \lambda C_\beta| = 0$,*
- (ii) *If $\lambda_1 = \frac{p^i}{q^j} \lambda_2$, then $HD(C_\alpha - \lambda_1 C_\beta) = HD(C_\alpha - \lambda_2 C_\beta)$. Moreover, $|C_\alpha - \lambda_1 C_\beta| = 0$ if and only if $|C_\alpha - \lambda_2 C_\beta| = 0$.*

Proof. (i). In this case, $\tan \theta = \frac{p(q-1)}{q(p-1)}$ and it is easy to check that $\Pi_\theta((1, 1 - \frac{1}{q})) = (\frac{1}{p}, 0)$. Thus, the number of intervals emerged from the projection of squares in the first step of the construction of $C_\alpha \times C_\beta$ is at most $\frac{3}{4} 2^{m_0+n_0}$. As above, the iterated function system corresponding to $\lambda = \frac{q(p-1)}{p(q-1)}$ consists of at most $\frac{3}{4} 2^{m_0+n_0}$ maps, that ensures its similarity dimension is smaller than $\log_{p^{m_0}} \frac{3}{4} 2^{m_0+n_0}$. Therefore, $HD(C_\alpha - \lambda C_\beta) < 1$. This completes the proof of (i).

(ii). The assertion is obtained since operators (2) are affine and the points on the vertical lines $s = \lambda_1$ pass over the vertical lines $s = \lambda_2$ with suitable compositions of them. \square

A Pisot number is a positive algebraic integer greater than 1 such that all its Galois conjugates are less than 1 in absolute value. Denote the set of all Pisot numbers by S . The set S forms a closed subset and has infinitely many limit points. The smallest element of S is Plastic number $\rho = 1.3247 \dots$ which is the unique real solution of the cubic equation $x^3 - x - 1 = 0$. Plastic number is isolated in S and the golden number $\gamma = \frac{\sqrt{5}+1}{2}$ satisfying $x^2 = x + 1$ is also the smallest limit point of S . The topological properties of the set S may be useful to characterize the structure of subset $\mathcal{P} := \{ (C_{\omega^{-n}}, C_{\omega^{-m}}) \in \mathcal{C} \times \mathcal{C} \mid \omega \in S, m, n \in \mathbb{N} \}$ of the pair of middle Cantor sets.

Corollary 4. *For each element $(C_\alpha, C_\beta) \in \mathcal{P}$, there exists a dense subfield $F \subset \mathbb{R}$ such that for each $\mu \in F$, the set $C_\alpha - \mu C_\beta$ contains an interval or has zero Lebesgue measure.*

Proof. Fix ω, m and n such that $\alpha = \omega^{-n}$ and $\beta = \omega^{-m}$ and consider dense subfield $F := \mathbb{Q}[\omega]$ of the real numbers. It is not hard to see that for each $\mu \in F$, there exists $k \in \mathbb{N}$ such that all a_μ appeared in Theorem 3 satisfying $b_\mu = -\frac{a_\mu}{\omega^{nm}} \in \frac{1}{k} \mathbb{Z}[\omega]$. When $HD(C_\alpha - \mu C_\beta) = 1$, then by using Theorems 2.9 and 1.3 of [9], we see that $(C_\alpha - \mu C_\beta)^\circ \neq \emptyset$. Thus, the set $C_\alpha - \mu C_\beta$ contains an interval which proves the corollary. \square

Suppose that affine Cantor sets K and K' are homogeneous, that means there exist real numbers p and q such that $\psi|_{I_i} = px + e_i$ and $\psi'|_{I'_j} = qx + f_j$. Also suppose that they satisfy $\frac{\log p}{\log q} = \frac{n_0}{m_0}$ as above. Let $\{a_k\}_{k=0}^{m_0-1}$ be a finite sequence of numbers $\frac{e_i}{p-1}$, and

$\{b_k\}_{k=0}^{n_0-1}$ be a finite sequence of numbers $\frac{f_i}{q-1}$. Then we can write Theorem 3 for K and K' instead of C_α and C_β . Thus,

Corollary 5. *There exists a dense subset \mathcal{D} of all the homogeneous pairs of Cantor sets such that for each $(K, K') \in \mathcal{D}$, the set $K - K'$ forms an attractor of the iterated function system.*

However, it seems that that we can not extend this assertion for affine Cantor sets. Even for affine Cantor sets K and K' with the simplest combinatorics respectively defined by maps

$$\phi(x) := \begin{cases} p_0x & x \in [0, \frac{1}{p_0}] \\ p_1x - p_1 + 1 & x \in [1 - \frac{1}{p_1}, 1] \end{cases} \quad \phi'(x) := \begin{cases} q_0x & x \in [0, \frac{1}{q_0}] \\ q_1x - q_1 + 1 & x \in [1 - \frac{1}{q_1}, 1] \end{cases},$$

we not only need $\frac{\log p_0}{\log q_0}, \frac{\log p_0}{\log q_1}, \frac{\log p_1}{\log q_0}, \frac{\log p_1}{\log q_1} \in \mathbb{Q}$, but also $\frac{\log p_0 - \log p_1}{\log q_0}, \frac{\log p_0 - \log p_1}{\log q_1} \in \mathbb{Z}$. For example, if $\frac{\log p_0 - \log p_1}{\log q_0} \notin \mathbb{Z}$, we can construct the sequence $\{T_n\}$ inductively as below. Suppose that we have chose T_n , then let $T_{n+1} = T_0$, if $p_0 \cdot \prod_1^n DT_i \notin \{q_0^n \mid n \in \mathbb{Z}\}$ and otherwise let $T_{n+1} = T_1$. Note that if $p_0 \cdot \prod_1^n DT_i$ and $p_1 \cdot \prod_1^n DT_i$ belongs to $\{q_0^n \mid n \in \mathbb{Z}\}$ then $\frac{\log p_0 - \log p_1}{\log q_0} \in \mathbb{Z}$. Consider the sequences $\{k_n\}$ of natural numbers, such that $\frac{\prod_1^n DT_i}{q_0^{k_n}}$ is bounded and away from zero. Hence

