Painlevé representation of Tracy-Widom$_\beta$ distribution for $\beta = 6$.

Igor Rumanov
Dept. of Applied Mathematics, CU Boulder, Boulder, CO
e-mail: igor.rumanov@colorado.edu

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Abstract

In [29], we found explicit Lax pairs for the soft edge of beta ensembles with even integer values of β. Using this general result, the case $\beta = 6$ is further considered here. This is the smallest even β, when the corresponding Lax pair and its relation to Painlevé II (PII) have not been known before, unlike cases $\beta = 2$ and 4. It turns out that again everything can be expressed in terms of the Hastings-McLeod solution of PII. In particular, a second order nonlinear ODE for the logarithmic derivative of Tracy-Widom distribution for $\beta = 6$ involving the PII function in the coefficients, is found, which allows one to compute asymptotics for the distribution function. The ODE is a consequence of a linear system of three ODEs for which the local Painlevé analysis yields series solutions with exponents in the set 4/3, 1/3 and $-2/3$.

1 Introduction and main result

Beta ensembles of random matrices introduced by Dyson [14] were originally defined as Coulomb gas (fluid) of particles-eigenvalues for general values of Dyson index β beyond the three most important cases $\beta = 1, 2, 4$ known as real orthogonal (OE), complex unitary invariant (UE) and symplectic (SE) ensembles, respectively. The importance of general β ensembles and the number of their applications grow fast in recent years due to the developments of Conformal Field Theory (CFT) [7] connections with other subjects, see e.g. [2, 30] and references therein on the relation with β-ensembles and linear PDEs of [7]. They include the AGT correspondence [3] relating CFT with supersymmetric quantum gauge field theories, and also condensed matter physics, e.g. electronic transport in wires disordered by impurities and quantum Hall effect. The tie of CFT itself with β-ensembles can be traced back to the times of its birth when the Coulomb gas representation of CFT correlation functions appeared in terms of Dotsenko-Fateev integrals [11]. There are also the genuine matrix ensembles with general β eigenvalue ditributions, first found in [13] for Gaussian and Laguerre distributions and later extended to other measures, see [22] and references therein. A comprehensive treatment of the available before the last several years results on β-ensembles and their applications is contained in [17].

The soft edge probability distributions (i.e. for the largest eigenvalue when the matrix size $n \to \infty$) for $\beta = 1, 2, 4$ have been known since the seminal works of Tracy and Widom [31, 32] in terms of Hastings-Mcleod solution [20] of Painlevé II. More recently, a one-parameter generalization of these distributions (describing e.g. the soft edge limit of certain *spiked* ensembles or ensembles with external source) was shown to satisfy a diffusion-drift PDE for general values of β [15, 26, 8]. For the three above special values its limit as the additional parameter x tends to ∞ is the corresponding Tracy-Widom distribution. However, the best available description up to date for the Tracy-Widom distributions of different beta ensembles is the mentioned Fokker-Planck PDE, eq. (1.1) below.

This article is a sequel to [29]. Current results are a further demonstration of classical integrable structure present for values of β beyond the three special ones where it was known or always expected. It should be somehow related to the quantum integrable structure of CFT with central charge $c \leq 1$ found in [6] but this is a matter of future investigation. We study the distribution function for the soft edge of (spiked) Dyson beta ensembles which satisfies the boundary value problem first considered by Bloemendal and Virag [8]:

$$\left(\partial_t + \frac{2}{\beta}\partial_{xx} + (t - x^2)\partial_x\right)\mathcal{F}^{(\beta)}(t,x) = 0. \tag{1.1}$$

The boundary conditions ensure that the solution $\mathcal{F}^{(\beta)}$ to the Fokker-Planck (FP) eq. (1.1) is a probability distribution function:

$$\mathcal{F}^{(\beta)}(t,x) \to 0 \quad \text{as } x \to -\infty, t < \infty, \qquad \mathcal{F}^{(\beta)}(t,x) \to 1 \quad \text{as } t, x \to \infty \text{ together},$$

$$\mathcal{F}^{(\beta)}(t,x) \to \mathcal{F}_0(t) \quad \text{as } x \to +\infty, t \text{ finite}. \tag{1.2}$$

The last function $\mathcal{F}_0(t)$ is the Tracy-Widom distribution (TW_β). Equation (1.1) can be rightfully called quantum Painlevé II, see [23], in imaginary time since besides the time derivative it contains operator which is the canonically quantized Painlevé II Hamiltonian with $2/\beta$ playing the role of Planck constant. It will be convenient for us to consider the rescaled eq. (1.1),

$$\left(\kappa\partial_t + \partial_{xx} + (t - x^2)\partial_x\right)\mathcal{F}(t,x) = 0, \tag{1.3}$$

i.e. eq. (1.1) with t and x rescaled as $x \to x/\kappa^{1/3}$, $t \to t/\kappa^{2/3}$, $\kappa = \beta/2$.

In [29] we found explicit Lax pairs

$$\partial_x \begin{pmatrix} \mathcal{F} \\ G \end{pmatrix} = L \begin{pmatrix} \mathcal{F} \\ G \end{pmatrix}, \qquad \partial_t \begin{pmatrix} \mathcal{F} \\ G \end{pmatrix} = B \begin{pmatrix} \mathcal{F} \\ G \end{pmatrix}, \tag{1.4}$$

describing the soft edge of (spiked) random matrix beta ensembles with all even integer Dyson indices β, such that $\mathcal{F}(t,x)$ is the first component of their eigenvector. For positive integer $\kappa = \beta/2$ we obtained

$$L = \begin{pmatrix} L_1 & L_+ \\ L_- & L_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-v + L_d) & L_+ \\ -\frac{1}{2L_+}(\kappa B_d + \partial_x L_d + L_d^2/2 + f_v) & \frac{1}{2}(-v - L_d) \end{pmatrix}, \tag{1.5}$$

$$B = \begin{pmatrix} B_1 & B_+ \\ B_- & B_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(-x + \frac{U(t)+t^2/2}{\kappa} - \frac{\phi'}{\phi} + B_d\right) & -\frac{\partial_x L_+}{\kappa} \\ -\frac{2L_- \partial_x L_+ + \kappa\partial_t L_d - \kappa\partial_x B_d}{2\kappa L_+} & \frac{1}{2}\left(-x + \frac{U(t)+t^2/2}{\kappa} - \frac{\phi'}{\phi} - B_d\right) \end{pmatrix}, \tag{1.6}$$

where

$$v = t - x^2, \tag{1.7}$$

is the drift function in the Fokker-Planck equation (1.3) (or (1.1)),

$$L_+ = \phi(t)\prod_{k=1}^{\kappa}(x - Q_k(t)), \tag{1.8}$$

$$L_d = -L_+ \cdot \sum_{k=1}^{\kappa} \frac{\kappa Q_k' - 2R_k}{(x - Q_k)\prod_{j\neq k}^{\kappa}(Q_k - Q_j)} = -\sum_{k=1}^{\kappa}(\kappa Q_k' - 2R_k)\prod_{j\neq k}^{\kappa}\frac{x - Q_j}{Q_k - Q_j}, \tag{1.9}$$

$$\kappa B_d = \kappa\phi'(t)/\phi + \sum_{k=1}^{\kappa}\frac{\kappa Q_k' - 2R_k}{x - Q_k}\left(\sum_{l=1}^{\kappa}\frac{\prod_{j\neq l}^{\kappa}(x - Q_j)}{\prod_{j\neq k}^{\kappa}(Q_k - Q_j)} - 1\right), \tag{1.10}$$

$$f_v(t,x) = \kappa B_t - \partial_x v - v^2/2 = -\left(\frac{x^4}{2} - tx^2 + (\kappa - 2)x - U(t) + \frac{\kappa\phi'(t)}{\phi}\right), \tag{1.11}$$

$B_t \equiv TrB$ and $U(t)$ is defined by

$$\kappa U'(t) = -\sum_{k=1}^{\kappa} Q_k^2. \tag{1.12}$$

Function $\phi(t)$ remains arbitrary, e.g. one can take $\phi(t) \equiv 1$.

Functions $Q_k(t)$ satisfy equations of motion for particles with Calogero interaction and additional time-dependent external force which would give Painlevé II equation without the interaction [29]. Considered together with eq. (1.12), they possess κ explicit first integrals found in [29], which we do not reproduce here because we find a different more convenient form of them in what follows.

