Hyperbolical geometric quantum phase and topological dual mass

Jian-Qi Shen *
Zhejiang Institute of Modern Physics and Department of Physics, Zhejiang University, Hangzhou 310027, P.R. China
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In this note we show the existence of the hyperbolical geometric quantum phase that is different from the ordinary trigonometric geometric quantum phase. Gravitomagnetic charge (dual mass) is the gravitational analogue of magnetic monopole in Electrodynamics; but, as will be shown here, it possesses more interesting and significant features, *e.g.*, it may constitute the dual matter that has different gravitational properties compared with mass. In order to describe the space-time curvature due to the topological dual mass, we construct the dual Einstein's tensor. Further investigation shows that gravitomagnetic potentials caused by dual mass are respectively analogous to the trigonometric and hyperbolic geometric phase. The study of the geometric phase and dual mass provides a valuable insight into the time evolution of quantum systems and the topological properties in General Relativity.

Keywords: Hyperbolical geometric quantum phase, topological dual mass

Both geometric phase [1] of wave function in Quantum Mechanics and gravitomagnetic charge (topological dual charge of mass) in the general theory of relativity reveal Nature's geometric or global properties. Differing from the dynamical phase, geometric phase depends only on the geometric nature of the pathway along which the quantum system evolves [2]. Geometric phase exists in time-dependent quantum systems or systems whose Hamiltonian possesses evolution parameters [3]. As is well known, the dynamical phase of wave function in Quantum Mechanics is dependent on dynamical quantities such as energy, frequency, coupling coefficients and velocity of a particle or a quantum system, while the geometric phase is immediately independent of these physical quantities. When Berry found that the wave function would give rise to a non-integral phase (Berry's phase) in quantum adiabatic process [1], geometric phase problems attract attention of many physicists in various fields such as gravity theory [4], differential geometry [2], atomic and molecular physics [5], nuclear physics [5], quantum optics [6], condensed matter physics [7] and molecular reaction (molecular chemistry) [5] as well. In many simple quantum systems such as an electron possessing intrinsic magnetic moment interacting with a time-dependent magnetic field (or a neutron spin interacting with the Earth's rotation [8]), a photon propagating inside the curved optical fiber [9], and the time-dependent Jaynes-Cummings model describing the interaction of the two-level atom with a radiation field [10], geometric phase is often proportional to $2\pi(1-\cos\theta)$, which equals the solid angle subtended by the curve with respect to the origin of parameter space[1]. This, therefore, implies that geometric phase differs from dynamical phase and it involves global and topological information on the time evolution of quantum systems. In addition to this trigonometric geometric phase, there exists the so-called hyperbolical geometric phase that is expressed by $2\pi(1-\cosh\theta)$ with the hyperbolical cosine $\cosh\theta = \frac{1}{2}\left[\exp(\theta)+\exp(-\theta)\right]$ in some time-dependent quantum systems, *e.g.*, the two-level atomic system with electric dipole-dipole interaction and the harmonic-oscillator system [11]. It is verified that the generators of the Hamiltonians of these quantum systems form the $SU(1,1)$ Lie algebra. Further analysis indicates that quantum systems, which possess the non-compact Lie algebraic structure (whose group parameters can be taken to be infinity) will present the hyperbolical geometric phase, while quantum systems with compact Lie algebraic structure will give rise to the trigonometric geometric phase. Since Lorentz group (describing the boosts of reference frames in the space direction) in the special theory of relativity is also a non-compact group, this leads us to consider the topological properties associated with space-time. We take into account the gravitational analogue of magnetic charge [12], *i.e.*, gravitomagnetic charge that is the source of gravitomagnetic field just as the case that mass (gravitoelectric charge) is the source of gravitoelectric field (*i.e.*, Newtonian gravitational field in the sense of weak-field approximation). In this sense, gravitomagnetic charge is also termed dual mass. It should be noted that the concept of the ordinary mass is of no physical significance for the gravitomagnetic charge; it is of interest to investigate the relativistic dynamics and gravitational effects as well as geometric properties of this topological dual mass (should such exist). From the point of view of dif-