$$\bigcap (T_n \circ T_{n-1} \circ \dots \circ T_1 \circ T_0^{k_n})^{-1} \left(\left\{ \frac{\prod_1^n DT_i}{q_0^{k_n}} \right\} \times \left[-\frac{\prod_1^n DT_i}{q_0^{k_n}}, 1 \right] \right)$$

contains a single point namely $(1, t_0)$ and consequently $t_0 \in K - K'$ while it seems that generically this point does not passes over $s = 1$ with any other compositions of the operators T_0 and T_1 too.

4 Weak stable intersection

In this section, we apply Theorem 3 to provide conditions that guarantee the existence of an interval in $f(K) - g(K')$ and then we bring some results in this direction. First we state a definition.

Definition 2. *The elements of the iterated function systems $\mathcal{S}_\lambda = \{S_\lambda^i\}$ corresponding to the pair (C_α, C_β) are regularly linked on $(m_1, m_2) \subset \mathbb{R}^*$ if*

- (i) *the set $\bigcup S_\lambda^i((-\lambda, 1))$ is connected, for each $\lambda \in (m_1, m_2)$,*
- (ii) *for each $1 \leq i, j \leq 2^{m_0+n_0}$, we have $S_\lambda^i((-\lambda, 1)) \cap S_\lambda^j((-\lambda, 1)) = \emptyset$. Otherwise there exists $d \in \mathbb{R}$ such that $d \in S_\lambda^i((-\lambda, 1)) \cap S_\lambda^j((-\lambda, 1))$, for every $\lambda \in (m_1, m_2)$.*

Geometrically, the part (i) expresses that on the first step of the structure of $C_\alpha \times C_\beta$, the projection of all squares overlap each other under the angle $\cot^{-1} \lambda$ for each $\lambda \in (m_1, m_2)$. Also, the part (ii) expresses that on the first step of the structure of $C_\alpha \times C_\beta$, if the projection of two subsquares intersect together under the angle $\cot^{-1} \lambda$ for any $\lambda \in (m_1, m_2)$, then there exists a point namely d such that for each $\lambda \in (m_1, m_2)$ the point d belongs to the projection of these two subsquares under the angle $\cot^{-1} \lambda$.

Proposition 2. *Suppose that the elements of the iterated function systems \mathcal{S}_λ corresponding to the pair $(C_\alpha, C_\beta) \in \mathcal{L}$ are regularly linked on (m_1, m_2) . Furthermore, suppose that $f, g : [0, 1] \rightarrow \mathbb{R}$ are defined in a way that for one point $(x_0, y_0) \in C_\alpha \times C_\beta$, functions f in a neighborhood x_0 and g in a neighborhood y_0 are of the class C^1 and $m_1 < \frac{g'(y_0)}{f'(x_0)} < m_2$. Then $f(C_\alpha) - g(C_\beta)$ contains an interval.*

Proof. As just mentioned in Section 2, the set $C_\alpha - \lambda C_\beta$ consists of all points that appear from the projection of $C_\alpha \times C_\beta$ under the angle $\cot^{-1} \lambda$. This fact together with the part (i) of Definition 2 give $C_\alpha - \lambda C_\beta = [-\lambda, 1]$, for each $\lambda \in (m_1, m_2)$. The second condition presented in the assumption of the proposition guarantees the existence of a square named C situated in (x_0, y_0) , in further steps of the construction $C_\alpha \times C_\beta$, such that for the family of curves $f(x) - g(y) = c$ which stay in C satisfying $m_2^{-1} < y' = \frac{f'(x)}{g'(y)} < m_1^{-1}$. This and the part (ii) of Definition 2 implies that if we take the projection map in the direction of the curves $f(x) - g(y) = c$ onto the line $y = y_0$, then the projection of all the subsquares of further construction situated in the square C overlap each other. We draw this situation in Figure 1 for $m_1 > 0$. Note that, a curve from the family

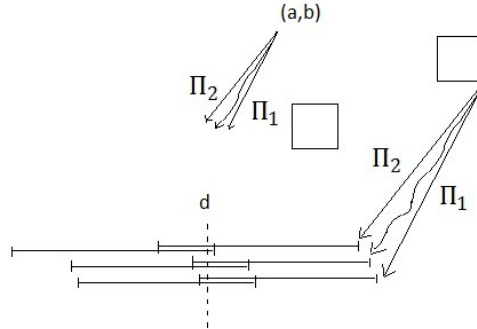


Figure 1: Π_1 and Π_2 project two sub squares of C under the angles $\cot^{-1} m_1$ and $\cot^{-1} m_2$, respectively. We observe that how the projection of these squares in the direction of curves $f(x) - g(y) = c$ are determined on the line $y = y_0$.