The main result of the paper is

Theorem 1. *The log-derivative of the rescaled Tracy-Widom distribution $\mathcal{F}_0(t)$ for $\beta = 6$, where the Tracy-Widom distribution is $F_{TW}^{(\beta=6)}(t) = \mathcal{F}_0(\kappa^{2/3}t) = \mathcal{F}_0(3^{2/3}t)$, can be written as*

$$3(\ln \mathcal{F}_0)' = u - \frac{q^2}{u} + \eta, \tag{1.13}$$

where q is the Hastings-McLeod solution of Painlevé II, $u = (q')^2 - tq^2 - q^4$ so that $u' = -q^2$, and $\eta(t)$ satisfies the second order ODE:

$$9\eta'' + 9\eta\eta' + \eta^3 - 4\left(3\left(\frac{q^2}{u} - u\right)' + t\right)\eta - 8\left(\frac{q^2}{u} - u\right)'' - 2 = 0. \tag{1.14}$$

Equivalently, function $h_- = \eta - \frac{q^2}{u} = \frac{\nu}{\mu_-}$ can be found from the linear system of ODEs,

$$3q^2\mu_+' = (q^2)'\mu_+ - q^2\nu, \tag{1.15}$$

$$3q^2\mu_-' = -(q^2)'\mu_- + q^2\nu, \tag{1.16}$$

$$3q^2\nu' = 2q^4\mu_- - 2u\mu_+. \tag{1.17}$$

The plan of the paper is the following. In section 2 we describe the Lax pair for $\kappa = 3$ and demonstrate how its polynomiality leads to polynomial ODEs and their first integrals. We analyze the system obtained in section 3 and find Painlevé II as well as the system (1.15)–(1.17) inside it. In section 4 we establish the connection (1.13), derive various equivalent forms of eq. (1.14) and find the asymptotics of $(\ln \mathcal{F}_0)'$ as $t \to \infty$. Section 5 presents the analysis of system (1.15)–(1.17) on Painlevé property with partly negative result, another interesting outcome. For completeness and comparison we present the known [8, 28] simplest cases of $\kappa = 1, 2$ (i.e. $\beta = 2, 4$) which lead to Lax pairs for (classical) Painlevé II in the Appendix, and proceed with the first nontrivial case beyond it, $\kappa = 3$ ($\beta = 6$).

2 The Lax pair for $\kappa = 3$

This is the first case beyond just Painlevé II, where the classical integrability and the Lax pair were not known before. The general formulas of section 1 give:

$$L_+ = \phi(t)(x - Q_1)(x - Q_2)(x - Q_3) = \phi(t)(x^3 - e_1 x^2 + e_2 x - e_3), \tag{2.1}$$

where we introduced the elementary symmetric functions of Q_k, e_j $(k, j = 1, 2, 3)$. Then

$$B_+ = -\frac{\partial_x L_+}{3} = -\phi(x^2 - 2e_1 x/3 + e_2/3), \tag{2.2}$$

$$B_t = -x + \frac{1}{3}\left(U + \frac{t^2}{2}\right) - \frac{\phi'}{\phi}. \tag{2.3}$$

Since L_d is now a quadratic polynomial in x (see eq. (1.9)), let

$$L_d = q_2 x^2 - q_1 x + q_0, \tag{2.4}$$

where we could write each $q_j(t)$ in terms of Q_k-variables explicitly by eq. (1.9) but we will not need this since, as we will see, q_j are more convenient variables which will lead to polynomial first integrals of the system unlike the Garnier-like variables Q_k in terms of which the first integrals are rational. Let also

$$B_d - \frac{\phi'}{\phi} = d_1 x - d_0 \tag{2.5}$$

(it is a linear function, see eq. (1.10)). Plugging the above expressions into the general equation

$$L_+ \cdot \kappa B_d + \partial_x L_+ \cdot L_d = \kappa \partial_t L_+ + \partial_{xx} L_+ \tag{2.6}$$

and equating the coefficients of the corresponding powers of x, we find d_1 and d_2 in terms of q_j as well as first order ODEs for e_j:

$$x^4: \qquad d_1 = -q_2, \tag{2.7}$$

$$x^3: \qquad d_0 = \frac{e_1 q_2}{3} - q_1, \tag{2.8}$$

$$x^2: \qquad 3e_1' + (e_1^2 - 2e_2)q_2 - e_1 q_1 + 3q_0 = 0, \tag{2.9}$$

$$x^1: \qquad 3(e_2' + 2) + (e_1 e_2 - 3e_3)q_2 - 2e_2 q_1 + 2e_1 q_0 = 0, \tag{2.10}$$

$$x^0: \qquad 3e_3' + 2e_1 + e_1 e_3 q_2 - 3e_3 q_1 + e_2 q_0 = 0. \tag{2.11}$$

Now we determine L_- from the corresponding component of general eq. (1.5), using also eq. (1.11),

$$L_- = -\frac{L_d^2/2 + \partial_x L_d + 3B_d + f_v}{2L_+}, \tag{2.12}$$

Besides L_-, this equation turns out to yield the new polynomial first integrals. Explicitly the numerator of eq. (2.12) reads:

$$L_d^2/2+\partial_x L_d+3B_d+f_v = \frac{q_2^2-1}{2}x^4 - q_2q_1x^3 + \frac{2q_2q_0+q_1^2+2t}{2}x^2 - (q_1q_0+q_2+1)x + \frac{q_0^2}{2} + U - e_1q_2 + 2q_1$$

$$= (x^3 - e_1x^2 + e_2x - e_3)\left(\frac{q_2^2-1}{2}(x+e_1) - q_2q_1\right) + x^2I_2(t) + xI_1(t) + I_0(t). \tag{2.13}$$

Clearing the denominator in eq. (2.12) and matching powers of x in the resulting equation implies that L_- is a polynomial and so equals

$$L_- = -\frac{(q_2^2-1)(x+e_1) - 2q_2q_1}{4\phi}, \tag{2.14}$$

and the remainder of the division of eq. (2.13) by L_+ is zero which gives the three polynomial first integrals:

$$I_2(t) \equiv (e_1^2 - e_2)\frac{q_2^2-1}{2} - e_1q_2q_1 + \frac{2q_2q_0+q_1^2+2t}{2} = 0, \tag{2.15}$$

$$I_1(t) \equiv (e_3 - e_1e_2)\frac{q_2^2-1}{2} + e_2q_2q_1 - q_1q_0 - q_2 - 1 = 0, \tag{2.16}$$

$$I_0(t) \equiv e_1e_3\frac{q_2^2-1}{2} - e_3q_2q_1 + \frac{q_0^2}{2} + U - e_1q_2 + 2q_1 = 0. \tag{2.17}$$

Now we determine the last entry, B_-, from the general formula, see eq. (1.6),

$$3B_- = -\frac{2L_-\partial_x L_+ + 3\partial_t L_d - 3\partial_x B_d}{2L_+}, \tag{2.18}$$

and again the polynomiality of eq. (2.18) multiplied by L_- implies that

$$B_- = \frac{q_2^2-1}{4\phi} \tag{2.19}$$

is given by the polynomial result of the division by L_+ and the remainder terms are equal to zero and thus give the three ODEs,

$$3q_2' - 2e_1(q_2^2-1) + 3q_2q_1 = 0, \tag{2.20}$$

$$3q_1' - (e_1^2 + e_2)(q_2^2 - 1) + 2e_1 q_2 q_1 = 0, \tag{2.21}$$

$$3(q_0' + q_2) - \frac{(e_1 e_2 + 3e_3)}{2}(q_2^2 - 1) + e_2 q_2 q_1 = 0. \tag{2.22}$$

Adding to the system of ODEs (2.9)–(2.11) and (2.20)–(2.22) the ODE (1.12) for the function U, rewritten as

$$3U' = -\sum_1^3 Q_k^2 = 2e_2 - e_1^2, \tag{2.23}$$

one can verify by tedious but straightforward calculation that

Lemma 1. *Equations (2.15)–(2.17) are first integrals of the system of ODEs (2.9)–(2.11) and (2.20)–(2.23).*

Thus, three of the ODEs, e.g. eqs. (2.9)–(2.11), can be considered as redundant. The lemma also shows consistency of our Lax pair construction for $\kappa = 3$ and gives the explicitly polynomial in x Lax matrices:

$$L = \begin{pmatrix} L_1 & L_+ \\ L_- & L_2 \end{pmatrix} = \begin{pmatrix} \frac{x^2-t}{2} + \frac{q_2x^2-q_1x+q_0}{2} & \phi(t)(x^3 - e_1x^2 + e_2x - e_3) \\ -\frac{(q_2^2-1)(x+e_1)-2q_2q_1}{4\phi} & \frac{x^2-t}{2} - \frac{q_2x^2-q_1x+q_0}{2} \end{pmatrix}, \tag{2.24}$$

$$B = \begin{pmatrix} B_1 & B_+ \\ B_- & B_2 \end{pmatrix} = \begin{pmatrix} \frac{-x+(U+t^2/2)/3-q_2x-e_1q_2/3+q_1}{2} & -\phi(t)(x^2 - 2e_1x/3 + e_2/3) \\ \frac{1}{4\phi}(q_2^2-1) & \frac{-x+(U+t^2/2)/3+q_2x+e_1q_2/3-q_1-\phi'/\phi}{2} \end{pmatrix}. \tag{2.25}$$