[1]The solid angle subtended by a curve shows the topological meanings of geometric phase. Geometric phase is absent in the quantum system when its Hamiltonian is independent of time. Geometric phase of many simple physical systems in the adiabatic process can be expressed in terms of the solid angle over the parameter space, *e.g.*, the wave function of a photon propagating inside the non-coplanar curved optical fiber obtains this topological phase, where the parameter space is just the momentum (*i.e.*, velocity) space of the photon.

ferential geometry, matter may be classified into two categories: gravitoelectric matter and gravitomagnetic matter. The former category possesses mass and constitutes the familiar physical world, while the latter possesses dual mass that would cause the non-analytical property of space-time metric. Einstein's field equation of gravitation in general theory of relativity governs the couplings of gravitoelectric matter (which possesses mass) to gravity (space-time); accordingly, we should have a field equation governing the interaction of dual matter with gravity. By making use of the variational principle, the gravitational field equation of gravitomagnetic matter can be obtained where the dual Einstein's tensor is denoted by $\frac{1}{2}\left(\epsilon_{\mu}{}^{\lambda\sigma\tau}\mathcal{R}_{\lambda\sigma\tau\nu}-\epsilon_{\nu}{}^{\lambda\sigma\tau}\mathcal{R}_{\lambda\sigma\tau\mu}\right)$ with $\epsilon_{\mu}{}^{\lambda\sigma\tau}$ and $\mathcal{R}_{\lambda\sigma\tau\nu}$ being the four-dimensional Levi-Civita completely antisymmetric tensor and the Riemann curvature tensor that describes the space-time curvature, respectively [13]. By exactly solving this field equation, one can show that the topological property of the solution $g_{t\varphi}(r,\theta)$ can be illustrated as follows: solid angle subtended by the curved C showing the topological property of the gravitomagnetic vector potential of a static gravitomagnetic charge at the origin of the spherical coordinate system. Take the gravitomagnetic vector potentials $\mathbf{g}=\left(0,0,\frac{2\mu}{4\pi}\cdot\frac{1-\cos\theta}{r\sin\theta}\right)$ in spherical coordinate system, then the loop integral, $\oint_{\mathrm{C}}\mathbf{g}\cdot\mathrm{d}\mathbf{l}=\mu(1-\cos\theta)$, is proportional to the solid angle subtended by the curved C with respect to the origin. The same situations arise in the adiabatic quantum geometric phase (Berry's quantum phase), which reflects the global or topological properties of time evolution (or parameters evolution) of quantum systems. Such property is in analogy with that of the geometric quantum phase in the time-dependent spin-gravity coupling (*i.e.*, the interaction between a spinning particle with gravitimagnetic field [8]) and other quantum adiabatic processes [4,9]. The topological properties of gravitomagnetic charge (dual mass) may be shown in terms of the global features of geometric quantum phase[2]. It follows that the expression of gravitomagnetic potential, $g_{t\varphi}(r,\theta)$, due to dual mass is exactly analogous to that of the trigonometric geometric phase. In the similar fashion, it is readily verified that the gravitomagnetic potential, $g_{\theta\varphi}(r,t)$, is similar to that of the hyperbolical geometric phase. This feature originates from the fact that the Lorentz group is a non-compact group. Although there is no evidence for the existence of this topological dual mass at present, it is still essential to consider this topological or global phenomenon in General Relativity. It is believed that there would exist formation (or creation) mechanism of gravitomagnetic charge in the gravitational interaction, just as some prevalent theories provide the theoretical mechanism of existence of magnetic monopole in various gauge interactions [14]. Magnetic monopole in electrodynamics and gauge field theory has been discussed and sought after for decades, and the existence of the 't Hooft-Polyakov monopole solution [14] has spurred new interest of both theorists and experimentalists [14]. As the topological gravitomagnetic charge in the curved space-time, dual mass is believed to give rise to such interesting situation similar to that of magnetic monopole. If it is indeed present in universe, dual mass will also lead to significant consequences in astrophysics and cosmology. We emphasize that although the gravitomagnetic vector potential produced by the gravitomagnetic charge is the classical solution to the field equation, this kind of topological gravitomagnetic monopoles may arise not as fundamental entities in gravity theory, *e.g.*, it will behave like a topological soliton. Gravitomagnetic charge has some interesting relativistic quantum gravitational effects [8], *e.g.*, the gravitational Meissner effect, which may serve as an interpretation of the smallness of the observed cosmological constant. In accordance with quantum field theory, vacuum possesses infinite zero-point energy density due to the vacuum quantum fluctuations; whereas according to Einstein's theory of General Relativity, infinite vacuum energy density yields the divergent curvature of space-time, namely, the space-time of vacuum is extremely curved. Apparently it is in contradiction with the practical fact, since it follows from experimental observations that the space-time of vacuum is asymptotically flat. In the context of quantum field theory a cosmological constant corresponds to the energy density associated with the vacuum and then the divergent cosmological constant may result from the infinite energy density of vacuum quantum fluctuations. However, a diverse set of observations suggests that the universe possesses a nonzero but very small cosmological constant [15]. How can we give a natural interpretation for the above paradox? Here, provided that vacuum matter is perfect fluid, which leads to the formal similarities between the weak-gravity equation in perfect fluid and the London's electrodynamics of superconductivity, we suggest a potential explanation by using the canceling mechanism via gravitational Meissner effect: the gravitoelectric field (Newtonian field of gravity) produced by the gravitoelectric charge (mass) of the vacuum quan-