$f(x) - g(y) = c$ which passes through the point (a, b) is forced to lie among the lines passing through the point (a, b) with slopes m_1^{-1} and m_2^{-1} . This process happens for all subsquares C . Putting these facts together, we conclude the existence of an interval in the set $f(C_\alpha) - g(C_\beta)$. \square

Example 1. Suppose that C is a middle- $\frac{1}{3}$ Cantor set. Then $C^2 + C^2$ and $\sin C + \cos C$ contain an interval. The projection of all the squares in the first step of the construction $C \times C$ cover each other, when $-1 < \lambda = \cot \theta < \frac{-1}{3}$. We can select numbers m_1 and m_2 such that $\frac{-1}{2} \in (m_1, m_2)$ and the elements of the iterated function systems corresponding to the pair (C, C) are regularly linked on (m_1, m_2) . let $f(x) = -g(x) = x^2$, hence the family of curves $x^2 + y^2 = c$ satisfying the differential equation $y' = \frac{x}{-y}$ and for the point $(\frac{2}{3}, \frac{1}{3}) \in C \times C$ we have $m_1 < \frac{-\frac{1}{3}}{3} < m_2$. Considering Proposition 2, the set $C^2 + C^2$ contains an interval. For the second one, when $\frac{1}{3} < \cot \theta < 1$, the projection of all the squares in the first step of the construction $C \times C$ cover each other. Moreover, the family of the curves $\sin x + \cos y = c$ satisfies $y' = \frac{\cos x}{\sin y}$ and for the point $(\frac{1}{3}, \frac{1}{3}) \in C \times C$ we have $\frac{1}{3} < \frac{\sin \frac{1}{3}}{\cos \frac{1}{3}} < 1$. The assertion is obtained by using Proposition 2, where m_1 and m_2 are enough close to $\frac{\sin \frac{1}{3}}{\cos \frac{1}{3}}$ and $\frac{1}{3} < m_1 < \frac{\sin \frac{1}{3}}{\cos \frac{1}{3}} < m_2 < 1$.

Although the first condition in Proposition 2 seems to be weak, this defect disappears in the second condition as we have seen in Example 1. We will also see this below and in Example 2 at the end of the Section 5.

Definition 3. Suppose that K and K' are two Cantor sets of the real numbers. We say that the pair (K, K') has weak stable intersection in the sense of topology C^r with $r \geq 1$, if $f(K) \cap g(K') \neq \emptyset$, for all diffeomorphisms of f and g in the C^r -neighborhood of the identity.

Thus, in the case K and K' have the weak stable intersection, the set $f(K) - g(K')$ contains an interval, for f and g selected in a C^r -neighborhood of the identity. Now, we can mention some remarkable points as follow.

Remark 1. (i) If the pair (K, K') has stable intersection, then we can take neighborhood \mathcal{U} of the identity, such that for the $f, g \in \mathcal{U}$, the sets $f(K)$ and $g(K')$ are regular Cantor sets, and $(f(K), g(K'))$ has stable intersection. Hence, the pair (K, K') has weak stable intersection too. Thus, an appropriate way to show that the Cantor sets K and K' does not have stable intersection is to introduce a sequence of diffeomorphisms $\{h_n\}$ near $h(x) = x$, such that the Lebesgue measure of $K - h_n(K')$ is zero.

(ii) If $HD(K) + HD(K') < 1$ and K and K' are the regular Cantor sets, then $f(K) - g(K')$ does not contain any interval for each $f, g \in C^1$, since

$$\begin{aligned}
HD(f(K) - g(K')) &\leq HD(f(K) \times g(K')) = HD((f, g)(K \times K')) \\
&= HD(K \times K') \\
&= HD(K) + HD(K') \\
&< 1.
\end{aligned}$$

Thus, the pair (K, K') does not have weak stable intersection.

(iii) Note that if we take arbitrary Cantor sets K and K' with $\dim_H K = \overline{\dim}_B K$ instead of regular Cantor sets K and K' in (ii), then we obtain the same assertion. Before we state a result about the existence of weak stable intersection of Cantor sets C_α and λC_β , we state following open problem;

Open Problem 1. Are there any $(C_\alpha, C_\beta) \in \Omega$ that has weak stable intersection while does not have stable intersection? what about regular Cantor sets (K, K') ? The pair

(C, C) with $\frac{1}{3}$ -middle Cantor set does not have weak stable intersection too. In fact, natural variations of Sannami's example [14], which follows from the results of [1], shows that there are central Cantor sets K which are diffeomorphic to C by diffeomorphisms C^∞ very close to the identity such that $K - K$ has empty interior with positive Lebesgue measure.

Note that $C - C = [0, 1]$ and that $\bigcup S_1^i((-1, 1)) = (-1, -\frac{1}{3}) \cup (-\frac{1}{3}, \frac{1}{3}) \cup (\frac{1}{3}, 1)$. But for the iterated function systems $\mathcal{S}_\lambda = \{S_\lambda^i\}$ corresponding to the pair (C_α, C_β) that satisfies $\bigcup S_1^i((-1, 1)) = (-1, 1)$, we can select the interval (m^{-1}, m) such that for every $1 \leq i, j \leq 2^{m_0+n_0}$ there exists $d \in \mathbb{R}$ if $d \in S_1^i((-1, 1)) \cap S_1^j((-1, 1))$, then $d \in S_\lambda^i((-\lambda, 1)) \cap S_\lambda^j((-\lambda, 1))$, for every $\lambda \in (m^{-1}, m)$. Let \mathcal{U} be a C^r -neighborhood of the identity map that $m^{-1} < \frac{f'(x)}{g'(y)} < m$ for every $x, y \in [0, 1]$ and $f, g \in \mathcal{U}$. By planning the arguments similar to what employed in the proof of Proposition 2, we observe that $f(C_\alpha) - g(C_\beta)$ contains an interval. Indeed, it is proved that the pair $(C_\alpha, C_\beta) \in \mathcal{L}$ has weak stable intersection. The following corollary can be viewed as an extension of the result.