3 The $\kappa = 3$ system.

Continuing to analyze the system obtained for $\kappa = 3$, we notice that the look of eqs. (2.15)–(2.17) and (2.20)–(2.23) can be significantly simplified if one introduces a new function $r(t)$ such that

$$e_1(q_2^2 - 1) = r(q_2^2 - 1) + 2q_2q_1. \tag{3.1}$$

Then the three first integrals eqs. (2.15)–(2.17) can be written as, respectively,

$$e_2(q_2^2 - 1) = re_1(q_2^2 - 1) + 2q_2q_0 + q_1^2 + 2t, \tag{3.2}$$

$$e_3(q_2^2 - 1) = re_2(q_2^2 - 1) + 2q_1q_0 + 2 + 2q_2, \tag{3.3}$$

$$0 = re_3(q_2^2 - 1) + q_0^2 + 2U - 2e_1 q_2 + 4q_1. \tag{3.4}$$

We also rewrite eqs. (2.20)–(2.22) as

$$6q_2' = (e_1 + 3r)(q_2^2 - 1), \tag{3.5}$$

$$3q_1' = (e_2 + re_1)(q_2^2 - 1), \tag{3.6}$$

$$6q_0' = (3e_3 + re_2)(q_2^2 - 1) - 6q_2. \tag{3.7}$$

We see that eqs. (3.1)–(3.4) have a nice "telescopic" structure and their linear combination $r^3 \cdot (3.1) + r^2 \cdot (3.2) + r \cdot (3.3) + (3.4)$ gives

$$(r^2 q_2 + rq_1 + q_0)^2 - r^4 + 2tr^2 + 2r + 2U + 2[(r - e_1)q_2 + 2q_1] = 0. \tag{3.8}$$

Anticipating what follows we introduce the new function u_r to replace U,

$$u_r = U + (r - e_1)q_2 + 2q_1 = U + \frac{r - e_1}{q_2} = U - \frac{2q_1}{q_2^2 - 1}, \tag{3.9}$$

where we used eq. (3.1) in the last two equalities. Thus, we can consider the new first integral

$$(r^2 q_2 + rq_1 + q_0)^2 - r^4 + 2tr^2 + 2r + 2u_r = 0 \tag{3.10}$$

as replacing eq. (3.4). Then we have to derive the ODE for u_r replacing eq. (2.23):

$$3u_r' = 3u' + (r - e_1) \cdot 3q_2' + 6q_1' + 3q_2 r' - q_2 \cdot 3e_1' = 3q_2(r' + q_0) + \frac{3r(e_1 + r)(q_2^2 - 1)}{2},$$

where we used eqs. (2.23), (3.5), (3.6), (2.9) and finally eq. (3.1) to come to the last expression. Using eq. (3.1) one more time we get

$$u_r' = q_2(r' + q_0) + \frac{r(e_1 + r)(q_2^2 - 1)}{2} = q_2(r' + r^2 q_2 + rq_1 + q_0) - r^2, \tag{3.11}$$

On the other hand, from the last expression on the right-hand side of eq. (3.9), we find, with the help of eqs. (2.20), (2.21) and (3.1),

$$3u_r' = 3U' - 2\left(\frac{3q_1}{q_2^2 - 1}\right)' = -e_1^2 - 2re_1 + \frac{2q_2 q_1}{q_2^2 - 1} \cdot (e_1 + 3r).$$

Finally, using eq. (3.1) again, we obtain

$$u_r' = -r^2, \tag{3.12}$$

justifying the introduction of u_r. Then eq. (3.11) implies (the special case $q_2 \equiv 0$ to be considered separately)

$$r' + r^2 q_2 + r q_1 + q_0 = 0, \tag{3.13}$$

and eq. (3.10) now means that

$$(r')^2 - r^4 + 2tr^2 + 2r + 2u_r = 0, \tag{3.14}$$

which, together with eq. (3.12), yields Painlevé II equation for the function r,

$$r'' = 2r^3 - 2tr - 1. \tag{3.15}$$

This is incidentally the same Painlevé II which satisfies the function $Q(t) = -q'(t)/q$ for $\kappa = 1$, see eq. (2.9). So we identify

$$r = -\frac{q'}{q}, \tag{3.16}$$

where q is the Hastings-McLeod solution of Painlevé II eq. (2.10).

Now we can eliminate q_0 expressing it from eq. (3.13) and substituting into the other equations. Then we are left with only two independent ODEs to resolve, with coefficients depending on the known function r. It is convenient to choose eqs. (3.5) and (3.6) as such and, after using eqs. (3.1) and (3.2) (besides eq. (3.13)) to eliminate e_1 and e_2, they become, respectively,

$$3q_2' = 2r(q_2^2 - 1) + q_2 q_1, \tag{3.17}$$

$$3q_1' = 2rq_2q_1 + q_1^2 + 2(t - r^2) - 2r'q_2. \tag{3.18}$$

We notice a combination $r_1 = 2rq_2 + q_1$ appearing in both the last equations, differentiating it we get

$$3r_1' = r_1^2 + 4r'q_2 + 2(t - 3r^2), \tag{3.19}$$

and introducing a new function χ such that

$$r_1 \equiv 2rq_2 + q_1 = -3\frac{\chi'}{\chi}, \tag{3.20}$$

we rewrite eq. (3.17) as

$$3(q_2\chi)' = -2r\chi. \tag{3.21}$$

In turn, eq. (3.18) after substituting eq. (3.20) becomes:

$$3(q_1\chi)' = -2[r'q_2\chi + (r^2 - t)\chi]. \tag{3.22}$$

We introduce now two new functions by

$$q_2 = \frac{\mu}{\chi}, \qquad q_1 = \frac{\nu}{\chi}, \tag{3.23}$$

which allows us to get a system of three linear equations equivalent to eqs. (3.17), (3.18):

$$3\chi' = -2r\mu - \nu, \tag{3.24}$$

$$3\mu' = -2r\chi, \tag{3.25}$$

$$3\nu' = -2(r'\mu + (r^2 - t)\chi). \tag{3.26}$$

(Eq. (3.19) is redundant being a consequence of them.) Let us express now everything in terms in terms of Painlevé transcendent q instead of r. Recall that

$$q' = -rq, \qquad r' = r^2 - t - 2q^2, \tag{3.27}$$

and let us introduce function u such that

$$u = (q')^2 - q^4 - tq^2, \qquad u' = -q^2. \tag{3.28}$$

Then

$$r' + r^2 - t = 2(r^2 - t - q^2) = 2\frac{u}{q^2}. \tag{3.29}$$

Introducing also

$$\mu_\pm = \mu \pm \chi \tag{3.30}$$

transforms eqs. (3.24)–(3.26) into the system (1.15)–(1.17) of the main theorem. The linear system (1.15)–(1.17) with coefficients depending on q^2 and u completely characterizes the $\kappa = 3$ ($\beta = 6$) case since all the important functions can be readily found from μ_+, μ_- and ν as we will see in the next section.

4 Tracy-Widom distribution for $\kappa = 3$ and auxiliary functions

Return to the Quantum Painlevé II – the Fokker-Planck equation (1.3). Let us consider the asymptotic expansion of $\mathcal{F}(t, x)$ as $x \to \infty$,

$$\mathcal{F}(t, x) = \sum_{n=0}^{\infty} \frac{\mathcal{F}_n(t)}{x^n}, \tag{4.1}$$

which agrees with the boundary conditions eq. (1.2) corresponding to the sought solution $\mathcal{F}(t, x)$ being a probability distribution function. The function $\mathcal{F}_0(t)$ is also a probability distribution function which is to be called (rescaled – recall going to eq. (1.3) from eq. (1.1)) Tracy-Widom-beta (TW_β in short) distribution, in accordance with the fact that it equals the known Tracy-Widom distributions [31, 32] for $\beta = 2, 1$, and 4. Substituting eq. (4.1) into eq. (1.3), one finds recursion relations for the expansion coefficients, i.e. since

$$\partial_t \mathcal{F} = \sum_{n=0}^{\infty} \frac{\mathcal{F}_n'(t)}{x^n}, \quad \partial_x \mathcal{F} = -\sum_{n=2}^{\infty} \frac{(n-1)\mathcal{F}_{n-1}(t)}{x^n}, \quad \partial_{xx} \mathcal{F} = \sum_{n=3}^{\infty} \frac{(n-1)(n-2)\mathcal{F}_{n-2}(t)}{x^n}, \tag{4.2}$$

one obtains

$$\mathcal{F}_1 = -\kappa \mathcal{F}_0', \qquad \mathcal{F}_2 = -\frac{\kappa}{2}\mathcal{F}_1' = \frac{\kappa^2}{2}\mathcal{F}_0'', \qquad \mathcal{F}_3 = \frac{t\mathcal{F}_1 - \kappa \mathcal{F}_2'}{3} = -\frac{\kappa^3 \mathcal{F}_0''' + 2t\kappa \mathcal{F}_0'}{6}, \tag{4.3}$$

and

$$(n+1)\mathcal{F}_{n+1} = -\kappa \mathcal{F}_n' + (n-1)t\mathcal{F}_{n-1} - (n-1)(n-2)\mathcal{F}_{n-2}, \qquad n \geq 3. \tag{4.4}$$