[2] Gravitomagnetic moment results from the mass current, which is also the dynamical physical quantity. Comparison between gravitomagnetic charge and geometric phase enables to show the topological properties of the former. The reason why the topological property is important lies in that the global description of the physical phenomena is essential to understand the world. It is of interest that dual matter may constitute a dual world where dual mass abides by their own dynamical and gravitational laws, which is somewhat different from the laws in our world; for example, dual mass is acted upon by a gravitomagnetic Lorentz force in Newton's gravitational (gravitoelectric) field, and the static dual mass produces the gravitomagnetic field rather than the Newton's gravitoelectric field.

tum fluctuations is exactly canceled by the gravitoelectric field due to the induced current of the gravitomagnetic charge of the vacuum quantum fluctuations; the gravitomagnetic field produced by the gravitomagnetic charge (dual mass) of the vacuum quantum fluctuations is exactly canceled by the gravitomagnetic field due to the induced current of the gravitoelectric charge (mass current) of the vacuum quantum fluctuations. Thus, at least in the framework of weak-field approximation, the extreme space-time curvature of vacuum caused by the large amount of the vacuum energy does not arise, and the gravitational effects of cosmological constant is eliminated by the contributions of the gravitomagnetic charge (dual mass). If gravitational Meissner effect is of really physical significance, then it is necessary to apply this effect to the early universe where quantum and inflationary cosmologies dominate the evolution of the universe. Study of the geometric property in quantum regimes is an interesting and valuable direction. Since it reveals the global and topological properties of evolution of quantum systems, geometric phase has many applications in various branches of physics, say, in the coupling of neutron spin to the Earth's rotation [8], a potential application may be suggested where the information on the Earth's variations of rotating frequency will be obtained by measuring the geometric phase of the oppositely polarized neutrons through the neutron-gravity interferometer experiment. The topological charge in curved space-time also deserves further investigation, since it reflects plentiful global or geometric properties hidden in the gravity theory. It is believed that both theoretical and experimental interest in this direction may enable people to understand the global phenomena of the physical world better.

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