Corollary 6. The pair $(C_\alpha, \lambda C_\beta)$ with condition $\bigcup S_\lambda^i((-\lambda, 1)) = (-\lambda, 1)$ has weak stable intersection.

5 Hausdorff dimension

The results of this section begin by observing the complication of calculating Hausdorff dimension $C_\alpha - \lambda C_\beta$ with the simplest non trivial choice of the middle Cantor sets C_α and C_β ; indeed, $\frac{\log \alpha}{\log \beta} = \frac{3}{2} \in \mathbb{Q}$, together with the special number λ . Take $\alpha := \frac{1}{p} := \frac{1}{\gamma^3}$ and $\beta := \frac{1}{q} := \frac{1}{\gamma^2}$, where here γ is the golden number. Also, let C_α and C_β be two middle Cantor sets with expanding maps ϕ_α and ϕ_β , respectively, as follows:

$$C_\alpha : \underbrace{\quad \frac{1}{p} \quad}_{\quad} \quad \underbrace{\quad \frac{1}{p} \quad}_{\quad} \quad C_\beta : \underbrace{\quad \frac{1}{q} \quad}_{\quad} \quad \underbrace{\quad \frac{1}{q} \quad}_{\quad}$$

$$\phi_\alpha(x) := \begin{cases} px & x \in [0, \frac{1}{p}] \\ px - p + 1 & x \in [1 - \frac{1}{p}, 1] \end{cases}, \quad \phi_\beta(x) := \begin{cases} qx & x \in [0, \frac{1}{q}] \\ qx - q + 1 & x \in [1 - \frac{1}{q}, 1] \end{cases}.$$

The pair $(C_\alpha, C_\beta) \in \Omega$, since $HD(C_\alpha) + HD(C_\beta) = \frac{5}{6} \log_\gamma 2 \cong 1.2003$ and $\tau(C_\alpha) \cdot \tau(C_\beta) = \frac{1}{3-\gamma} \cong 0.7236$.

Proposition 3. *The iterated function system corresponding to $C_\alpha - \frac{2}{\gamma} C_\beta$ is of finite type and Hausdorff dimension of its attractor is smaller than one.*

Proof. Firstly, it is easy to check that

$$\begin{cases} \gamma^2 = \gamma + 1, & \gamma^3 = 2\gamma + 1, & \gamma^4 = 3\gamma + 2, & \gamma^5 = 5\gamma + 3, & \gamma^6 = 8\gamma + 5, & \dots \\ \frac{1}{\gamma} = \gamma - 1, & \frac{1}{\gamma^2} = 2 - \gamma, & \frac{1}{\gamma^3} = 2\gamma - 3, & \frac{1}{\gamma^4} = 5 - 3\gamma, & \dots \end{cases}$$

Consider Π_θ with $\cot \theta := \frac{q(p-1)}{p(q-1)} = \frac{2}{\gamma} = \sqrt{5} - 1$, (see part (I) of Corollary 3). letting $m_0 = 2$, $n_0 = 3$ and $\lambda = \frac{2}{\gamma}$ in Theorem 3, then we obtain 21 maps $T = \{T_i \mid T_i(t) = \gamma^6 t + a_i\}_{i=1}^{21}$ on \mathbb{R} , that are return maps to the vertical line $s = \frac{2}{\gamma}$. As we mentioned in Section 2, we can easily find a_i 's as below:

- $(w_1, w_2) = \left(0, (1 - \frac{1}{q})(1 + \frac{1}{q} + \frac{1}{q^2})\right) \Rightarrow a_1 = -(6\gamma + 4)(4\gamma - 8) = 8\gamma + 8,$
- $(w_1, w_2) = \left(0, (1 - \frac{1}{q})(1 + \frac{1}{q})\right) \Rightarrow a_2 = -(6\gamma + 4)(\gamma - 3) = 8\gamma + 6,$
- $(w_1, w_2) = \left((1 - \frac{1}{p})\frac{1}{p}, (1 - \frac{1}{q})(1 + \frac{1}{q} + \frac{1}{q^2})\right) \Rightarrow a_3 = -(6\gamma + 4)(6\gamma - 11) = 6\gamma + 8,$
- $(w_1, w_2) = \left((1 - \frac{1}{p})\frac{1}{p}, (1 - \frac{1}{q})(1 + \frac{1}{q})\right) \Rightarrow a_4 = -(6\gamma + 4)(3\gamma - 6) = 6\gamma + 6,$
- $(w_1, w_2) = \left(0, (1 - \frac{1}{q})\right) \Rightarrow a_5 = -(6\gamma + 4)(0 - 1) = 6\gamma + 4,$
- $(w_1, w_2) = \left((1 - \frac{1}{p})\frac{1}{p}, (1 - \frac{1}{q})(1 + \frac{1}{q^2})\right) \Rightarrow a_6 = -(6\gamma + 4)(5\gamma - 9) = 4\gamma + 6,$
- $(w_1, w_2) = \left((1 - \frac{1}{p})\frac{1}{p}, (1 - \frac{1}{q})\right) \Rightarrow a_7 = -(6\gamma + 4)(2\gamma - 4) = 4\gamma + 4.$

Note that, the numbers in the right side of the notion \Rightarrow are the a_i 's which are related to the square of the first step of the construction $C_\alpha \times C_\beta$ appeared in the point presented in the left side. Surley, we can find other a_i 's as follows:

- for every $7 < i \leq 14$, we use the relation $a_i = a_{i-7} - p(p-1) = a_{i-7} - (6\gamma + 4)$ and we get $a_8 = 2\gamma + 4$, $a_9 = 2\gamma + 2$, $a_{10} = 4$, $a_{11} = 2$, $a_{12} = 0$, $a_{13} = -2\gamma + 2$ and $a_{14} = -2\gamma$.
- for every $14 < i \leq 21$, we use the relation $a_i = a_{i-7} - p(p-1) = a_{i-7} - (6\gamma + 4)$ and we get $a_{15} = -4\gamma$, $a_{16} = -4\gamma - 2$, $a_{17} = -6\gamma$, $a_{18} = -6\gamma - 2$, $a_{19} = -6\gamma - 4$, $a_{20} = -8\gamma - 2$ and $a_{21} = -8\gamma - 4$.