Thus, all the functions $\mathcal{F}_n(t)$ can be recursively found in terms of $\mathcal{F}_0(t)$ and its derivatives. Recall that $\mathcal{F}(t,x)$ also satisfies [29] a first order ODE,

$$\kappa \partial_t \mathcal{F} + P(t,x)\partial_x \mathcal{F} + b(t,x)\mathcal{F} = 0, \tag{4.5}$$

where $P(t,x)$ and $b(t,x)$ are explicitly known for integer κ in terms of the entries of the Lax matrices eqs. (1.5), (1.6):

$$P(t,x) = -\kappa \frac{B_+}{L_+} = \sum_{k=1}^{\kappa} \frac{1}{x - Q_k(t)}, \tag{4.6}$$

$$b(t,x) = \frac{1}{2}\sum_{k=1}^{\kappa} \frac{\kappa Q_k' + t - Q_k^2 - 2R_k}{x - Q_k} - \frac{1}{2}\left(\frac{t^2}{2} + U(t) + \sum_{k=1}^{\kappa} Q_k\right), \tag{4.7}$$

where

$$R_k = \sum_{j \neq k}^{\kappa} \frac{1}{Q_k - Q_j}. \tag{4.8}$$

One can expand eq. (4.5) at large x as well for any κ and we will explore the full consequences of this elsewhere. For our current purposes we need only the first terms of this expansion, the limit of eq. (4.5) as $x \to \infty$, which yields

$$\kappa \mathcal{F}_0' - \frac{1}{2}\left(\frac{t^2}{2} + U(t) + e_1\right)\mathcal{F}_0 = 0, \tag{4.9}$$

where we used that

$$\sum_{k=1}^{\kappa} Q_k = e_1 \tag{4.10}$$

for integer κ by definition, however, eq. (4.9) holds for every κ, integer or not, with e_1 and U defined from expansion of $P(t,x)$ and $b(t,x)$ at large x which is valid and has the same form for all κ, unlike eq. (4.10). Now for the case at hand, $\kappa = 3$, eq. (4.9) says:

$$3(\ln \mathcal{F}_0)' = \frac{1}{2}\left(\frac{t^2}{2} + U(t) + e_1\right), \tag{4.11}$$

which implies a simple connection of $\mathcal{F}_0$ with functions considered in the previous sections. First, from eqs. (3.1) and (3.23) we have

$$e_1 = r + \frac{2q_2q_1}{q_2^2 - 1} = r + \frac{2\mu\nu}{\mu^2 - \chi^2}, \tag{4.12}$$

and, using eqs. (3.27) and (3.30),

$$e_1 = -\frac{q'}{q} + \frac{\nu}{\mu_+} + \frac{\nu}{\mu_-} \equiv -\frac{g'}{2g} + \frac{\nu}{\mu_+} + \frac{\nu}{\mu_-}. \tag{4.13}$$

Here and further on we denote $g = q^2$. Next, from eq. (3.9), (3.23) and (3.30) we find

$$U = u_r + \frac{2q_1}{q_2^2 - 1} = u_r + \frac{2\chi\nu}{\mu^2 - \chi^2} = u_r + \frac{\nu}{\mu_-} - \frac{\nu}{\mu_+}. \tag{4.14}$$

At last, using eqs. (3.14), (3.27) and (3.28), we express u_r as

$$u_r = -\frac{(r')^2}{2} + \frac{r^4}{2} - tr^2 - r = 2u + \frac{q'}{q} - \frac{t^2}{2} = 2u + \frac{g'}{2g} - \frac{t^2}{2}. \tag{4.15}$$

Substituting eqs. (4.13)–(4.15) into eq. (4.11) we finally obtain

$$3(\ln \mathcal{F}_0)' = u + \frac{\nu}{\mu_-}. \tag{4.16}$$

For further convenience, let us denote

$$h_-(t) = \frac{\nu}{\mu_-}, \qquad h_+(t) = \frac{\nu}{\mu_+}. \tag{4.17}$$

Consider again system (3.31)–(3.33). If we introduce functions

$$\sigma_- = g\mu_-^3, \qquad \sigma_+ = \frac{\mu_+^3}{g}, \tag{4.18}$$

then eqs. (3.31) and (3.32) can be rewritten as, respectively,

$$\sigma_+' = -h_+\sigma_+, \qquad \sigma_-' = h_-\sigma_-. \tag{4.19}$$

Taking cube of eqs. (4.17) and using eqs. (4.18), we get the relation

$$g^2 \frac{\sigma_+}{\sigma_-} \left(\frac{h_+}{h_-} \right)^3 = 1. \tag{4.20}$$

Eq. (3.33) can be rewritten in two ways, using eq. (4.17),

$$3g(h_+\mu_+)' = 2g^2\mu_- - 2u\mu_+, \tag{4.21}$$

$$3g(h_-\mu_-)' = 2g^2\mu_- - 2u\mu_+. \tag{4.22}$$

Dividing eq. (4.21) by μ_+ and eq. (4.22) by μ_-, and using eqs. (3.31), (3.32) and definitions (4.17), yields equations for h_+ and h_-, respectively,

$$3gh_+' + g'h_+ - gh_+^2 + 2u = 2g^2 \frac{h_+}{h_-}, \tag{4.23}$$

$$3gh_-' - g'h_- + gh_-^2 - 2g^2 = -2u \frac{h_-}{h_+}. \tag{4.24}$$

Multiplying eq. (4.23) by h_-, eq. (4.24) by h_+ and subtracting them gives equation which can be integrated to the already known relation (4.20). Expressing h_+ from eq. (4.24) and substituting into eq. (4.23) yields a second order ODE for h_- only with coefficients depending on g and u,

$$\frac{2uh_- - 3(3gh_-' - g'h_- + gh_-^2 - 2g^2)'}{3gh_-' - g'h_- + gh_-^2 - 2g^2} - h_- + 2\frac{g'}{g} - 3\frac{g}{u} = 0. \tag{4.25}$$

Using eqs. (3.28) rewritten as

$$u = \frac{(g')^2}{4g} - g^2 - tg, \qquad u' = -g, \tag{4.26}$$

and their consequence in the form

$$g'' = 6g^2 + 4tg + 2u = \frac{(g')^2}{g} + 2g^2 - 2u, \tag{4.27}$$

eq. (4.25) can be brought to its final form,

$$9h_-'' + 9\left(h_- + \frac{g}{u}\right) h_-' + h_-^3 + 3\frac{g}{u}h_-^2 - \left(3\frac{g'}{u} + 12g + 4t\right) h_- - 8g' - \frac{6g^2}{u} = 0. \tag{4.28}$$

Similarly, expressing h_- from eq. (4.23) and substituting into eq. (4.24) gives a second order ODE for h_+, which, after using eqs. (4.26) and (4.27), finally becomes

$$9g^2h_+'' - 3g^2h_+h_+' - g^2h_+^3 + 2gg'h_+^2 - [(g')^2 - 6g^3 + 4ug]h_+ - 6g^2 - 8ug' = 0. \tag{4.29}$$

One can derive the ODE for $\Phi \equiv 3(\ln \mathcal{F}_0)' = h_- + u$ from eq. (4.28):

$$9\Phi'' + 9\left(\Phi + \frac{g}{u} - u\right)\Phi' + \Phi^3 + 3\left(\frac{g}{u} - u\right)\Phi^2 +$$
$$+ \left(3u^2 - 3\frac{g'}{u} - 9g - 4t\right)\Phi + 4g' + \frac{3g^2}{u} + 4tu + 6ug - u^3 = 0. \tag{4.30}$$

It is more convenient for finding asymptotics of Φ as $t \to +\infty$.

The simplest form of the final equation is reached, however, if one uses $\eta = h_- + g/u$ as the dependent variable. Then eq. (4.28) acquires the form

$$9\eta'' + 9\eta\eta' + \eta^3 - 4(3\Gamma' + t)\eta - 8\Gamma'' - 2 = 0, \tag{4.31}$$

(which is the eq. (1.14) of the main theorem) where

$$\eta = h_- + \frac{g}{u}, \qquad \Phi = \eta - \Gamma, \qquad \Gamma = \frac{g}{u} - u. \tag{4.32}$$

To derive it we used that

$$\left(\frac{g}{u}\right)' = \frac{g'}{u} + \left(\frac{g}{u}\right)^2, \qquad \left(\frac{g}{u}\right)'' = 3\frac{gg'}{u^2} + 2\left(\frac{g}{u}\right)^3 + 6\frac{g^2}{u} + 4t\frac{g}{u} + 2. \tag{4.33}$$

However, eq. (4.31) turns out to be not convenient for finding the asymptotics of Φ as $t \to \pm\infty$.

4.1 Asymptotics of $\Phi(t)$ as $t \to +\infty$

When $t \to +\infty$ it is best to use eq. (4.30). One can verify that Φ has the following asymptotic expansion:

$$\Phi = \frac{e^{-4t^{3/2}/3}}{t^4} \sum_{n=0}^{\infty} \frac{\phi_n}{t^{3n/2}} \qquad \text{as } t \to +\infty. \tag{4.34}$$

Due to the exponential factor eq. (4.30) linearizes in this limit (as is the case for the involved Painlevé II itself) and takes form (after multiplying by u)

$$9u\Phi'' + 9g\Phi' - (3g' + 4tu)\Phi + 4ug' + 3g^2 + 4tu^2 = 0, \tag{4.35}$$

which can be solved by the series (4.34). The Painlevé functions expand in this limit as

$$q = \frac{e^{-2t^{3/2}/3}}{t^{1/4}} \sum_{n=0}^{\infty} \frac{C_n}{t^{3n/2}}, \qquad g = \frac{e^{-4t^{3/2}/3}}{t^{1/2}} \sum_{n=0}^{\infty} \frac{g_n}{t^{3n/2}}, \qquad u = \frac{e^{-4t^{3/2}/3}}{t} \sum_{n=0}^{\infty} \frac{u_n}{t^{3n/2}}, \tag{4.36}$$

where the coefficients are related by $g_0 = 2u_0$,

$$n \geq 1 : g_n = 2u_n + \frac{3n-1}{2}u_{n-1}, \quad g_n = \sum_{l=0}^{n} C_l C_{n-l}, \; C_{n+1} = -\frac{(1+6n)(5+6n)}{48(n+1)}C_n, \tag{4.37}$$

where $C_0 = \frac{1}{2\sqrt{\pi}}$ is known e.g. from [16]. Then we get

$$g' = e^{-4t^{3/2}/3} \sum_{n=0}^{\infty} \frac{(g')_n}{t^{3n/2}}, \qquad g^2 = \frac{e^{-8t^{3/2}/3}}{t} \sum_{n=0}^{\infty} \frac{(g^2)_n}{t^{3n/2}}, \tag{4.38}$$

where

$$(g')_0 = -2g_0, n \geq 1 : (g')_n = -(2g_n + (3n/2 - 1)g_{n-1}), \qquad (g^2)_n = \sum_{l=0}^{n} g_l g_{n-l}, \tag{4.39}$$

Substituting everything into eq. (4.35) one verifies that the first two orders in powers of t cancel identically, i.e.

$$4(ug')_0 + 3(g^2)_0 + 4(u^2)_0 = 0, \qquad 4(ug')_1 + 3(g^2)_1 + 4(u^2)_1 = 0, \tag{4.40}$$

(since $g_0 = 2u_0$, $g_1 = 2u_1 + u_0$, $(g')_0 = -2g_0$ etc.) and the others recursively determine coefficients ϕ_n, $n \geq 0$:

$$(32u_0 - 3(g')_0 - 18g_0)\phi_n = 4g_0\phi_n = -(4ug' + 3g^2 + 4u^2)_{n+2} -$$

$$-\sum_{l=0}^{n-1}[32u_{n-l} - 3(g')_{n-l} - 18g_{n-l} + 27(3+2l)u_{n-1-l} - 9(5+3l)g_{n-1-l}/2]\phi_l +$$

$$+9\sum_{l=0}^{n-2}(1+3l/2)(2+3l/2)u_{n-2-l}\phi_l = 0, \quad n \geq 0. \tag{4.41}$$

Thus one obtains e.g.

$$4g_0\phi_0 = -4(u_0(g')_2 + u_1(g')_1 + u_2(g')_0) - 3(2g_0g_2 + g_1^2) - 8(2u_0u_2 + u_1^2), \tag{4.42}$$

and, using the above relations between coefficients, $\phi_0 = (3g_1 + g_0)/8 = (6C_0C_1 + C_0^2)/8 = 3C_0^2/64 = 3/(256\pi)$. Upon dividing by 3 and rescaling back $t \to 3^{2/3}t$ this matches predictions from [18, 10].

4.2 Asymptotics of $\Phi(t)$ as $t \to -\infty$

It is convenient to use eq. (4.28) for the function h_- here, we multiply it by u to clear denominators. The expansions for Painlevé functions in this limit are

$$q = \sum_{n=0}^{\infty} C_n \left(-\frac{t}{2}\right)^{-3n+1/2}, \quad g = \sum_{n=0}^{\infty} g_n \left(-\frac{t}{2}\right)^{-3n+1}, \quad u = \sum_{n=0}^{\infty} u_n \left(-\frac{t}{2}\right)^{-3n+2}, \tag{4.43}$$

where the coefficients are related by

$$g_n = \left(1 - \frac{3n}{2}\right) u_n, \qquad g_n = \sum_{l=0}^{n} C_l C_{n-l},$$

$$4C_n = \frac{36(n-1)^2 - 1}{16} C_{n-1} - \sum_{k+l+m=n; k,l,m\geq 0}^{k,l,m\leq n-1} 2C_k C_l C_m, \qquad C_0 = 1. \tag{4.44}$$

Then

$$g' = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{(g')_n}{(-t/2)^{3n}}, \qquad g^2 = \frac{t^2}{4}\sum_{n=0}^{\infty} \frac{(g^2)_n}{(-t/2)^{3n}}, \qquad ug' = -\frac{t^2}{8}\sum_{n=0}^{\infty} \frac{(ug')_n}{(-t/2)^{3n}}, \tag{4.45}$$

where

$$(g')_n = \frac{(3n-1)(3n-2)}{2} u_n, \qquad (g^2)_n = \sum_{l=0}^{n} g_l g_{n-l}, \qquad (ug')_n = \sum_{l=0}^{n} u_l (g)'_{n-l}. \tag{4.46}$$

Substituting the series solutions of the form $h_- = \tilde{h}_0(-t)^\alpha + \dots$, where α is the leading exponent, into eq. (4.28), one finds that there are two possibilities, $\alpha = 1/2$ and $\alpha = -1$. To describe the Tracy-Widom distribution, one has to pick $\alpha = 1/2$ (unlike we did in the first version of the paper) to match the results obtained by other methods, see below[1]. Then eq. (4.28) gives $\tilde{h}_0^2 = 2$, and again there are two choices and the right one is $\tilde{h}_0 = -\sqrt{2}$. This leads to the solution series such that

$$h_- = -(-2t)^{1/2}\sum_{n=0}^{\infty} \frac{h_n}{(-t/2)^{3n/2}}, \qquad h'_- = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(h')_n}{(-t/2)^{(3n+1)/2}}, \qquad (h')_n = (1-3n)h_n, \tag{4.47}$$

$$h''_- = \frac{1}{8}\sum_{n=0}^{\infty} \frac{(h'')_n}{(-t/2)^{3(n+1)/2}}, \qquad (h'')_n = (1-9n^2)h_n. \tag{4.48}$$

Substituting everything into eq. (4.28) one finds $-u_0 h_0^3 + 3u_0 g_0 h_0 - 2u_0 h_0 = 0$ for $n = 0$ and, since $u_0 = g_0 = 1$, one gets $h_0^2 = 1$ and chooses now $h_0 = 1$. Then also $(h')_0 = (h'')_0 = h_0 = 1$ by eqs. (4.50), (4.51). The general recursion relation for the coefficients is

$$8[-(uh_-^3)_n + 3(ugh_-)_n - 2(uh)_n] + 4[-\frac{9}{4}(uh_- h'_-)_{n-1} + 3(gh_-^2)_{n-1} + (g'u)_{(n-1)/2} - \frac{3}{2}(g^2)_{(n-1)/2}] +$$

[1] We are very grateful to Peter Forrester for pointing out the discrepancies to the author immediately after the first version of the paper appeared online.

$$+\frac{9}{8}(uh''_-)_{n-2}+\frac{9}{2}(gh'_-)_{n-2}-3(g'h_-)_{n-2}=0, \tag{4.49}$$

where it is implied that

$$(fh_-^k)_n=\sum_{j=0}^{n} f_{(n-j)/2}(h_-^k)_j, \quad (h_-^k)_j=\sum_{m_1+\dots+m_k=j} h_{m_1}\dots h_{m_k}, \quad f=u,g,g',ug,$$

$$(uh_-h'_-)_n=\sum_{j=0}^{n} u_{(n-j)/2}\sum_{k=0}^{j} h_k(h')_{j-k}, \quad (uh''_-)_n=\sum_{j=0}^{n} u_{(n-j)/2}(h'')_j, \quad (gh'_-)_n=\sum_{j=0}^{n} g_{(n-j)/2}(h')_j,$$

and all the quantities with half-integer indices are zero. E.g. for $n=1$ we have

$$8[-u_0(h_-^3)_1+3u_0g_0h_1-2u_0h_1]+4[-\frac{9}{4}u_0h_0(h')_0+3g_0h_0^2+(g')_0u_0-\frac{3}{2}g_0^2]=0, \tag{4.50}$$

and, using that $(h'')_0=(h')_0=h_0=u_0=g_0=(g')_0=1$, $(h_-^3)_1=3h_0^2h_1=3h_1$, we obtain $h_1=1/16$. Thus, the first terms of expansion for Φ are

$$\Phi=u+h_-=\frac{t^2}{4}-\frac{1}{8t}+\dots-(-2t)^{1/2}-(-2t)^{1/2}\frac{1}{16(-t/2)^{3/2}}+\dots=$$

$$=\frac{t^2}{4}-(-2t)^{1/2}+\frac{1}{8t}+\dots, \tag{4.51}$$

and, after taking into account that $(\ln\mathcal{F}_0)'=\Phi/3$ and rescaling back $t\to 3^{2/3}t$, see the main result, this matches the known results [9], see also formula (2.16) in [19],

$$\ln F_{TW}^{\beta}=-\beta\frac{|t|^3}{24}+\frac{\sqrt{2}(\beta/2-1)}{3}|t|^{3/2}+\frac{\beta/2+2/\beta-3}{8}\ln|t|+\dots=-\frac{|t|^3}{4}+\frac{2\sqrt{2}}{3}|t|^{3/2}+\frac{1}{24}\ln|t|+\dots$$

5 Painlevé Analysis of $\kappa=3$ system.

It is interesting and illuminating to verify if the system (1.15)–(1.17) satisfies the Painlevé property, i.e. if its solutions are single-valued. The author would like to thank M. Ablowitz for the suggestion to do it.