The first assertion is obtained by using Theorem 2.9 of [9], since $b_i = -\frac{a_i}{p^2} \in \mathbb{Z}[\gamma]$. For the second, Theorem 3 implies that $C_\alpha - \frac{2}{\gamma}C_\beta = \bigcap_{i \in \mathbb{N}} T^{-i}([\frac{-2}{\gamma}, 1])$. Now we are going to describe a scheme to estimate a suitable upper bound of its Hausdorff dimension. To do this, we split the interval $[\frac{-2}{\gamma}, 1]$ by using the return maps $\{T_j^{-1}\}_{j=1}^{21}$ as follows:

- $G_1 := [T_2^{-1}(\frac{-2}{\gamma}), T_1^{-1}(1)] = [\frac{-10\gamma-4}{p^2}, \frac{-8\gamma-7}{p^2}]$, $G_2 := [T_4^{-1}(\frac{-2}{\gamma}), T_3^{-1}(1)] = [\frac{-8\gamma-4}{p^2}, \frac{-6\gamma-7}{p^2}]$,
- $G_3 := [T_5^{-1}(\frac{-2}{\gamma}), T_4^{-1}(1)] = [\frac{-8\gamma-2}{p^2}, \frac{-6\gamma-5}{p^2}]$, $G_4 := [T_7^{-1}(\frac{-2}{\gamma}), T_6^{-1}(1)] = [\frac{-6\gamma-2}{p^2}, \frac{-4\gamma-5}{p^2}]$,
- $R_1 := [T_3^{-1}(\frac{-2}{\gamma}), T_2^{-1}(1)] = [\frac{-8\gamma-6}{p^2}, \frac{-8\gamma-5}{p^2}]$, $R_2 := [T_6^{-1}(\frac{-2}{\gamma}), T_5^{-1}(1)] = [\frac{-6\gamma-4}{p^2}, \frac{-6\gamma-3}{p^2}]$,
- $G_i := G_{i-4} + \frac{p-1}{p}$ $4 < i \leq 8$, and $G_i := G_{i-4} + \frac{p-1}{p}$ $8 < i \leq 12$,
- $R_i := R_{i-2} + \frac{p-1}{p}$ $2 < i \leq 4$, and $R_i := R_{i-2} + \frac{p-1}{p}$ $4 < i \leq 6$,
- $H_1 := (T_7^{-1}(1), T_8^{-1}(\frac{-2}{\gamma})) = (\frac{-4\gamma-3}{p^2}, \frac{-4\gamma-2}{p^2})$, and $H_2 := (T_{14}^{-1}(1), T_{15}^{-1}(\frac{-2}{\gamma})) = (\frac{2\gamma+1}{p^2}, \frac{2\gamma+2}{p^2})$,
- $Z_1 := ch(G_2 \cup G_3) \setminus G_2 \cup G_3$, and $Z_2 := ch(G_6 \cup G_7) \setminus G_6 \cup G_7$, and $Z_3 := ch(G_{10} \cup G_{11}) \setminus G_{10} \cup G_{11}$,

where $ch(A)$ is the shorthand of the convex hull $A \subseteq \mathbb{R}$. Let $\{Y_i\}_{i=1}^{12}$ be the intervals that situated between two subsequent G_i and R_j , respectively, and $\{X_i\}_{i=1}^6$ are connected components of the complement of the above sets on the interval $[\frac{-2}{\gamma}, 1]$, respectively, see Figure 2.

$$\cdot \underline{X_1} \quad \underline{\quad} \quad \underline{\quad} \quad \underline{R_2} \quad \underline{\quad} \quad \underline{H_1} \quad \underline{\quad} \quad \underline{G_6} \quad \underline{Y_7} \quad \underline{\quad} \quad \underline{H_2} \quad \underline{\quad} \quad \underline{Z_3} \quad \underline{\quad} \quad \cdot$$

Figure 2: Projection of the emerged squares in the first step of the construction on $[\frac{-2}{\gamma}, 1]$ has been shown. For instance, we have determined the intervals X_1 , Y_7 , Z_3 , H_1 , H_2 , G_6 and R_2 .