As is well-known, see e.g. [1, 16], the only singularities of all solutions of Painlevé II in the complex plane are simple poles, and all poles of the Hastings-McLeod solution q lie in two symmetric sectors of angle $\pi/3$ around imaginary axis with the vertex at the origin [24]. Every solution of Painlevé II q always has a Laurent expansion around a point $t=t_0$,

$$q = z^l \sum_{n=0}^{\infty} a_n z^n, \tag{5.1}$$

where $z = t - t_0$. All its poles and zeros are simple, see e.g. [1, 16], so the exponent l can be -1, 0 or 1. Consider the corresponding expansions for the functions entering the coefficients of eqs. (1.15)–(1.17). As follows from eq. (5.1),

$$g \equiv q^2 = z^{2l} \sum_{n=0}^{\infty} g_n z^n, \qquad g' = z^{2l-1} \sum_{n=0}^{\infty} (2l+n) g_n z^n, \tag{5.2}$$

and, using also that $u' = -q^2 = -g$, we find that

$$u = u_0 - z^{2l+1} \sum_{n=0}^{\infty} \frac{g_n}{2l+1+n} z^n, \tag{5.3}$$

Then we consider expansions

$$\mu_+ = z^{m_+} \sum_{n=0}^{\infty} K_n z^n, \qquad \mu_- = z^{m_-} \sum_{n=0}^{\infty} M_n z^n, \qquad \nu = z^{m_\nu} \sum_{n=0}^{\infty} S_n z^n. \tag{5.4}$$

Substituting all the expansions into the eqs. (1.15)–(1.17), we obtain, respectively,

$$3z^{m_+-1} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}(j+m_+)K_j = z^{m_+-1} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} (2l+n-j) g_{n-j} K_j - z^{m_\nu} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j} S_j, \tag{5.5}$$

$$3z^{m_--1} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}(j+m_-)M_j = -z^{m_--1} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} (2l+n-j) g_{n-j} M_j + z^{m_\nu} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j} S_j, \tag{5.6}$$

$$3z^{m_\nu+2l-1} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}(j+m_\nu)S_j = 2z^{m_-+4l} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} (g^2)_{n-j} M_j -$$
$$-2u_0 z^{m_+} \sum_{n=0}^{\infty} K_n z^n + 2z^{m_++2l+1} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \frac{g_{n-j}}{2l+1+n-j} K_j, \tag{5.7}$$

(It follows from eqs. (5.5), (5.6) that $m_\nu \geq m_+ - 1$ and $m_\nu \geq m_- - 1$ in general.) It is convenient to proceed from here considering separately the cases when q has pole ($l = -1$) and q has zero ($l = 1$). As for the case of a regular point of q ($l = 0$), the solutions of linear ODEs are always regular at the regular points of their coefficients.

5.1 Local behavior near a pole of q.

At a simple pole of q, one has $a_0 = \pm 1$, $a_1 = 0$, therefore $g_0 = a_0^2 = 1$ and $g_1 = 2a_0a_1 = 0$. Now eqs. (5.5)–(5.7) read, respectively,

$$z^{m_\nu}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}S_j = z^{m_+-1}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}(-3m_+ - 2 + n - 4j)K_j, \tag{5.8}$$

$$z^{m_\nu}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}S_j = z^{m_--1}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}(3m_- - 2 + n + 2j)M_j. \tag{5.9}$$

$$3z^{m_\nu-1}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}(j+m_\nu)S_j = 2z^{m_--2}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} (g^2)_{n-j}M_j -$$

$$-2u_0 z^{m_++2}\sum_{n=0}^{\infty} K_n z^n + 2z^{m_++1}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \frac{g_{n-j}}{n-j-1}K_j. \tag{5.10}$$

Analyzing their first terms, one can conclude that there are two different possibilities for the values of the exponents: either $m_+ = m_- = m$, $m_\nu = m-1$, or $m_+ = m_\nu = -2/3$, $m_- = 1/3$.

Case $m_+ = m_- = m$, $m_\nu = m - 1$.

Eq. (5.9) gives coefficients S_n in terms of M_n recursively (we use the facts that $g_0 = 1$ and $g_1 = 0$):

$$S_0 = (3m-2)M_0, \qquad S_1 = (3m+1)M_1, \tag{5.11}$$

$$n \ge 2: \quad S_n = (3(n+m)-2)M_n + \sum_{j=0}^{n-2} g_{n-j}[(n-2+3m+2j)M_j - S_j]. \tag{5.12}$$

Eq. (5.8) gives similar relations between coefficients S_n and K_n, and comparing the two outcomes, we get the recursive expression for coefficients K_n in terms of M_n,

$$(3(n+m)+2)K_n = -(3(n+m)-2)M_n - \sum_{j=0}^{n-2} g_{n-j}[(n-2+3m+2j)M_j - (n-2-3m-4j)K_j], \tag{5.13}$$

which is valid for all $n \ge 0$ if the terms with negative indices or sum with upper limit less than the lower are understood as absent. Finally, we write down the corresponding outcome of eq. (5.10) (again understood in the above sense and thus valid for all $n \ge 0$):

$$3(n+m-1)S_n = 2M_n - 2K_{n-3} +$$

$$+\sum_{j=0}^{n-2}\left[g_{n-j}(4M_j-3(j+m-1)S_j)+2(g^2)_{n-4-j}M_j+2\frac{g_{n-4-j}}{n-5-j}K_j\right]. \tag{5.14}$$

Now we substitute S_n from eq. (5.12) into eq. (5.14) and obtain the recursion relation which determines the coefficients M_n,

$$9(n+m-4/3)(n+m-1/3)M_n=\sum_{j=0}^{n-2}g_{n-j}[3(n-j)S_j+(4-3(n+m-1)(3(n+m)-2-2(n-j)))M_j]$$

$$-2K_{n-3}+2\sum_{j=0}^{n-4}\left[(g^2)_{n-4-j}M_j+\frac{g_{n-3-j}}{n-4-j}K_l\right]. \tag{5.15}$$

Putting $n=0$ in eq. (5.15), since the first coefficient M_0 is non-zero by definition, we get

$$(m-4/3)(m-1/3)=0,$$

i.e. the possible exponents m are $m=4/3$ or $m=1/3$. Putting $n=1$ instead gives

$$(m-1/3)(m+2/3)M_1=0,$$

which means that either $m=1/3$ and M_1 remains undetermined or, if $m=4/3$, then $M_1=0$. One can see that in both cases $m=4/3$ and $m=1/3$ all the other coefficients M_n, $n\geq 2$, as well as all coefficients S_n and K_n are uniquely determined from eq. (5.15) together with eqs. (5.12) and (5.13).

Case $m=4/3$: only one constant M_0 is free, $M_1=0$ (which entails also $S_1=K_1=0$) and the other coefficients are recursively determined by

$$S_n=(3n+2)M_n+\sum_{j=0}^{n-2}g_{n-j}[(n+2+2j)M_j-S_j]. \tag{5.16}$$

$$3(n+2)K_n=-(3n+2)M_n-\sum_{j=0}^{n-2}g_{n-j}[(n+2+2j)M_j-(n-6-4j)K_j], \tag{5.17}$$

$$9n(n+1)M_n=\sum_{j=0}^{n-2}g_{n-j}[3(n-j)S_j+(4-(3n+1)(n+2+2j))M_j]-2K_{n-3}+$$

$$+2\sum_{j=0}^{n-4}\left[(g^2)_{n-4-j}M_j+\frac{g_{n-3-j}}{n-4-j}K_j\right]. \tag{5.18}$$

Case $m = 1/3$: two constants, M_0 and M_1 are free and the other coefficients are recursively determined by

$$S_n = (3n-1)M_n + \sum_{j=0}^{n-2} g_{n-j}[(n-1+2j)M_j - S_j]. \tag{5.19}$$

$$3(n+1)K_n = -(3n-1)M_n - \sum_{j=0}^{n-2} g_{n-j}[(n-1+2j)M_j - (n-3-4j)K_j], \tag{5.20}$$

$$9n(n-1)M_n = \sum_{j=0}^{n-2} g_{n-j}[3(n-j)S_j + (4-(3n-2)(n-1+2j))M_j] - 2K_{n-3} +$$

$$+2\sum_{j=0}^{n-4}\left[(g^2)_{n-4-j}M_j + \frac{g_{n-3-j}}{n-4-j}K_j\right]. \tag{5.21}$$

Case $m_+ = m_\nu = -2/3$, $m_- = 1/3$.
Then in eqs. (5.8), (5.9) the constant K_0 remains free (undetermined). Eq. (5.9) yields $S_0 = -M_0$, and, for $n \geq 1$,

$$S_n = (3n-1)M_n + \sum_{j=0}^{n-1} g_{n-j}[(n+2j-1)M_j - S_j]. \tag{5.22}$$

The difference of eqs. (5.8) and (5.9) gives recursion ($n \geq 0$)

$$3(n+1)K_{n+1} = \sum_{j=0}^{n}[(n-4j+1)g_{n+1-j}K_j - (n+2j-1)g_{n-j}M_j] \tag{5.23}$$

(e.g. $3K_1 = M_0$). At last, eq. (5.10) leads to

$$(3n-2)S_n = 2M_n + \sum_{j=0}^{n-1}[2(g^2)_{n-j}M_j - (3j-2)g_{n-j}S_j] - 2u_0K_{n-3} + 2\sum_{j=0}^{n-2}\frac{g_{n-2-j}}{n-3-j}K_j, \tag{5.24}$$

valid for $n \geq 3$ and

$$S_0 = -M_0, \qquad S_1 = 2M_1, \qquad 2S_2 = M_2 + 2g_2M_0 + g_2S_0 - K_0. \tag{5.25}$$

The first two relations in eq. (5.25) are consistent with eq. (5.22), while the third, when compared to its $n = 2$ case, determines M_2 (or K_0) by

$$K_0 = -3(3M_2 + g_2M_0). \tag{5.26}$$

The coefficients M_n for $n \geq 3$ are recursively determined from substituting eq. (5.22) into eq. (5.24),

$$9n(n-1)M_n = \sum_{j=0}^{n-1}[(2(g^2)_{n-j} - (3n-2)(n+2j-1)g_{n-j})M_j - 3(n-j)g_{n-j}S_j] -$$

$$-2u_0K_{n-3} + 2\sum_{j=0}^{n-2}\frac{g_{n-2-j}}{n-3-j}K_j. \qquad (5.27)$$

Thus, we obtain the series solution with three free constants, e.g. M_0, M_1 and K_0.