Regardless of the indexes, we have

$$|H| = \frac{|X|}{2} = |Y| = |R| = \frac{1}{\gamma^6}, \quad |G| = \frac{2\gamma-3}{\gamma^6}, \quad |Z| = |X| - |G|. \quad (3)$$

Also, it is straightforward to show that

- $X_1 = T_1^{-1}((ch\{R_6, 1\})^c)$, $X_2 = T_7^{-1}((ch\{\frac{-2}{\gamma}, R_1\})^c)$,
- $Y_1 = T_2^{-1}((ch\{\frac{-2}{\gamma}, R_1\} \cup ch\{R_4, 1\})^c)$, $Y_2 = T_3^{-1}((ch\{\frac{-2}{\gamma}, R_3\} \cup ch\{R_6, 1\})^c)$,
- $Y_3 = T_5^{-1}((ch\{\frac{-2}{\gamma}, R_1\} \cup ch\{R_4, 1\})^c)$, $Y_4 = T_6^{-1}((ch\{\frac{-2}{\gamma}, R_3\} \cup ch\{R_6, 1\})^c)$,
- $Z_1 = T_4^{-1}((ch\{\frac{-2}{\gamma}, R_1\} \cup ch\{R_6, 1\})^c)$,

where c is the complement on interval $[\frac{-2}{\gamma}, 1]$. Other sets have similar relations.

The intervals $S^i[-\lambda, 1]$ overlap each other just on G_i or R_i . It is important to know that in each G_i or R_i , the inverse of G_s (also R_s) under the maps T is either the same, or there is not any intersection between them and they situate symmetrical as Figure 3.



Figure 3: Projection of the emerged squares in the second step of the construction on G, R , respectively.

Henceforth, we show that in each stage of the construction $C_\alpha \times C_\beta$, there exists an attractor namely F , with the minimum number of the contractions, that satisfies

$$\Pi_\theta(F) = \Pi_\theta(C_\alpha \times C_\beta) = C_\alpha - \frac{2}{\gamma}C_\beta \quad (5)$$

and then we see that $HD(F) < 1$ for the 6th step. In the first step, we have the iterated function system which consists of 32 contractions

$$S^1 := \left\{ S \mid S(x, y) = \frac{1}{p^2}(x, y) + \left(\left(1 - \frac{1}{p}\right) \sum_{k=0}^1 \frac{a_k}{p^k}, \left(1 - \frac{1}{q}\right) \sum_{k=0}^2 \frac{b_k}{q^k} \right) \quad a_k, b_k = 0 \text{ or } 1 \right\}$$

on the square $I \times I$. An equivalence relation on S^1 defines as follows:

$$S_1 \sim S_2 \quad \equiv \quad \Pi_\theta \circ S_1 = \Pi_\theta \circ S_2 \quad \forall S_1, S_2 \in S^1. \quad (6)$$

On our selection of number γ and angle θ , we can take $\mathcal{F} := \{S_1, \dots, S_{21}\} \subset S^1$, in this condition that $S_i \approx S_j$ for each $1 \leq i \neq j \leq 21$. Let F be the attractor of the family \mathcal{F} on $I \times I$. Then \mathcal{F} satisfies the relation (5) and we have $HD(F) = \log_{\gamma^6}^{21}$. On the n -step, let $S^n := \left\{ S = S_{i_1} \circ S_{i_2} \circ \dots \circ S_{i_n} \mid S_{i_j} \in S^1, 1 \leq j \leq n \right\}$. Again, for elements of S^n , we use the equivalence relation (6). If $1 \leq i \leq 6$, then we define $\mathcal{X}_i := \left\{ [S] \mid S(I \times I) \cap \Pi_\theta^{-1}(X_i) \neq \emptyset \right\}$. For $1 \leq i \leq 12$ the set \mathcal{Y}_i , and for $1 \leq i \leq 3$ the set \mathcal{Z}_i have been defined similarly. Moreover, if $1 \leq i \leq 12$, then we define $\mathcal{G}_i := \left\{ [S] \mid S(I \times I) \subset \Pi_\theta^{-1}(G_i) \right\}$. For $1 \leq i \leq 6$ the set \mathcal{R}_i has been defined similarly. Regardless of the index i , we take $x_n := |\mathcal{X}_i|$. Numbers y_n, z_n, g_n and r_n are defined similarly. Now we claim that

$$\begin{bmatrix} x_n \\ y_n \\ z_n \\ g_n \\ r_n \end{bmatrix} = \begin{bmatrix} 5 & 11 & 3 & 11 & 5 \\ 2 & 6 & 2 & 6 & 2 \\ 4 & 10 & 3 & 10 & 4 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 8 & 2 & 6 & 5 \end{bmatrix}^{n-2} \begin{bmatrix} 19 \\ 10 \\ 17 \\ 2 \\ 10 \end{bmatrix} =: A^{n-2} \begin{bmatrix} 19 \\ 10 \\ 17 \\ 2 \\ 10 \end{bmatrix}, \quad n \geq 2.$$

The assertion holds for $n = 2$. Indeed, we know that the elements of S^2 are as follows:

$$S(x, y) := \frac{1}{p^4}(x, y) + \left(\left(1 - \frac{1}{p}\right) \sum_{k=0}^3 \frac{a_k}{p^k}, \left(1 - \frac{1}{q}\right) \sum_{k=0}^5 \frac{b_k}{q^k} \right) \quad a_k, b_k = 0 \text{ or } 1.$$

Consider contractions $S_i(x, y) = \frac{1}{p^4}(x, y) + a_i, 1 \leq i \leq 4$ of S^2 , where

- $a_1 := \left((1 - \frac{1}{p})(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}), (1 - \frac{1}{q})(\frac{1}{q^2} + \frac{1}{q^5}) \right),$
- $a_2 := \left((1 - \frac{1}{p})(1 + \frac{1}{p}), (1 - \frac{1}{q})(\frac{1}{q^3} + \frac{1}{q^4} + \frac{1}{q^5}) \right),$
- $a_3 := \left((1 - \frac{1}{p})(1 + \frac{1}{p}), (1 - \frac{1}{q})(\frac{1}{q^2} + \frac{1}{q^3} + \frac{1}{q^4} + \frac{1}{q^5}) \right),$
- $a_4 := \left((1 - \frac{1}{p})(1 + \frac{1}{p^2} + \frac{1}{p^3}), (1 - \frac{1}{q})(\frac{1}{q^3} + \frac{1}{q^5}) \right).$

Because of the characteristics of γ , we have $\Pi_\theta \circ S_1 = \Pi_\theta \circ S_2$ and $\Pi_\theta \circ S_3 = \Pi_\theta \circ S_4$. Regarding this fact and the relation (3), it is not hard to see that $g_2 = 2$ and $r_2 = 10$, see Figures (3) and (4).

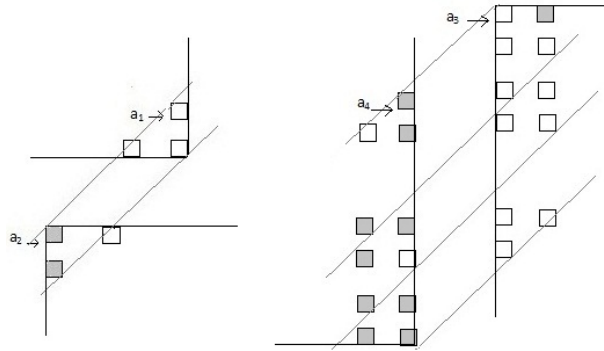


Figure 4:

Figure 4: The left (right) figure demonstrates all squares in the second step of the construction $C_\alpha \times C_\beta$ that entirely lie in the strip $\Pi_\theta^{-1}(G)$ ($\Pi_\theta^{-1}(R)$). We filled the squares which should be counted.

For $1 \leq i \neq j \leq 21$, we have $S^i[-\lambda, 1] \cap S^j[-\lambda, 1] \cap X_k = \emptyset$ and just the projection of two squares in the first step of the construction $C_\alpha \times C_\beta$ entirely stay in $ch\{R_6, 1\}$ so $x_2 = 19$, see the relation (4) and Figure (2). Similarly $y_2 = 10$ and $z_2 = 17$.

The general case is obtained by using induction method and the relation

$$[x_n, y_n, z_n, g_n, r_n]^T = A[x_{n-1}, y_{n-1}, z_{n-1}, g_{n-1}, r_{n-1}]^T.$$

Now we select the elements of $\mathcal{F} \subset S^n$, that are just in a class and F is its attractor on the square $I \times I$. Therefore, the relation (5) is valid. To calculate $HD(F)$ we define

$$k_n := |\mathcal{F}| = \langle (6, 12, 3, 12, 6), (x_n, y_n, z_n, g_n, r_n) \rangle,$$

that gives $HD(F) = \log_{(\gamma^6)^n} k_n = \log_{\gamma^6} \sqrt[n]{k_n}$. Using Maple program one gets

$$n = 2 \Rightarrow HD(F) = \log_{\gamma^6}^{19.2093\dots} > 1,$$

Similarly, we have

- $n = 3 \Rightarrow HD(F) = \log_{\gamma^6}^{18.5246\dots} > 1$,
- $n = 4 \Rightarrow HD(F) = \log_{\gamma^6}^{18.1817\dots} > 1$,
- $n = 5 \Rightarrow HD(F) = \log_{\gamma^6}^{17.9782\dots} > 1$,
- $n = 6 \Rightarrow HD(F) = \log_{\gamma^6}^{17.8437\dots} < 1$,

Thus, on the 6th step of the construction $C_\alpha \times C_\beta$, we see that $HD(F) < 0.9982$, since $\gamma^6 \cong 17.9442$. This completes the proof of the proposition. \square

Proposition 3 not only yields that Lebesgue measure of the set $C_\alpha - \frac{2}{\gamma}C_\beta$ is zero, but also says that the above Cantor sets with $\lambda = \frac{2}{\gamma}$ are good candidates for the following sense:

$$HD(C_\alpha - \lambda C_\beta) < \min\{1, HD(C_\alpha) + HD(C_\beta)\} \quad (7)$$

In general, when $\lambda = 1$, it has been showed that the equality holds in (7), where $\frac{\log \alpha}{\log \beta} \notin \mathbb{Q}$. In our example equality holds since $C_\alpha - C_\beta = [-1, 1]$. Also, it is obvious that if $HD(C_\alpha) + HD(C_\beta) < 1$ and $\frac{\log \alpha}{\log \beta} \in \mathbb{Q}$, then (7) is always valid [11]. Moreover, $\mathcal{H}^{HD(C_\alpha)+HD(C_\beta)}(C_\alpha + C_\beta) = 0$ ([5]).

Although, the iterated function system $\{S_i\}_{i=1}^{21}$ corresponding to the attractor $C_\alpha - \frac{2}{\gamma}C_\beta$ is of finite type and $HD(C_\alpha - \lambda C_\beta)$ could be calculated by characterizing the incidence matrix corresponding to this 21 maps, (we can not do it). But we find an easier way to do this by using the fact that the attractor F of the iterated function system of finite type satisfies $0 < \mathcal{H}^s(F) < 1$, where $s = \dim_H(F)$ [9]. Take A as in Proposition 3, one gets

Proposition 4. *The Hausdorff dimension of $C_\alpha - \frac{2}{\gamma}C_\beta$ is $\log_{p^2} \lambda$, where λ is the largest eigenvalue of the matrix A. Moreover, this number is the Box dimension of $C_\alpha - \frac{2}{\gamma}C_\beta$.*