We always have three linearly independent solutions of the system (1.15)–(1.17) locally around each simple pole of a function q, the solution of Painlevé II without constant term.

5.2 Local behavior near a zero of q.

Now $l = 1$ and from Painlevé II and eq. (3.28) we find that

$$u_0 = g_0, \qquad g_1 = a_1 = 0. \qquad (5.28)$$

Now in place of eqs. (5.8)–(5.10) we have

$$z^{m_\nu}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}S_j = z^{m_+-1}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}(2-3m_+ + n - 4j)K_j, \qquad (5.29)$$

$$z^{m_\nu}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}S_j = z^{m_--1}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}(3m_- + 2 + n + 2j)M_j. \qquad (5.30)$$

$$3z^{m_\nu+1}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} g_{n-j}(j+m_\nu)S_j = 2z^{m_-+4}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} (g^2)_{n-j}M_j -$$

$$-2u_0z^{m_+}\sum_{n=0}^{\infty} K_n z^n + 2z^{m_++3}\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n}\frac{g_{n-j}}{n-j+3}K_j. \qquad (5.31)$$

One also has two cases here to consider separately, either $m_+ = m_- = m$, $m_\nu = m-1$ as the first case for poles, or $m_+ = 1/3$, $m_\nu = m_- = -2/3$.

Case $m_+ = m_- = m$, $m_\nu = m-1$.

Eq. (5.29) yields

$$g_0S_n = g_0(2-3(n+m))K_n + \sum_{j=0}^{n-1} g_{n-j}[(2-3m+n-4j)K_j - S_j], \qquad (5.32)$$

and similarly the difference of eqs. (5.29) and (5.30) gives

$$g_0(3(n+m)+2)M_n = g_0(2-3(n+m))K_n + \sum_{j=0}^{n-1} g_{n-j}[(2-3m+n-4j)K_j - (2+3m+n+2j)M_j]. \tag{5.33}$$

At last, eq. (5.31) yields

$$3g_0(n+m-1)S_n = -2u_0K_n - 3\sum_{j=0}^{n-1} g_{n-j}(j+m-1)S_j + 2\sum_{j=0}^{n-3} \frac{g_{n-3-j}}{n-j}K_j + 2\sum_{j=0}^{n-4}(g^2)_{n-4-j}M_j, \tag{5.34}$$

and substituting eq. (5.32) into eq. (5.34) finally results in the recursion for K_n,

$$[3g_0(n+m-1)(2-3(n+m))+2u_0]K_n = 3\sum_{j=0}^{n-1} g_{n-j}[(n-j)S_j - (n+m-1)(2-3m+n-4j)K_j]+$$

$$+2\sum_{j=0}^{n-3} \frac{g_{n-3-j}}{n-j}K_j + 2\sum_{j=0}^{n-4}(g^2)_{n-4-j}M_j, \tag{5.35}$$

Again, putting $n = 0$ in eq. (5.35) implies (since $K_0 \neq 0$) that the possible exponents are $m = 4/3$ or $m = 1/3$. But now, using eq. (5.28), the $n = 1$ component of recursion (5.35) reads:

$$3g_0(m-1/3)(m+2/3)K_1 = -g_1[S_0 + 3m(m-1)K_0] = 0,$$

so either $m = 4/3$ and $K_1 = 0$ or $m = 1/3$ and K_1 remains undetermined. Eq. (5.32) with $n = 0$ gives

$$S_0 = (2-3m)K_0. \tag{5.36}$$

If $m = 4/3$, then eq. (5.35) becomes

$$9g_0n(n+1)K_n = -3\sum_{j=0}^{n-1} g_{n-j}[(n-j)S_j + (n+1/3)(4j+2-n)K_j]+$$

$$-2\sum_{j=0}^{n-3} \frac{g_{n-3-j}}{n-j}K_j - 2\sum_{j=0}^{n-4}(g^2)_{n-4-j}M_j. \tag{5.37}$$

If $m = 1/3$, then, after using also eq. (5.28), eq. (5.35) becomes

$$-9g_0n(n-1)K_n = 3\sum_{j=0}^{n-1} g_{n-j}[(n-j)S_j - (n+m-1)(2-3m+n-4j)K_j]+$$

$$+2\sum_{j=0}^{n-3}\frac{g_{n-3-j}}{n-j}K_j+2\sum_{j=0}^{n-4}(g^2)_{n-4-j}M_j, \tag{5.38}$$

Thus, the solution with $m=4/3$ has one free constant K_0 and the solution with $m=1/3$ has two free constants K_0 and K_1. Combined, they give a solution with three free constants.

Case $m_+=1/3$, $m_\nu=m_-=-2/3$. Eqs. (5.29)–(5.31) give the following recursion relations:

$$g_0S_n=g_0(1-3n)K_n+\sum_{j=0}^{n-1}g_{n-j}[(1+n-4j)K_j-S_j], \tag{5.39}$$

for $n\ge 1$ and $S_0=K_0$,

$$3g_0nM_n=\sum_{j=0}^{n-1}g_{n-j}[(1+n-4j)K_j-(n+2j)M_j], \tag{5.40}$$

for $n\ge 1$ and M_0 is undetermined;

$$g_0(3n-2)S_n=-2u_0K_n-\sum_{j=0}^{n-1}g_{n-j}(3j-2)S_j+2\sum_{j=0}^{n-3}\frac{g_{n-3-j}}{n-j}K_j+2\sum_{j=0}^{n-4}(g^2)_{n-4-j}M_j. \tag{5.41}$$

Substituting eq. (5.39) into eq. (5.41) and using $u_0=g_0$ gives the recursion for K_n,

$$9g_0n(n-1)K_n=\sum_{j=0}^{n-1}g_{n-j}[(3n-2)(1+n-4j)K_j-3(n-j)S_j]-2\sum_{j=0}^{n-3}\frac{g_{n-3-j}}{n-j}K_j-2\sum_{j=0}^{n-4}(g^2)_{n-4-j}M_j.$$

Again, since $g_1=0$, K_0 and K_1 remain undetermined and here we have another solution with three free constants M_0, K_0 and K_1.

Thus, all the solutions of the system have non-integer exponents, so, strictly speaking, the system does not have the Painlevé property since it has movable algebraic branch points. This is in common with Garnier systems [21], and in fact the functions Q_k from eqs. (1.8)–(1.12) have many properties of apparent singularities of Fuchsian ODEs making the linear problem for the Garnier systems, see [21, 30]. However, the cubes of our functions $\mu_\pm$ and ν are meromorphic in the complex plane as are their ratios, therefore we have the Painlevé property for them.

A nonlinear integrable ODE without the Painlevé property appeared recently in [12] where it described isomonodromic deformation dynamics with respect to a parameter in equation P_I^2, the second equation in Painlevé I hierarchy, a fourth order ODE which universally appears under scaling around generic gradient catastrophe points of hyperbolic nonlinear PDEs. It seems that these two examples are just tips of a large array of integrable systems without Painlevé property yet to be identified.

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Appendix: Lax pairs for $\kappa = 1, 2$

$\kappa = 1$.

Then, according to the above formulas of section 1, we have

$$L_+ = \phi(t)(x - Q(t)), \qquad B_+ = -\frac{\partial_x L_+}{\kappa} = -\phi(t), \tag{A1}$$

$$B_t = -\int_0^x \partial_t v dx + b_t(t) = -x + b_t(t), \qquad b_t(t) = \frac{t^2}{2} + U(t) - \frac{\phi'(t)}{\phi}, \tag{A2}$$

$$L_d = -Q'(t), \qquad B_d = \frac{\phi'(t)}{\phi}, \tag{A3}$$

and therefore

$$L_1 = -\frac{v + Q'}{2} = \frac{x^2 - t}{2} - \frac{Q'}{2}, \qquad L_2 = -\frac{v - Q'}{2} = \frac{x^2 - t}{2} + \frac{Q'}{2}, \tag{A4}$$

$$B_1 = -\frac{x}{2} + \frac{1}{2}\left(\frac{t^2}{2} + U\right), \qquad B_2 = -\frac{x}{2} + \frac{1}{2}\left(\frac{t^2}{2} + U\right) - \frac{\phi'(t)}{\phi}. \tag{A5}$$

Substituting the expression for L_-, we obtain

$$L_- = -\frac{\kappa B_d + \partial_x L_d + L_d^2/2 + V(v)}{2L_+} = \frac{x^4 - 2tx^2 - 2x - (Q')^2 - 2U}{4\phi(x - Q)} =$$

$$= \frac{x^3 + Qx^2 + (Q^2 - 2t)x + Q(Q^2 - 2t) - 2}{4\phi} + \frac{Q^2(Q^2 - 2t) - 2Q - (Q')^2 - 2U}{4\phi(x - Q)},$$

however, the only nonpolynomial term with would-be pole is in fact equal to zero: equations

$$U = \frac{Q^4}{2} - tQ^2 - Q - \frac{(Q')^2}{2}, \qquad U' = -Q^2, \tag{A6}$$

give the right Painlevé II equation for Q:

$$Q'' = 2Q^3 - 2tQ - 1, \tag{A7}$$

which is satisfied by $Q = -q'/q$, q being the Hastings-McLeod solution of Painlevé II with free parameter zero:

$$q'' = 2q^3 + tq. \tag{A8}$$

As we will see, this cancellation of terms which would make L_- have poles in x is the simplest example of the general phenomenon for the above Lax pairs constructed in [29]. Thus,

$$L_- = \frac{x^3 + Qx^2 + (Q^2 - 2t)x + Q(Q^2 - 2t) - 2}{4\phi}, \tag{A9}$$

and now, substituting the explicit expression for B_-, we meet another example of similar cancellation of polar terms:

$$B_- = -\frac{2\partial_x L_+ L_- + \kappa\partial_t L_d - \kappa\partial_x B_d}{2L_+} = -\frac{x^4 - 2tx^2 - 4Q(Q^2 - t)x + Q^2(3Q^2 - 2t)}{4\phi(x - Q)^2},$$

but $x^4 - 2tx^2 - 4Q(Q^2 - t)x + Q^2(3Q^2 - 2t) = (x - Q)^2(x^2 + 2Qx + 3Q^2 - 2t)$, so finally

$$B_- = -\frac{x^2 + 2Qx + 3Q^2 - 2t}{4\phi}. \tag{A10}$$

It is convenient to express everything in terms of q instead of $Q = -q'/q$, and to choose the arbitrary function $\phi(t)$ as $\phi = -q$, then

$$L_+ = -qx - q', \qquad B_+ = q, \qquad L_d = -\left(\frac{q'}{q}\right)^2 + 2q^2 + t, \qquad B_d = \frac{q'}{q}, \tag{A11}$$

and, introducing function $u(t)$ as in eq. (3.28) we get

$$U + \frac{t^2}{2} = 2u + \frac{q'}{q}. \tag{A12}$$

Finally we obtain the Lax pair for $\kappa = 1$ in terms of Hastings-McLeod $q(t)$:

$$L = \begin{pmatrix} L_1 & L_+ \\ L_- & L_2 \end{pmatrix} = \begin{pmatrix} \frac{x^2-t}{2} + \frac{1}{2}\left(-\left(\frac{q'}{q}\right)^2 + 2q^2 + t\right) & -qx - q' \\ -\frac{x^3 - q'x^2/q + ((q'/q)^2 - 2t)x - q'((q'/q)^2 - 2t)/q - 2}{4q} & \frac{x^2-t}{2} - \frac{1}{2}\left(-\left(\frac{q'}{q}\right)^2 + 2q^2 + t\right) \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 & B_+ \\ B_- & B_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-x + 2u + \frac{q'}{q}) & q \\ \frac{x^2 - 2q'x/q + 3(q'/q)^2 - 2t}{4q} & \frac{1}{2}(-x + 2u - \frac{q'}{q}) \end{pmatrix}.$$

We note that it is different from the Baik-Rains [5] pair appeared in this context in [4, 8]:

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & q \\ q & -x \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix},$$

$$\partial_x \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} q^2 & -qx - q' \\ -qx + q' & x^2 - t - q^2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

$\kappa = 2.$

Here we have $Q_1(t) = -Q_2(t) = Q(t)$ (Q is now different from the previous section, see below). From general formulas of section 1 we get

$$L_+ = \phi(t)(x - Q_1)(x - Q_2) = \phi(x^2 - Q^2), \qquad B_+ = -\frac{\partial_x L_+}{2} = -\phi(t)x, \tag{A13}$$

$$B_t = -\int_0^x \partial_t v dx + b_t(t) = -x + b_t(t), \qquad b_t(t) = \frac{1}{2}\left(\frac{t^2}{2} + U(t)\right) - \frac{\phi'(t)}{\phi}, \tag{A14}$$

$$L_d = -\left(2\frac{Q'}{Q} - \frac{1}{Q^2}\right)x, \qquad B_d = \frac{\phi'(t)}{\phi} + 2\frac{Q'}{Q} - \frac{1}{Q^2}, \tag{A15}$$

and therefore

$$L_1 = \frac{x^2 - t}{2} - \left(2\frac{Q'}{Q} - \frac{1}{Q^2}\right)\frac{x}{2}, \qquad L_2 = \frac{x^2 - t}{2} + \left(2\frac{Q'}{Q} - \frac{1}{Q^2}\right)\frac{x}{2}, \tag{A16}$$

$$B_1 = -\frac{x}{2} + \frac{1}{4}\left(\frac{t^2}{2} + U\right) + \frac{1}{2}\left(2\frac{Q'}{Q} - \frac{1}{Q^2}\right), \; B_2 = -\frac{x}{2} + \frac{1}{4}\left(\frac{t^2}{2} + U\right) - \frac{1}{2}\left(2\frac{Q'}{Q} - \frac{1}{Q^2}\right) - \frac{\phi'(t)}{\phi}. \tag{A17}$$

$$f_v = -\frac{x^4}{2} + tx^2 + U - \frac{2\phi'}{\phi}, \tag{A18}$$

Substituting the expression for L_- from eq. (1.5), we obtain

$$L_- = \frac{x^4/2 - (2(Q' - 1/2Q)^2 + tQ^2)\,x^2/Q^2 - U - 2Q'/Q + 1/Q^2}{2\phi(x^2 - Q^2)} =$$

$$= \frac{x^2 + Q^2 - 2t - (2Q'/Q - 1/Q^2)^2}{4\phi} + \frac{Q^2(Q^2 - 2t) - 4(Q')^2 + 1/Q^2 - 2U}{4\phi(x^2 - Q^2)},$$

but again the last fraction is equal to zero: equations

$$U = \frac{(Q^2)^2}{2} - tQ^2 - \frac{((Q^2)')^2}{2Q^2} + \frac{1}{2Q^2}, \qquad U' = -Q^2, \tag{A19}$$

lead to the following Painlevé II equation for Q^2:

$$2Q^2(Q^2)'' - ((Q^2)')^2 = 2(Q^2)^2(Q^2 - t) - 1, \tag{A20}$$

which is satisfied by $Q^2 = 2q^2 + 2q' + t$ with the same q satisfying eq. (A8). Thus

$$L_- = \frac{x^2 + Q^2 - 2t - (2Q'/Q - 1/Q^2)^2}{4\phi}, \tag{A21}$$

and now, substituting the explicit expression for B_- from eq. (1.6),

$$2B_- = \frac{-x\left(x^2 + Q^2 - 2t - (2Q'/Q - 1/Q^2)^2\right)/2 + x(2Q'/Q - 1/Q^2)'}{\phi(x^2 - Q^2)},$$

and, using eqs. (A19) and (A20), this simplifies to just

$$B_- = -\frac{x}{4\phi}. \tag{A22}$$

It is now convenient to choose the arbitrary function $\phi(t)$ so that

$$\phi'/\phi = -q,$$

then express everything in terms of q instead of Q, $Q^2 = 2q^2 + 2q' + t$. To this end, we record

$$(Q^2)' = 2qQ^2 + 1, \qquad \Longrightarrow \qquad 2\frac{Q'}{Q} - \frac{1}{Q^2} = 2q,$$

$$L_+ = \phi(t)(x^2 - 2q^2 - 2q' - t), \qquad B_+ = -\phi(t)x,$$

$$L_d = -2qx, \qquad B_d = q, \qquad L_- = \frac{x^2 + 2q' - t - 2q^2}{4\phi} \tag{A23}$$

and, again introducing function $u(t)$ given by eq. (3.28), we get

$$U + \frac{t^2}{2} = 2(u - q), \qquad B_t = -x + u. \tag{A24}$$

Finally we obtain the Lax pair for $\kappa = 2$ in terms of Hastings-McLeod $q(t)$, which is exactly the pair we derived in [28] by the hard-to-soft edge limit transition from Lax pair for quantum Painlevé III [23] describing the hard edge for beta ensembles [25, 27]:

$$L = \begin{pmatrix} L_1 & L_+ \\ L_- & L_2 \end{pmatrix} = \begin{pmatrix} \frac{x^2-t}{2} - qx & \phi(t)(x^2 - 2q^2 - 2q' - t) \\ \frac{x^2+2q'-t-2q^2}{4\phi} & \frac{x^2-t}{2} + qx \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 & B_+ \\ B_- & B_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-x + u + q) & -\phi(t)x \\ -\frac{x}{4\phi} & \frac{1}{2}(-x + u - q) \end{pmatrix}.$$

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