Proof. By the same notations used in Proposition 3 and $s := HD(C_\alpha - \frac{2}{\gamma}C_\beta)$. Noting to the scaling property of the s-dimensional Hausdorff measure \mathcal{H}^s , we obtain

$$\begin{bmatrix} p^{2s} \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap X) \\ p^{2s} \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap Y) \\ p^{2s} \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap Z) \\ p^{2s} \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap G) \\ p^{2s} \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap R) \end{bmatrix} = \begin{bmatrix} 5 & 11 & 3 & 11 & 5 \\ 2 & 6 & 2 & 6 & 2 \\ 4 & 10 & 3 & 10 & 4 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 8 & 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap X) \\ \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap Y) \\ \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap Z) \\ \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap G) \\ \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap R) \end{bmatrix},$$

which is equivalent to

$$(p^{2s}I - A) \begin{bmatrix} \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap X) \\ \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap Y) \\ \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap Z) \\ \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap G) \\ \mathcal{H}^s((C_\alpha - \frac{2}{\gamma}C_\beta) \cap R) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

As $C_\alpha - \frac{2}{\gamma}C_\beta$ is a s-set, $\det(p^{2s}I - A) = 0$. On the other hand, roots of the characteristic polynomial

$$x^5 - 20x^4 + 50x^3 - 28x^2 - 3x = x(x-1)(x^3 - 19x^2 + 31x + 3)$$

corresponding to the matrix A are 0, 1, $-0.0943\dots$, $1.8434\dots$, $17.2508\dots$. Due to these facts s has to be $\log_{p^2} 17.2508 = 0.9863\dots$. The second assertion is a direct result of Theorem 1.1 of [9]. \square

Regarding to our discussion, we convince that the pair (C_α, C_β) has a better structure among other members of \mathcal{L} . For instance, (C_α, C_β) belongs to \mathcal{P} . Hence, by using Corollary 4 we observe that for each $\mu \in \mathbb{Q}[\gamma] = \{c\gamma + d \mid c, d \in \mathbb{Q}\}$, the set $C_\alpha - \mu C_\beta$ contains an interval or has zero Lebesgue measure. Also, it is easy to find λ 's that $C_\alpha - \lambda C_\beta$ contains an interval. This can be useful in providing the assumptions of Proposition 2, for given f and g . For instance, by selecting $f(x) = g(x) = \sqrt{x}$, we express the below example.

Example 2. *The set $\sqrt{C_\alpha} - \sqrt{C_\beta}$ contains an interval. On the first step of the structure of $C_\alpha \times C_\beta$, the projection under the angle θ on all squares overlap each other, when $1 = \frac{(1-\frac{1}{q})-\frac{1}{q}}{\frac{1}{p}-0} < \tan \theta < \frac{\frac{1}{q^2}}{\frac{1}{p}(1-\frac{2}{p})} = \frac{q}{p-2}$. We can select a basic square in the next structures of $C_\alpha \times C_\beta$ situated in (x_0, y_0) between lines $y = x$ and $y = (\frac{q}{p-2})^2 x$ (there are plenty of squares close to the point $(1, 1)$) and so $1 < \frac{y_0}{x_0} < (\frac{q}{p-2})^2$. On the other hand, we see that the family of curves $\sqrt{x} - \sqrt{y} = c$ satisfies $y' = \sqrt{\frac{y}{x}}$. Now we can select $\frac{p-2}{q} < m_1 < \sqrt{\frac{x_0}{y_0}} < m_2 < 1$ such that the elements of the iterated function systems corresponding to (C_α, C_β) be regularly linked on (m_1, m_2) . Regarding to the Proposition 2, the set $\sqrt{C_\alpha} - \sqrt{C_\beta}$ contains an interval.*

Our example is also different from Solomyak's example in this point that we can not put it in a continuous curve

$$\begin{aligned} r : [\alpha_1, \alpha_2] &\mapsto \Omega \times \mathbb{R}^* \\ \alpha &\rightarrow (C_\alpha, C_{\beta(\alpha)}, \lambda(\alpha)) \end{aligned}$$

with condition $HD(C_\alpha - \lambda(\alpha)C_{\beta(\alpha)}) < 1$, for each $\alpha \in [\alpha_1, \alpha_2]$. Note that, Solomyak's curves that can be written in the form $r_{m_0, n_0}(\alpha) = (C_\alpha, C_{\frac{m_0}{\alpha n_0}}, -1)$ stay close to $\{(C_\alpha, C_\beta, -1) \mid HD(C_\alpha) + HD(C_\beta) = 1\}$, of course the case $m_0 = n_0$ is an exception, (see Remark and Figure 2 of [15]). Another example of this kind is the family of the curves $r_{m_0, n_0}(\alpha) = (C_\alpha, C_{\frac{m_0}{\alpha n_0}}, \frac{1-\alpha}{1-\alpha \frac{m_0}{n_0}})$ that can be obtained from the part (I) of Corollary 3. They stay close to the boundary $\Omega \times \mathbb{R}^*$ too.

In contrast, it seems that there exists a sequence $\{\lambda_i\}$ of real numbers with $\lambda_i > 1$ convergent to one which $C_\alpha - \lambda_i C_\beta$ have zero Lebesgue measure, (recall that $|C_\alpha - \frac{2}{\gamma}C_\beta| =$

0 and $C_\alpha - 1C_\beta = [-1, 1]$). A positive answer to this not only rejects the below problem but also gives $\{\lambda \mid HD(C_\alpha - \lambda C_\beta) < 1\}' \neq \{0\}$.

Open Problem 2. Does the pair (C_α, C_β) have stable intersection, what about weak?

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